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TRANSPORT IN THERMAL EQUILIBRIUM, GAPLESS MODES, AND ANOMALIES*

by Jürg FRÖHLICH

1. Recollections of good, old times

When I was a young man fate was very generous towards me. I got my education as a theoretical physicist at ETH-Zürich, where I had excellent teachers. After graduating from ETH, I had the great opportunity to go out into the world. After a year at the University of Geneva I became a postdoc at Harvard, a little time after one of the golden ages of theoretical physics had begun. I then received an offer of an assistant professorship at the mathematics department of Princeton University. Of course, I was not and am not a pure mathematician. But, at Princeton, mathematical physics had jobs in the physics- *and* in the mathematics department – like at ETH – and I happened to get my job from the mathematicians (who, incidentally, appear to have played some rôle whenever I received a good job offer). By the way, one of the great things about IHÉS is that it does not have departments – it's unified!

That fate was very generous with me did, unfortunately, not prevent me from missing lots of opportunities. For example, I did not understand that, sometimes, it is better to spend time on learning something new from erudite colleagues, especially in an environment like Harvard or Princeton, rather than on writing too many not quite important papers. And I did not properly realize that it is during one's younger years that one ought to become a real professional in one's field of interest. I was very lucky in coming across a certain number of reasonably interesting problems from *physics* that, often in collaboration with some of my friends, could be solved *mathematically*, but without requiring broad, professional knowledge of mathematics; (this little note may be an example). – Unfortunately, there is no guarantee that this kind of luck persists in later years, when natural talent and creative juices are declining.

There were quite a few opportunities that I did *not* miss, though. I still remember an afternoon in the spring of 1977 when Pierre Deligne visited me in my office, on top of the

* This note was written while the author was visiting the School of Mathematics of the IAS at Princeton, NJ.

Fine Hall tower. He brought the mysterious message that the director of IHÉS, Professor Kuiper, would like to meet me at Princeton. We agreed on a date – and Niko Kuiper showed up at our apartment in the junior faculty ghetto, as promised. I remember that I was slightly disappointed when I first saw him. I had imagined that a mathematician directing the IHÉS would be tall and impressive looking and have worldly manners. – However, what Niko brought along was most impressive: an offer of a permanent position at IHÉS! My wife and I were dreaming. When we were assured that this offer was not just a dream, but was real we asked some friends to take care of our two daughters; and off we went for a splendid week-end in New York City. – I did not miss *this* opportunity, namely to accept the offer from IHÉS, after some polite hesitation.

Of course, the offer from IHÉS had something to do with the circumstance that one of my teachers, supporters and friends from ETH, Res Jost, was a member of the "comité scientifique" at that time. Sometimes one has got to have good friends and to be lucky.

In January 1978, perhaps the happiest period of my adult life started: We moved to Bures-sur-Yvette! Life at Bures was simple and pleasant. The atmosphere at the institute was inspiring, yet quite intimate and human. My research efforts were reasonably successful – I maintained close ties to my friends from Princeton days. And Paris is only a forty minutes RER-ride away.

Had I not missed some of those opportunities and developed a little more self-confidence, we would most probably not have left Bures, anymore, and certainly not after just four and a half years. As one grows older and a little more detached from the successes and failures of the day, life at an institute like IHÉS would certainly feel good; (such institutes are great for young researchers at the peak of their creativity and for elder statesmen, but not great for people in between). – Perhaps, the best things in life are not meant to last for ever.

At IHÉS, I had delightful colleagues. Some of them are married and have wonderful wives, who greatly enriched social life at Bures. It was natural that we tried to contribute our share, and we liked to do so.

I was collaborating with various colleagues; among others, with David Brydges, Erhard Seiler and, in particular, with Tom Spencer. (I found it somewhat difficult to get in closer touch with French colleagues from neighboring institutions.) Some of our joint work of that period was, I believe, pretty good.

The IHÉS is unique as an institution where mathematical and theoretical physicists, on one hand, and pure mathematicians, on the other hand, meet each other daily, at lunch, at tea, in the discussion room, during games of Volley ball, in seminars. They discuss science together in a relaxed atmosphere – to find out how different their perspectives and their way of thinking tend to be and how difficult it is to understand each other; (Pierre likes to complain that, not only do physicists usually not prove their claims, but they fail to come up with precise definitions). But we have all learnt that, under favorable circumstances, such a dialogue can be tremendously inspiring and fruitful! I have made the experience personally, at IHÉS

It is with thankfulness that I remember the great times at Bures, and it is with gratitude that I contribute this little note to the "Festschrift" commemorating the 40^{th} anniversary of IHÉS This institute has been a focal point of extraordinary mathematical creativity. I cannot think of a more successful institute of fundamental research. I wish the IHÉS a vigorous continuation of its unique success story into the future – and generous support!

2. Quantized conductances (1)

The original motivation of the work described in this note has been to provide simple and conceptually clear explanations of various formulae for quantized conductances, which have been encountered in the analysis of experimental data. Here are some typical examples.

Example 1. Consider a quantum Hall device with, e.g., an annular (Corbino) geometry. Let V denote the voltage drop in the radial direction, between the inner and the outer edge, and let I_H denote the total Hall current in the azimuthal direction. The Hall conductance, G_H, is defined by

$$G_H = I_H/V. \tag{2.1}$$

One finds that *if the longitudinal resistance vanishes* (i.e., if the two-dimensional electron gas in the device is "incompressible") then G_H is a *rational multiple* of $\frac{e^2}{h}$, i.e.,

$$G_H = \frac{n}{d} \cdot \frac{e^2}{h}, \quad n = 0,1,2,..., \quad d = 1,2,3,... \tag{2.2}$$

In (2.2), e denotes the elementary electric charge and h denotes Planck's constant. Well established Hall fractions, $\sigma_H = \frac{n}{d}$, in the range $0 < \sigma_H \leqslant 1$ are listed in Fig. 1; (see [1]; and [2, 3] for general background).

Example 2. In a ballistic (quantum) wire, i.e., in a pure, very thin wire without back scattering centers, one finds that the conductance $G_W = I/V$ (I: current through the wire, V: voltage drop between the two ends of the wire) is given by

$$G_W = 2N\frac{e^2}{h}, \quad N = 0,1,2,..., \tag{2.3}$$

under suitable experimental conditions ("small" V, temperature not "very small", "adiabatic gates"); see [4, 5].

Example 3. In measurements of *heat conduction* in quantum wires, one finds that the heat current is an *integer multiple* of a "fundamental" current which depends on the temperatures of the two heat reservoirs at the ends of the wire.

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(1) The material sketched in this note is the result of collaboration with A. Alekseev and V. Cheianov [0], in continuation of earlier work with T. Kerler, U. Studer and E. Thiran [1].

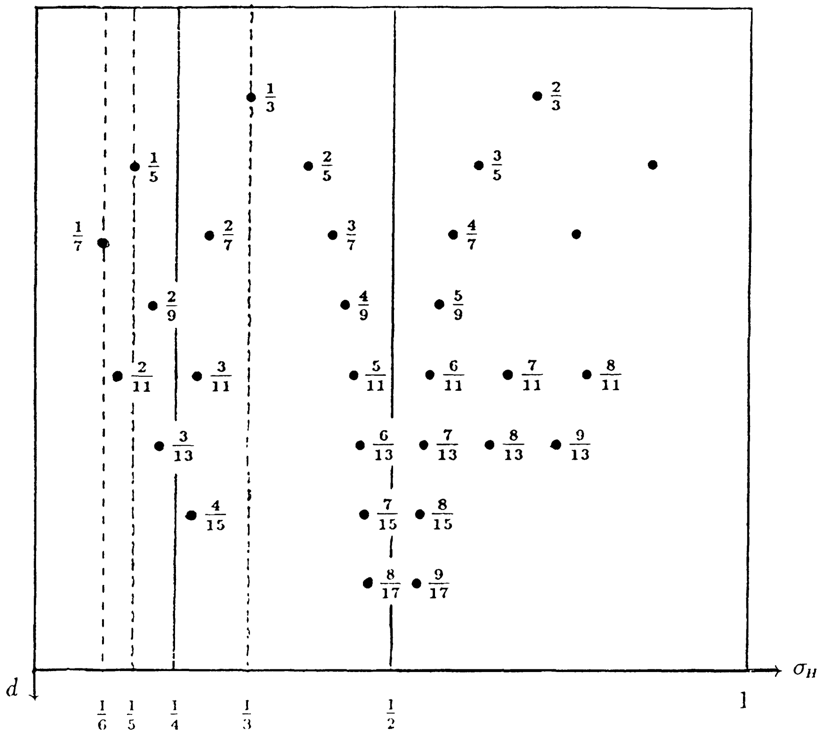

Fig. 1

If electromagnetic waves are sent through an "adiabatic hole" connecting two half-spaces one approximately finds an "integer quantization" of electromagnetic energy flux.

Our task is to attempt to provide a theoretical explanation of these remarkable experimental discoveries; hopefully one that enables us to predict further related effects.

Conductance quantization is observed in a rather wide temperature range. It appears that it is only found in systems without dissipative processes. When it is observed it is insensitive to small changes in the parameters specifying the system and to details of sample preparation; i.e., it has *universality properties.* – It will turn out that the key feature of systems exhibiting conductance quantization is that they have *conserved chiral charges*; (such conservation laws will only hold approximately, i.e., in slightly idealized systems). Once one has understood this point, the right formulae follow almost automatically, and one arrives at natural generalizations.

3. Transport in thermal equilibrium through gapless modes

In this section we prepare the ground for a theoretical explanation of the effects described in Sect. 2. We consider a quantum-mechanical system $\mathcal{S}$ whose dynamics is determined by a Hamiltonian H, which is a selfadjoint operator on the Hilbert space $\mathcal{H}$ of pure state vectors of $\mathcal{S}$ with discrete energy spectrum. It is assumed that the system obeys conservation laws described by some conserved "charges" $N_1, \ldots, N_L$. Hence

$$[H, N_l] = 0, \quad [N_l, N_k] = 0, \quad l, k = 1, \ldots, L, \tag{3.1}$$

(e.g. in the sense that the spectral projections of H and of N_l, N_k commute with each other, for all k and l). The system $\mathcal{S}$ is coupled to L reservoirs, $\mathcal{R}_1, \ldots, \mathcal{R}_L$, with the property that the expectation value of the conserved charge N_l in a stationary state of $\mathcal{S}$ can be tuned to some fixed value through exchange of "quasi-particles" between $\mathcal{S}$ and $\mathcal{R}_l$, i.e., through a current between $\mathcal{S}$ and $\mathcal{R}_l$ that carries "N_l-charge", for all $l = 1, \ldots, L$.

We are interested in describing a thermal equilibrium state of $\mathcal{S}$, coupled to $\mathcal{R}_1, \ldots, \mathcal{R}_L$, at a temperature $T = (k_B \beta)^{-1}$. According to Gibbs, we should work in the grand-canonical ensemble. The reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_L$ then enter the description of the thermal equilibrium of $\mathcal{S}$ only through their *chemical potentials* $\mu_1, \ldots, \mu_L$. The chemical potential μ_l is a thermodynamic parameter canonically conjugate to the charge N_l; in particular, the dimension of $\mu_l \cdot N_l$ is that of an energy. According to Landau and von Neumann, the thermal equilibrium state of $\mathcal{S}$ at temperature $(k_B \beta)^{-1}$ in the grand-canonical ensemble, with fixed values of $\mu_1, \ldots, \mu_L$, is given by the density matrix

$$\rho_{\beta, \underline{\mu}} = \Xi_{\beta, \underline{\mu}}^{-1} \exp\left[-\beta\left(H - \sum_{l=1}^{L} \mu_l N_l\right)\right], \tag{3.2}$$

where the grand partition function $\Xi_{\beta, \underline{\mu}}$ is determined by the requirement that

$$\operatorname{tr} \rho_{\beta, \underline{\mu}} = 1. \tag{3.3}$$

(It is assumed here that $\exp\left[-\beta\left(H - \Sigma \mu_l N_l\right)\right]$ is a trace-class operator on $\mathcal{H}$, for all $\beta > 0$; we are studying a system in a compact region of physical space.) The equilibrium expectation of a bounded operator, a, on $\mathcal{H}$ is defined by

$$\langle a \rangle_{\beta, \underline{\mu}} := \operatorname{tr}\left(\rho_{\beta, \underline{\mu}}\, a\right). \tag{3.4}$$

Let $\mathcal{J}(x) = \left(\mathcal{J}^0(x), \vec{\mathcal{J}}(x)\right)$ be a conserved current density of $\mathcal{S}$, where $x = (\underline{x}, t)$, t is time and $\underline{x}$ is a point of physical space contained inside $\mathcal{S}$. We are interested in calculating the expectation values of products of components of $\mathcal{J}$ in the state $\rho_{\beta, \underline{\mu}}$; in particular, we should like to calculate $\langle \vec{\mathcal{J}}(x) \rangle_{\beta, \underline{\mu}}$. Of course, if the dimension of space is larger than one $\langle \vec{\mathcal{J}}(x) \rangle_{\beta, \underline{\mu}}$ vanishes unless rotation invariance is broken by some external field. If $\vec{\mathcal{J}}(x)$ is a vector current then $\langle \vec{\mathcal{J}}(x) \rangle_{\beta, \underline{\mu}}$ vanishes unless the state $\rho_{\beta, \underline{\mu}}$ is *not* invariant under space-reflection and time reversal. This happens if some of the charges $N_1, \ldots, N_L$ are not invariant under space-reflection and time reversal, i.e., if they are *chiral.*

To say that $\mathcal{J}$ is conserved means that it satisfies the continuity equation

$$\partial_\mu \mathcal{J}^\mu = 0, \tag{3.5}$$

where $x^0 = t$ denotes time, and $\partial_\mu = \partial/\partial x^\mu$. If the space-time of the system $\mathcal{S}$ is topologically trivial ("star-shaped") then eq. (3.5) implies that there is a globally defined vector field $\vec{\phi}(x)$ such that

$$\mathcal{J}^0(x) = e \operatorname{div} \vec{\phi}(x), \quad \vec{\mathcal{J}}(x) = -e \frac{\partial}{\partial t} \vec{\phi}(x). \tag{3.6}$$

Let us suppose that $\vec{\phi}(x)$ is an operator-valued distribution on $\mathcal{H}$, whose time-dependence is determined by the formal Heisenberg equation

$$\frac{\partial}{\partial t} \vec{\phi}(x) = \frac{i}{\hbar} \left[\mathrm{H}, \vec{\phi}(x)\right]. \tag{3.7}$$

[Technically, we are treading on somewhat slippery ground here; but we shall proceed formally, in order to explain the key ideas on a few pages.] From (3.6) and (3.7) we derive that

$$\langle \vec{\mathcal{J}}(x) \rangle_{\beta, \underline{\mu}} = \frac{ie}{\hbar} \langle [\mathrm{H}, \vec{\phi}(x)] \rangle_{\beta, \underline{\mu}}. \tag{3.8}$$

Formally, the r.s. of (3.8) *vanishes*, because $\langle (\cdot) \rangle_{\beta, \underline{\mu}}$ is a time-translation invariant state. However, the field $\vec{\phi}$ turns out to have ill-defined *zero-modes*, and it is not legitimate to pretend that $[\mathrm{H}, \vec{\phi}(x)] = \mathrm{H}\vec{\phi}(x) - \vec{\phi}(x)\mathrm{H}$, because both terms on the r.s. are divergent, due to the zero-modes of $\vec{\phi}$. What *is* legitimate is to claim that

$$\frac{\partial}{\partial t} \vec{\phi}(x) = \frac{i}{\hbar} \left[\mathrm{H} - \sum_{l=1}^{\mathrm{L}} \mu_l \mathrm{N}_l, \vec{\phi}(x)\right] + \frac{i}{\hbar} \sum_{l=1}^{\mathrm{L}} \mu_l \left[\mathrm{N}_l, \vec{\phi}(x)\right], \tag{3.9}$$

and that the expectation value

$$\left\langle \left[\mathrm{H} - \sum_{l=1}^{\mathrm{L}} \mu_l \mathrm{N}_l, \vec{\phi}(x)\right] \right\rangle_{\beta, \underline{\mu}}$$

vanishes. This can be seen by replacing the Hamiltonian H by a *regularized* Hamiltonian $\mathrm{H}^{(\varepsilon)}$ generating a dynamics that eliminates the zero-modes of $\vec{\phi}$. One replaces the state $\rho_{\beta, \underline{\mu}}$ by a regularized state $\rho^{(\varepsilon)}_{\beta, \underline{\mu}}$ proportional to $\exp\left[-\beta\left(\mathrm{H}^{(\varepsilon)} - \Sigma\, \mu_l \mathrm{N}_l\right)\right]$, and we set

$$\langle a \rangle^{(\varepsilon)}_{\beta, \underline{\mu}} := \operatorname{tr}\left(\rho^{(\varepsilon)}_{\beta, \underline{\mu}}\, a\right),$$

for any bounded operator a on $\mathcal{H}$. Then

$$\begin{aligned} \langle \vec{\mathcal{J}}(x) \rangle_{\beta, \underline{\mu}} &= \lim_{\varepsilon \to 0} \frac{ie}{\hbar} \left\langle \left[\mathrm{H}^{(\varepsilon)} - \sum_{l=1}^{\mathrm{L}} \mu_l \mathrm{N}_l, \vec{\phi}(x)\right] \right\rangle^{(\varepsilon)}_{\beta, \underline{\mu}} \\ &+ \lim_{\varepsilon \to 0} \sum_{l=1}^{\mathrm{L}} \frac{ie\mu_l}{\hbar} \langle [\mathrm{N}_l, \vec{\phi}(x)] \rangle^{(\varepsilon)}_{\beta, \underline{\mu}}. \end{aligned} \tag{3.10}$$

Obviously

$$\left\langle \left[\mathrm{H}^{(\varepsilon)} - \sum_{l=1}^{\mathrm{L}} \mu_l \mathrm{N}_l, \vec{\phi}(x) \right] \right\rangle_{\beta, \underline{\mu}}^{(\varepsilon)} = 0, \tag{3.11}$$

and one might be tempted to expect that $\lim_{\varepsilon \to 0} \langle [\mathrm{N}_l, \vec{\phi}(x)] \rangle_{\beta, \underline{\mu}}^{(\varepsilon)}$ vanishes, for all l, because the charges N_l are conserved. However, as long as the regularization is present ($\varepsilon \neq 0$), these charges are *not* conserved, and there is no guarantee that the second term on the r.s. of (3.10) vanishes!

We conclude that

$$\begin{aligned} \langle \vec{\mathcal{J}}(x) \rangle_{\beta, \underline{\mu}} &= \lim_{\varepsilon \to 0} \sum_{l=1}^{\mathrm{L}} \frac{ie\mu_l}{\hbar} \langle [\mathrm{N}_l, \vec{\phi}(x)] \rangle_{\beta, \underline{\mu}}^{(\varepsilon)} \\ &=: \sum_{l=1}^{\mathrm{L}} \frac{ie\mu_l}{\hbar} \langle [\mathrm{N}_l, \vec{\phi}(x)] \rangle_{\beta, \underline{\mu}}. \end{aligned} \tag{3.12}$$

Eq. (3.12) might be called a *current sum rule.*

Let us assume that the conserved charges N_l, $l = 1, 2, \ldots$, are given as integrals of the 0-components of conserved currents over space. Then the current sum rule (3.12) implies that if $\langle \vec{\mathcal{J}}(x) \rangle_{\beta, \underline{\mu}} \neq 0$ there must be *gapless modes* in the system. The proof (see [0]) is analogous to the proof of the *Goldstone theorem* in the theory of broken continuous symmetries. [We omit a more detailed discussion.]

The sum rule (3.12) is the main result of this section. A careful derivation of equation (3.12) and of our analogue of the Goldstone theorem could be given by using the *operator-algebra approach* to quantum statistical mechanics [6], which, under the good influence of David Ruelle, had a strong presence at IHÉS, during the sixties, and, mainly through the fundamental work of Alain Connes, has made inroads into pure mathematics. But, in order to reach our punch line on a reasonable number of pages, we refrain from entering into a careful technical discussion.

Instead, we turn to a story that may remind Louis Michel of some good old times: *current algebra,* and...

4. Anomalous commutators

We start by considering a one-dimensional system of chiral fermions, e.g. an electron liquid in a quantum wire. The system has a conserved vector current $\mathcal{J}$, with

$$\partial_\mu \mathcal{J}^\mu = 0. \tag{4.1}$$

Since the fermions are assumed to be chiral, there must also exist a conserved pseudo-vector (axial) current, $\mathcal{J}_a^\mu$, with

$$\partial_\mu \mathcal{J}_a^\mu = 0. \tag{4.2}$$

In 1+1 space-time dimensions, $\mathcal{J}^\mu$ and $\mathcal{J}_a^\mu$ are related to each other by

$$\mathcal{J}_a^\mu = \varepsilon^{\mu\nu} \mathcal{J}_\nu, \tag{4.3}$$

where $\varepsilon^{00} = \varepsilon^{11} = 0,\ \varepsilon^{01} = -\varepsilon^{10} = 1$. As explained after eq. (3.6), eq. (4.1) can be solved by introducing a scalar field $\varphi(x)$ such that

$$\mathcal{J}^\mu(x) = \frac{q}{2\pi} \varepsilon^{\mu\nu} (\partial_\nu \varphi)(x), \tag{4.4}$$

where q is the charge of a fermion; ($q = -e$, where e is the elementary electric charge, for ordinary electrons). Eq. (4.3) then implies that

$$\mathcal{J}_a^\mu(x) = -\frac{q}{2\pi} (\partial^\mu \varphi)(x), \tag{4.5}$$

and, consequently, (4.2) yields

$$\Box \varphi(x) = 0. \tag{4.6}$$

Thus, if the vector- and the pseudo-vector currents are conserved then the potential φ of the vector current is a *massless free field*. This is an example of a Lagrangian field theory. It has an action functional, S, given by

$$S(\varphi) = \frac{1}{4\pi} \int (\partial_\mu \varphi)(x) (\partial^\mu \varphi)(x)\, d^2x. \tag{4.7}$$

As usual, we define the momentum, $\pi(x)$, canonically conjugate to $\varphi(x)$ by

$$\pi(x) = \delta S(\varphi)/\delta(\partial_0 \varphi(x)) = \frac{1}{2\pi} \frac{\partial}{\partial t} \varphi(x). \tag{4.8}$$

After quantization, φ and π become operator-valued distributions satisfying the equal-time canonical commutation relations

$$\left[\pi(\underline{x}, t), \varphi\left(\underline{y}, t\right)\right] = -i\delta\left(\underline{x} - \underline{y}\right). \tag{4.9}$$

In view of (4.4), (4.5) and (4.8), we find that

$$\left[\mathcal{J}^0(\underline{x}, t), \varphi\left(\underline{y}, t\right)\right] = 0, \tag{4.10}$$

but

$$\left[\mathcal{J}_a^0(\underline{x}, t), \varphi\left(\underline{y}, t\right)\right] = iq\delta\left(\underline{x} - \underline{y}\right). \tag{4.11}$$

Using that $\mathcal{J}^0(x) = q(\partial_{\underline{x}}\varphi)(x)$, we see that eq. (4.11) is equivalent to the well known *anomalous commutator*

$$\left[\mathcal{J}^0(\underline{x}, t), \mathcal{J}_a^0(\underline{y}, t)\right] = i\frac{q^2}{2\pi} \delta'\left(\underline{x} - \underline{y}\right). \tag{4.12}$$

It may be useful to recall how eqs. (4.3) through (4.7) are related to the *chiral anomaly* in two space-time dimensions. We imagine that the system is put into an external electric field $E(x)$. In two dimensions, the electromagnetic vector potential $A_\mu(x)$ (in the Coulomb gauge) is related to $E(x)$ by

$$\left(\frac{\partial}{\partial t} A_1\right)(x) = E(x), \quad A_0(x) = 0. \tag{4.13}$$

The action functional $S(\varphi)$ defined in eq. (4.7) must now be replaced by

$$\begin{aligned} S(\varphi, A) &= \frac{1}{4\pi}\int (\partial_\mu\varphi)(x)\,(\partial^\mu\varphi)(x)\,d^2x + \int \mathcal{J}^\mu(x)\,A_\mu(x)\,d^2x \\ &= \frac{1}{4\pi}\int \left\{(\partial_\mu\varphi)(x)\,(\partial^\mu\varphi)(x) - 2\,q\,\varepsilon^{\mu\nu}\,(\partial_\nu\varphi)(x)\,A_\mu(x)\right\}\,d^2x\,. \end{aligned} \tag{4.14}$$

The field (Euler-Lagrange) equation derived from (4.14) is

$$\Box\,\varphi(x) = q\,E(x). \tag{4.15}$$

The standard equation for the *chiral anomaly* in two space-time dimensions is

$$\partial_\mu\,\mathcal{J}^\mu_a(x) = \frac{q^2}{2\pi}\,E(x). \tag{4.16}$$

Equations (4.15) and (4.16) are *equivalent* if and only if

$$\mathcal{J}^\mu_a = \varepsilon^{\mu\nu}\,\mathcal{J}_\nu = \frac{q}{2\pi}\,\partial^\mu\,\varphi\,(x),$$

which is equation (4.3), (and eqs. (4.1) through (4.3) ultimately lead to (4.12)).

Apparently, according to (4.16), the usual pseudo-vector current $\mathcal{J}^\mu_a$ of a system of chiral fermions in two space-time dimensions fails to be conserved when the system is put into an external electric field. However, one may define a modified pseudo-vector current, $\widehat{\mathcal{J}}^\mu_a$, by setting

$$\widehat{\mathcal{J}}^\mu_a(x) := \mathcal{J}^\mu_a(x) - \frac{q^2}{2\pi}\,\varepsilon^{\mu\nu}\,A_\nu(x). \tag{4.17}$$

Then

$$\partial_\mu\,\widehat{\mathcal{J}}^\mu_a = 0,$$

but $\widehat{\mathcal{J}}^\mu_a$ fails to be gauge-invariant. Nevertheless, the *conserved charge* associated with $\widehat{\mathcal{J}}^\mu_a$ *is* gauge-invariant. We notice that the anomalous commutator between $\mathcal{J}^0$ and $\widehat{\mathcal{J}}^0_a$ is still given by (4.12).

From the conserved currents $\mathcal{J}^\mu$ and $\widehat{\mathcal{J}}^\mu_a$ we can construct two conserved charges

$$N_l := \frac{1}{2}\int \left(\mathcal{J}^0\,(\underline{x}, t) - \widehat{\mathcal{J}}^0_a\,(\underline{x}, t)\right)\,d\underline{x}, \tag{4.18}$$

$$N_r := \frac{1}{2}\int \left(\mathcal{J}^0\,(\underline{x}, t) + \widehat{\mathcal{J}}^0_a\,(\underline{x}, t)\right)\,d\underline{x}, \tag{4.19}$$

measuring the total electric charge of "left-moving" – and of "right-moving" modes, respectively. Moreover, normal-ordered exponentials of integrals of $\mathcal{J}^0 \pm \widehat{\mathcal{J}}_a^0$, i.e., vertex operators, represent left- or right-moving *charged fields.*

There is a similar story about anomalous commutators and their relation to the chiral anomaly in four- and higher-even-dimensional space times. It dates from the late sixties and early seventies. An excellent reference is [7]. We just summarize the main results.

We consider a relativistic quantum theory of charged, massless fermions. Such a theory is expected to have a conserved vector current, $\mathcal{J}^\mu$, and a conserved axial current, $\mathcal{J}_a^\mu$. Suppose that the fermions are now coupled to an external electromagnetic field with field tensor $\mathrm{F} = (\mathrm{F}_{\mu\nu})$. Then if $\mathcal{J}_a^\mu$ is chosen to be gauge-invariant it is not conserved:

$$\partial_\mu \mathcal{J}_a^\mu = \frac{e^2}{16\pi^2} \mathrm{F}_{\mu\nu} \widetilde{\mathrm{F}}^{\mu\nu}, \tag{4.20}$$

where $\widetilde{\mathrm{F}}$ is the *dual* field tensor. One says that the axial current has an "anomalous divergence" (*chiral anomaly*); while

$$\partial_\mu \mathcal{J}^\mu = 0. \tag{4.21}$$

See [7] for a discussion and derivation of (4.20).

Introducing a *non-gauge-invariant* axial current, $\widehat{\mathcal{J}}_a^\mu$, by setting

$$\widehat{\mathcal{J}}_a^\mu := \mathcal{J}_a^\mu - \frac{e^2}{16\pi^2} \varepsilon^{\mu\nu\alpha\beta} \mathrm{A}_\nu \partial_\alpha \mathrm{A}_\beta, \tag{4.22}$$

one finds that

$$\partial_\mu \widehat{\mathcal{J}}_a^\mu = 0; \tag{4.23}$$

but the price to be paid is a loss of gauge-invariance!

The second term on the r.s. of (4.22) is dual to the *Chern-Simons 3-form,* which plays a fundamental rôle in the theory of secondary characteristic classes and differential characters. Mathematically, eq. (4.20) is related to the fact that the argument of the determinant

$$\text{“}\det\left(i\, \not{D}_{\mathrm{A}} \frac{1-\gamma_5}{2}\right)\text{”},$$

where $\not{D}_{\mathrm{A}}$ is the covariant Dirac operator, is not gauge-invariant. Its variation under gauge transformations is given by the gauge variation of the five-dimensional Chern-Simons form integrated over a five-dimensional manifold whose boundary is the four-dimensional space-time of the system considered here. Eq. (4.20) is also related to the *anomalous commutators*

$$\left[\mathcal{J}^0(\underline{x}, t), \widehat{\mathcal{J}}_a^0(\underline{y}, t)\right] = \frac{ie}{4\pi^2} \vec{\mathrm{B}}(\underline{y}, t) \cdot \left(\vec{\nabla}_x \delta\right)(\underline{x} - \underline{y}), \tag{4.24}$$

$$\left[\mathcal{J}^0(\underline{x}, t), \widehat{\mathcal{J}}_a^j(\underline{y}, t)\right] = \frac{ie}{8\pi^2} \widetilde{\mathrm{F}}^{jk}(\underline{y}, t) \cdot (\partial_k \delta)(\underline{x} - \underline{y}). \tag{4.25}$$

See [7] for a derivation of (4.24), (4.25).

Next, we recall that $\mathcal{J}^0(\underline{x}, t) = \text{div}\, \vec{\phi}(\underline{x}, t)$, where $\vec{\phi}$ is the "vector potential" of $\mathcal{J}^\mu$ introduced in the last section. Then eq. (4.24) and the fact that $\text{div}\, \vec{B} = 0$ imply that

$$\left[\vec{\phi}(\underline{x}, t), \widehat{\mathcal{J}}_a^0(\underline{y}, t)\right] = \frac{ie}{4\pi^2}\, \vec{B}\left(\underline{y}, t\right)\, \delta\left(\underline{x} - \underline{y}\right) + \text{curl}\, \vec{\Pi}\left(\underline{x} - \underline{y}, t\right), \tag{4.26}$$

for some vector field $\vec{\Pi}$. [The reasoning leading to (4.26) outlined here is somewhat cavalier, but our main application of (4.26) will be correct, nevertheless!]

In the next section we shall use the theory of the chiral anomaly and anomalous commutators, together with the general results of Sect. 3, to develop a general *"theory of transport in thermal equilibrium"*. It is worth mentioning that, besides the chiral anomaly, one can also consider the gravitational-, the conformal- and mixed anomalies and work out their consequences for our general theory of transport in thermal equilibrium; but we shall not pursue this idea here.

5. What ballistic wires and stars may have in common

In this section, we combine the results of Sects. 3 and 4 to arrive at physically interesting statements.

We start by considering a ballistic wire, i.e., a very thin, clean one-dimensional conductor without back scattering centers. The ends of the wire are connected to two reservoirs at chemical potentials μ_l and μ_r, respectively, with

$$\mu_l - \mu_r = V, \tag{5.1}$$

where V is the voltage drop.

A real wire is a three-dimensional object with a tiny cross section in the two directions transversal to its axis. For simplicity, we may assume that the wire is arbitrarily long. At low enough temperature, the three-dimensional nature of a real wire can be retained in a theoretical description by saying that the electrons in the wire form several *independent, but interacting Luttinger liquids* [8], e.g. K such liquids, each of which has a *conserved vector current*, $\mathcal{J}_i^\mu$, and a *conserved axial current*, $\widehat{\mathcal{J}}_{a,\,i}^\mu$ $i = 1, \ldots, K$. The charge of an elementary quasi-particle in the i^{th} liquid is given by

$$q_i = Q_i\, e, \tag{5.2}$$

where Q_i is some real number, and $Q_i = -1$ if the quasi-particle is an *electron*. The total electric current operator, $\mathcal{J}^\mu$, is given by

$$\mathcal{J}^\mu = \sum_{i=1}^{K} \mathcal{J}_i^\mu, \tag{5.3}$$

and the total axial current operator, $\mathcal{J}_a^{\mu}$, by

$$\widehat{\mathcal{J}}_a^{\mu} = \sum_{i=1}^{K} \widehat{\mathcal{J}}_{a,\,i}^{\mu}\,. \tag{5.4}$$

The total electric charge operator of the left- (right-) moving modes in the wire is the operator N_l (N_r) introduced in eq. (4.18) (eq. (4.19)). Its expectation value in a thermal equilibrium state of the wire is tuned by the chemical potential μ_l (μ_r) of the reservoir at the right (left) end of the wire. We suppose that the wire is kept at an inverse temperature β. [Our description captures the basic physics of a ballistic wire only if β^{-1} and eV are very small compared to an intrinsic energy scale of the wire.] Our goal is to calculate the total electric current through the wire, given the potential drop V.

The basic current sum rule (3.12) says that

$$\begin{aligned} I &= \langle \mathcal{J}^1(x) \rangle_{\beta,\,\underline{\mu}} \\ &= \frac{ie}{\hbar} \left\{ \mu_l \langle\, [N_l, \varphi(x)]\, \rangle_{\beta,\,\underline{\mu}} + \mu_r \langle\, [N_r, \varphi(x)]\, \rangle_{\beta,\,\underline{\mu}} \right\}, \end{aligned} \tag{5.5}$$

where φ is the potential of the current $\mathcal{J}^{\mu}$. Since all the currents $\mathcal{J}_i^{\mu}$, $i = 1, \ldots, K$, are conserved, every one of them can be derived from a potential, and

$$\varphi = \sum_{i=1}^{K} Q_i\, \varphi_i\,, \text{ where } \frac{e}{2\pi}\, Q_i\, \varepsilon^{\mu\nu}\, \partial_\nu\, \varphi_i = \mathcal{J}_i^{\mu}. \tag{5.6}$$

Using eqs. (5.6), (5.5), (4.18), (4.19) and the key equation (4.11) for the anomalous commutator, we find that

$$\begin{aligned} I &= \sum_{i=1}^{K} \langle \mathcal{J}_i^1(x) \rangle_{\beta,\,\underline{\mu}} \\ &= \sum_{i=1}^{K} \frac{ie\,Q_i}{h} \left\{ \mu_l \langle\, [N_{l,\,i}, \varphi_i(x)]\, \rangle_{\beta,\,\underline{\mu}} + \mu_r \langle\, [N_{r,\,i}, \varphi_i(x)]\, \rangle_{\beta,\,\underline{\mu}} \right\} \\ &= \frac{e^2}{h} \left(\sum_{i=1}^{K} Q_i^2 \right) (\mu_l - \mu_r) \\ &= \frac{e^2}{h} \left(\sum_{i=1}^{K} Q_i^2 \right) V, \end{aligned} \tag{5.7}$$

where we have used that

$$\left[N_{l,\,i}, \varphi_j(x) \right] = -\, ie\, Q_i\, \delta_{ij}, \tag{5.8}$$

and

$$\left[N_{r,\,i}, \varphi_j(x) \right] = ie\, Q_i\, \delta_{ij}. \tag{5.9}$$

Thus we have derived the formula

$$G_W = \frac{e^2}{h}\left(\sum_{i=1}^{K} Q_i^2\right). \tag{5.10}$$

In a conventional ballistic wire, all the quasi-particles are *electrons.* Hence

$$Q_i = -1, \quad i = 1, \ldots, K. \tag{5.11}$$

Moreover, every electron state in the wire is doubly degenerate, because of *electron spin,* (assuming the wire is not in a magnetic field). Thus, conserved currents come in pairs (spin up and spin down), so that K is *even,* i.e.,

$$K = 2N, \quad N = 0, 1, 2, \ldots \tag{5.12}$$

In conclusion,

$$G_W = 2N\,\frac{e^2}{h}, \quad N = 0, 1, 2, \ldots \tag{5.13}$$

This is what was claimed in eq. (2.3).

The theory of the *Hall conductance* of an incompressible Hall fluid in a Hall sample with e.g. the Corbino geometry is very similar to the theory of ballistic wires outlined above, with the following differences. Let V denote the potential drop between the outer and the inner edge. We assume that eV and $k_B T$ are very small compared to $\hbar\Omega_C$, where Ω_C is the cyclotron frequency. Under these conditions, the Hall current in an incompressible Hall fluid with the Corbino geometry is the sum of the diamagnetic edge currents at the two edges of the sample. These edge currents are *chiral.* If we define the left and the right currents by

$$\mathcal{J}_l^\mu := \frac{1}{2}\left(\mathcal{J}^\mu - \widehat{\mathcal{J}}_a^\mu\right), \; \mathcal{J}_r^\mu := \frac{1}{2}\left(\mathcal{J}^\mu + \widehat{\mathcal{J}}_a^\mu\right) \tag{5.14}$$

then $\mathcal{J}_l^\mu$ is localized at one edge and $\mathcal{J}_r^\mu$ is localized at the other edge. The two edges are macroscopically separated, and the probability for tunnelling of charges from one edge to the other one vanishes for all practical purposes. The conservation of the charges N_l and N_r is therefore valid with very high accuracy.

As in the example of the ballistic wire, we make the ansatz

$$\mathcal{J}_l^\mu = \sum_{i=1}^{K} \mathcal{J}_{l,i}^\mu, \quad \mathcal{J}_r^\mu = \sum_{i=1}^{K} \mathcal{J}_{r,i}^\mu, \tag{5.15}$$

with

$$\mathcal{J}_{l,i}^\mu = \frac{e}{4\pi} Q_i \left(\varepsilon^{\mu\nu}\partial_\nu \varphi_i + \partial^\mu \varphi_i\right), \tag{5.16}$$

and similarly for $\mathcal{J}^{\mu}_{r,\,i}$. Calculations very similar to the ones shown above yield the equation

$$
\begin{aligned}
I_H &= \sum_{i=1}^{K} \langle \mathcal{J}^1_{l,\,i}(\theta) + \mathcal{J}^1_{r,\,i}(\theta) \rangle_{\beta,\,\underline{\mu}} \\
&= \frac{e^2}{h} \left(\sum_{i=1}^{K} Q_i^2 \right) (\mu_l - \mu_r) \\
&= \frac{e^2}{h} \left(\sum_{i=1}^{K} Q_i^2 \right) V,
\end{aligned}
\tag{5.17}
$$

where μ_l and μ_r are the chemical potentials of electrons at the outer edge and inner edge, respectively; (the variable θ is an angular variable parametrizing the outer *and* the inner edge).

It is notorious that incompressible Hall fluids may exhibit quasi-particles (at the edges) of fractional charge and fractional statistics. Therefore one cannot claim that $Q_i = \pm 1$, for all i. In ref. [1], a *connection between electric charge and quantum statistics* ({electric charge = even (odd) integer multiple of e} $\Longleftrightarrow$ {quantum statistics = Bose (Fermi) statistics}) has enabled the authors to prove that $Q := (Q_1, \ldots, Q_K)$ must be a *visible vector in a lattice dual to an odd-integral lattice of rank* K. It follows from this result that

$$
\sigma_H := \sum_{i=1}^{K} Q_i^2 \quad \underline{\text{is a rational number}}. \tag{5.18}
$$

Of course, the question immediately arises which pairs (Γ, Q), with Γ an odd-integral lattice and Q a visible vector in Γ^*, may be encountered in the study of *real* incompressible Hall fluids. Progress in answering this question arose when some of the authors of [1] were visiting the IHÉS, back in 1993, and were able to bother Louis Michel and several visiting mathematicians with questions on lattice theory. Although Louis discovered that the authors of [1] were lacking mathematical culture in the area of lattice theory (at least one of them still is), he was willing and able to provide very valuable advice and help; help that (somewhat in contrast to the one offered by some mathematicians) was so useful, because it was very concrete. – Unfortunately, the story about how lattice theory solves some of the puzzles posed by the fractional quantum Hall effect is neither very simple, nor very short. We therefore refer the reader to the literature, [1, 3] and refs. given there; see also [9].

We have completed our sketch of the theories underlying Example 1 and Example 2 described in Section 2. The theory underlying Example 3 (quantized heat currents) is quite similar; it is known among specialists in the field. So let us proceed to systems in *four space-time dimensions.* For example, imagine that we study a young, rotating star containing a dense, relativistic electron(-positron) plasma. It may be legitimate to neglect the mass of electrons (and positrons) and assume that *"handedness"* is (approximately) conserved. The plasma then has two (approximately) conserved charges

$$N_{l/r} = \frac{1}{2}\int \left(\mathcal{J}^0(\underline{x},t) \mp \mathcal{J}^0_a(\underline{x},t)\right) d^3x \,. \tag{5.19}$$

The expectation values of N_l and N_r in the state of the star are determined by the chemical potentials, μ_l and μ_r, of left- and right-handed electrons, respectively. Since weak interactions may play some rôle in the genesis of the plasma, it is plausible to suppose that $\mu_l \neq \mu_r$. Moreover, it is likely that there is a reasonably strong magnetic field, $\vec{B}$, inside the star.

Making use of our basic current sum rule (3.12), using expressions (5.19) for the conserved charges, and applying eq. (4.26), integrated over all of space, we find the equation

$$\vec{j}(\underline{x},t) = \frac{e^2}{2\pi h}(\mu_l - \mu_r)\,\vec{B}(\underline{x},t)\,, \tag{5.20}$$

for the electric current density

$$\vec{j}(\underline{x},t) := \langle \vec{\mathcal{J}}(\underline{x},t)\rangle_{\beta,\underline{\mu}}, \tag{5.21}$$

in a thermal equilibrium state, inside the star. Current conservation then leads to the prediction of *surface currents* on the surface of the star. [I hope that Thibault Damour will soon explain to me whether eq. (5.20) describes some *virtual* reality about stars, or some significant effect in *real* stars.] Equation (5.20) is the 3+1 dimensional analogue of the basic equation

$$I_H = \frac{e^2}{h}\sigma_H V, \tag{5.22}$$

describing the Hall effect of an incompressible Hall fluid; (recall that σ_H is the dimensionless Hall conductivity). Equation (5.22) can be viewed as coming from the equations

$$j^\mu(\underline{x},t) = \frac{e^2}{h}\sigma_H\,\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}(\underline{x},t) \tag{5.23}$$

in (2 + 1) space-time dimensions, which summarize some of the transport theory for incompressible quantum Hall fluids; (see [1, 3] and refs.). Likewise, equation (5.20) can be viewed as coming from the equations

$$j^\mu(\underline{x},t) = \frac{e^2}{16\pi h}\varepsilon^{\mu\nu\lambda\alpha\beta}F_{\nu\lambda}(\underline{x},t)\,F_{\alpha\beta}(\underline{x},t) \tag{5.24}$$

in (4 + 1) space-time dimensions, (with $F_{5j} = 0,\ j = 1,2,3$). They would lead to an interpretation of the difference of the chemical potentials of left- and right-handed charged particles, $\mu_l - \mu_r$, as originating from a *non-vanishing electric field* ($\propto F_{04}$) pointing in a direction transversal to the four visible dimensions of our world.

Equations (5.23) and (5.24) can be derived from the three- and five-dimensional, abelian *Chern-Simons actions*, respectively. Their non-abelian cousins have been recognized to play a fundamental rôle in algebraic topology, in particular in knot theory. It is regrettable that there is no space, here, to describe this story, which, in what concerns my own, modest contributions, would be a nice illustration of the usefulness of interactions between physicists and mathematicians at IHÉS (see [10]).

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