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Publications Mathématiques de l'IHÉS
Volume 0 (0), pp. 97–141
https://doi.org/10.5802/pmihes.2

Publication membre du Centre Mersenne pour l'édition scientifique ouverte
https://www.centre-mersenne.org/
e-ISSN : 1618-1913

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ABSTRACT

We prove a priori bounds for Feigenbaum quadratic polynomials, i.e., infinitely renormalizable polynomials $f_c : z \mapsto z^2 + c$ of bounded type. It implies local connectivity of the corresponding Julia sets $J(f_c)$ and MLC (local connectivity of the Mandelbrot set $\mathcal{M}$) at the corresponding parameters c. It also yields the scaling Universality, dynamical and parameter, for the corresponding combinatorics. The MLC Conjecture was open for the most classical period-doubling Feigenbaum parameter as well as for the complex tripling renormalizations. Universality for the latter was conjectured by Goldberg–Khanin–Sinai in the early 1980s.

CONTENTS

1. Introduction

1.1. *Brief history.* — The MLC Conjecture on the local connectivity of the Mandelbrot set $\mathcal{M}$, put forward by Douady and Hubbard in the mid 1980s [7, 6], is the central problem of contemporary Holomorphic Dynamics. It would imply a precise topological model for $\mathcal{M}$ and the Fatou Conjecture on the density of hyperbolic maps in $\mathcal{M}$, and it is intimately related to the Mostow–Thurston Rigidity phenomenon in 3D Hyperbolic Geometry.

Around 1990 Yoccoz proved MLC at any parameter $c \in \mathcal{M}$ that is not infinitely renormalizable (in the quadratic-like sense) thus linking the problem tightly to the Quadratic-like Renormalization Theory (see [16, 29]). First advances in this direction appeared in [24], where MLC was established for infinitely renormalizable parameters

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Acknowledgements: We would like to thank Jeremy Kahn for many stimulating discussions. We are grateful to the referee for a comprehensive reading of the manuscript. A number of our pictures are made with W. Jung's program *Mandel*. Results of this paper were first announced in 2021 at the Quasiworld Seminar and at the CIRM conference "Advancing Bridges in Complex Dynamics", see the talk by the second author [26].

Fundings: The first author was partially supported by the NSF grant DMS 2055532 and the ERC grant "HOLOGRAM". The second author has been partly supported by the NSF, the Hagler and Clay Fellowships, the Institute for Theoretical Studies at ETH (Zurich), and MSRI (Berkeley).

https://doi.org/10.5802/pmihes.2

of *high type* satisfying the Secondary Limb Condition (SLC) and the general problem was reduced, under SLC, to the problem of a priori bounds.

Quadratic-like renormalization appears in two flavors, *primitive* and *satellite*. The above results were concerned with the primitive case. Next breakthrough in this direction appeared in the work of Jeremy Kahn [18] who established a priori bounds, and hence MLC, for all infinitely renormalizable parameters of *bounded primitive* type (using the Covering Lemma [22]). It followed up with the work [20, 21] handling the *definitely primitive case*.

It took more than 10 years to prove MLC at some infinitely renormalizable parameters of *bounded satellite* type [10] (based upon the Pacman Renormalization Theory developed in [11]). However, the most prominent parameter, the *period-doubling Feigenbaum point* corresponding to the cascade of doubling renormalizations (see Figure 1) was not covered by this result. In this paper we are filling in this gap by proving MLC at the Feigenbaum point, and in fact, at *all infinitely renormalizable parameters of bounded type* (for which we will still preserve the same name).

Remark **1.1.** — The universality of the period-doubling cascade was discovered in the mid 1970s by Feigenbaum [13, 14] (the parameter part, see Figure 1) and independently by Coullet and Tresser [4, 5] (the dynamical part). It is intimately related to the renormalization phenomenon in the Quantum Field Theory and Statistical Mechanics; the discovery opened up a new universality paradigm in Dynamical Systems.

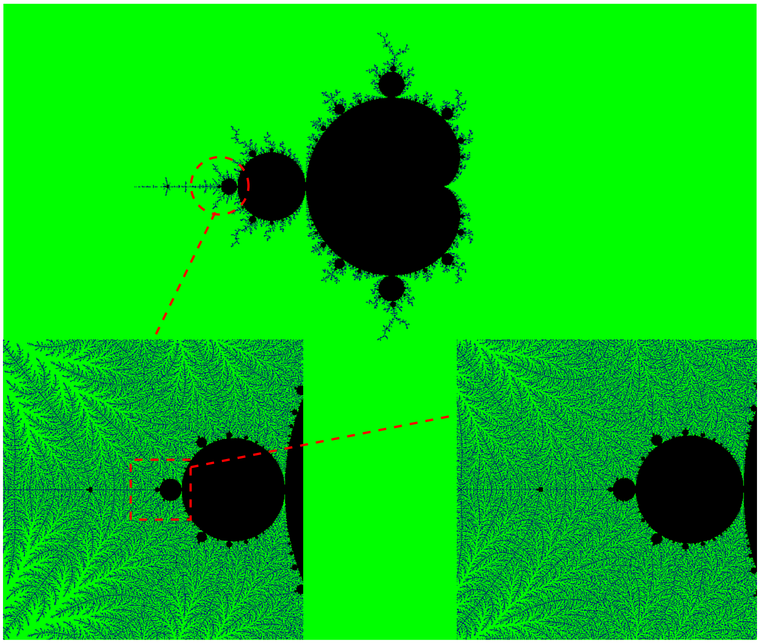

FIGURE 1. — Self-similarity of the Mandelbrot set at the period-doubling Feigenbaum parameter: $\mathcal{M}$ scales almost linearly under $w \mapsto 4.6692\ldots w + o(|w|^{1+\varepsilon})$.

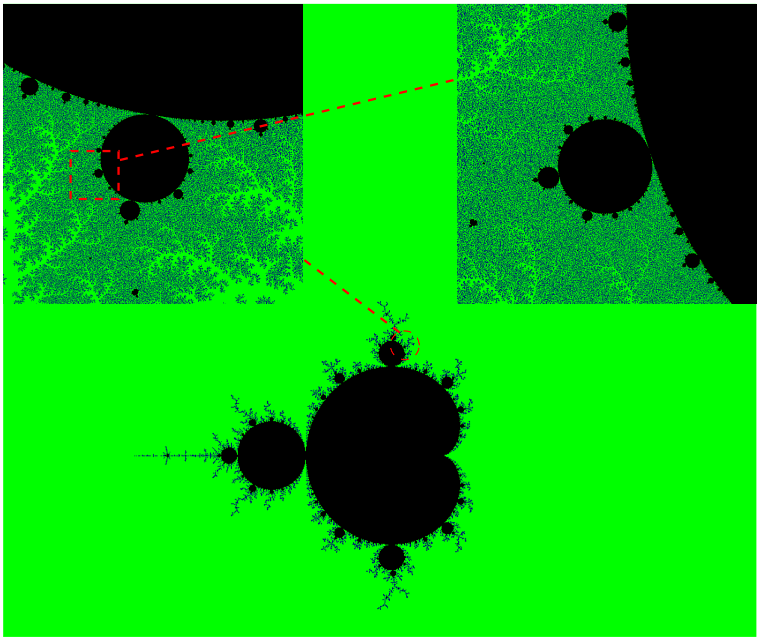

FIGURE 2. — Illustration to the Goldberg–Khanin–Sinai Conjecture: the Mandelbrot set scales almost linearly at the parameter of the period-tripling bifurcation. (The first zoom magnifies a deep renormalization level.)

1.2. *Statement of the main result and its consequences.* — A *Feigenbaum map* is an infinitely renormalizable quadratic-like map $f : U \to V$ with bounded combinatorics, i.e., all renormalization periods of f are bounded by some $\bar{p}$ (see Section 2.2.5 for precise definitions). One says that such an f has a priori bounds if its quadratic-like renormalizations $\mathcal{R}^n f : U^n \to V^n$ can be selected so that

$$\operatorname{mod}(V^n \setminus U^n) \geq \mu > 0.$$

The bounds are called *beau* if μ depends only on the combinatorial bound $\bar{p}$ as long as n is big enough (depending on $\operatorname{mod}(V \setminus U)$).

Theorem **A.** — *Any Feigenbaum quadratic-like map has a priori beau bounds.*

Together with the Rigidity Theorem of [24], it implies MLC at the corresponding parameters:

Theorem **B.** — *The Mandelbrot set is locally connected at any Feigenbaum parameter.*

The *Renormalization Conjecture* asserts that for any combinatorial bound $\bar{p}$ the renormalization transformation $\mathcal{R}$ has a renormalization horseshoe $\mathcal{A}_{\bar{p}}$ of Feigenbaum maps of type $\bar{p}$ on which it acts hyperbolically with one-dimensional unstable foliation. Together with [25] (see also [2, 28, 32]) we obtain:

Theorem **C.** — *For any combinatorial bound $\bar{p}$, the Renormalization Conjecture is valid in the space of quadratic-like maps.*

Let us consider the set $\mathcal{I}_{\bar{p}}$ of Feigenbaum parameters $c \in \mathcal{M}$ of type $\bar{p}$. Theorem B implies that each $c \in \mathcal{I}_{\bar{p}}$ is the intersection of the nest $\mathcal{M}^n(c)$ of little $\mathcal{M}$-copies corresponding to the n-fold renormalizations of f_c. Let us say that Mandelbrot set is *self-similar* over $\mathcal{I}_{\bar{p}}$ if the straightening map $\chi : \mathcal{M}^{n+1}(c) \to \mathcal{M}^n(\chi(c))$ (see Section 2.2.3) is $C^{1+\delta}$-conformal at any $c \in \mathcal{I}_{\bar{p}}$ (with uniform constants).

The Renormalization Conjecture (Theorem C) implies:

Theorem **D.** — *For any $\bar{p}$, the Mandelbrot set is self-similar over $\mathcal{I}_{\bar{p}}$.*

For the *satellite triplings*, the last two theorems confirm the *Goldberg–Sinai–Khanin Conjecture* [15] from the 1980s, see Figure 2.

1.3. *Rough outline of the proof.* — The historical developments of ideas behind the proof are summarized in [26]. For a ql map $f : U \to V$, its width is

$$\mathcal{W}(f) = \mathcal{W}(V \setminus \mathfrak{K}_f) = \frac{1}{\operatorname{mod}(V \setminus \mathfrak{K}_f)},$$

where $\mathfrak{K}_f$ is the filled Julia set of f. Our goal is to bound $\mathcal{W}(f)$ from above. We will first review the main ideas for the bounded primitive case, then we will discuss how to complete the argument in the bounded satellite case.

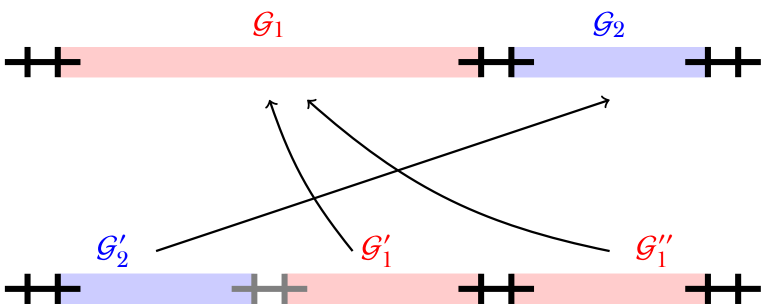

FIGURE 3. — Primitive pull-off in the airplane combinatorics. If $\mathcal{W}^{\text{ver}}_{\text{loc}}(\mathfrak{K}_j) \ll \mathcal{W}_{\text{loc}}(\mathfrak{K}_j)$, then most curves of the horizontal rectangles $\mathcal{G}_1, \mathcal{G}_2$ overflow their preimages $\mathcal{G}_1', \mathcal{G}_2', \mathcal{G}_1''$ as illustrated. Such a configuration is impossible by the Grötzsch inequality; i.e., (1.4) holds. Peripheral preimages are omitted.

1.3.1. *Pull-off in the primitive case* [18, 22]. — Consider the n^{th} renormalization $f_n = \mathcal{R}^n f$ of $f = f_0$, and let $\mathfrak{K}_0, \mathfrak{K}_1, \dots, \mathfrak{K}_{p-1}$ be the periodic cycle of little filled Julia sets of f_n in the dynamical plane of $f_0 : U \to V$. Assuming that the renormalization is primitive, all $\mathfrak{K}_i$ are pairwise disjoint.

Consider the thin-thick decomposition of $V' := V \setminus \bigcup_i \mathfrak{K}_i$: there are finitely many pairwise disjoint rectangles $\mathcal{G}_i$ between components of $\partial V'$ such that, up to $O_p(1)$

of width, the $\mathcal{G}_i$ contain all non-peripheral curves in V' with endpoints in $\partial V'$, see Figure 9 for an illustration of a thin-thick decomposition. A rectangle $\mathcal{G}_i$ is called

- *vertical* if it connects the outer boundary $\partial^{\text{out}} V' := \partial V$ of V' to one of its inner boundary components;
- *horizontal* if it connects two inner (i.e., non-outer) boundary components of V'.

Below we assume that $\mathcal{W}(f_n) \gg_p 1$. We have:

$$\mathcal{W}(f) \geq \sum_{\text{vertical } \mathcal{G}_i} \mathcal{W}(\mathcal{G}_i) \tag{1.1}$$

and, for every $\mathfrak{K}_j$:

$$\mathcal{W}(f_n) \asymp \mathcal{W}_{\text{loc}}(\mathfrak{K}_j) := \sum_{\substack{\mathcal{G}_i \text{ adjacent} \\ \text{to } \mathfrak{K}_j}} \mathcal{W}(\mathcal{G}_i) + O_p(1), \tag{1.2}$$

i.e. the sum in (1.2) is taken over all rectangles $\mathcal{G}_i$ adjacent to $\mathfrak{K}_j$. (A "ψ-ql renormalization" is used to achieve (1.2), see Sections 3.4 and 3.5.2, and (3.23).) Set also

$$\mathcal{W}^{\text{ver}}_{\text{loc}}(\mathfrak{K}_j) := \sum_{\substack{\text{vertical } \mathcal{G}_i \\ \text{adjacent to } \mathfrak{K}_j}} \mathcal{W}(\mathcal{G}_i). \tag{1.3}$$

A fundamental fact is that the horizontal $\mathcal{G}_i$ are eventually (after several restrictions of the domain) aligned with the Hubbard tree T_f of f as shown on Figure 3. Applying the Grötzsch inequality to the horizontal rectangles and their pullbacks (see the caption of Figure 3), we experience a definite loss of the horizontal weight in favor of the vertical one. The created definite vertical weight can then be pushed forward by means of the Covering Lemma [22]. Since all the local weights are comparable[1], we obtain:

$$\mathcal{W}^{\text{ver}}_{\text{loc}}(\mathfrak{K}_j) \asymp \mathcal{W}_{\text{loc}}(\mathfrak{K}_j) \quad \text{for every } \mathfrak{K}_j, \tag{1.4}$$

where "$\asymp$" is independent of p.

We stress that a key combinatorial ingredient used for (1.4) is the absence of periodic horizontal rectangles; i.e. a rectangle that has an iterated lift homotopic to itself rel small Julia sets, see Sections 1.3.2, 2.4, and 3.5.8.

Combining (1.1), (1.4), (1.2), and assuming $p \gg 1$, we obtain:

$$\begin{aligned} \mathcal{W}(f) &\geq \sum_{\text{vertical } \mathcal{G}_i} \mathcal{W}(\mathcal{G}_i) \\ &= \sum_{j=1}^{p} \mathcal{W}^{\text{ver}}_{\text{loc}}(\mathfrak{K}_j) \asymp \sum_{j=1}^{p} \mathcal{W}_{\text{loc}}(\mathfrak{K}_j) \asymp p\mathcal{W}(f_n) \gg 2\mathcal{W}(f_n); \end{aligned} \tag{1.5}$$

i.e., (since "$\gg$" dominates "$\asymp$"):

$$\mathcal{W}(f) \geq 2\mathcal{W}(f_n) \tag{1.6}$$

[1] This observation goes back to McMullen [27, Theorem 9.3].

implying a priori bounds in the primitive case. (For instance, by the *Record Argument*: by selecting "record levels", on which the degeneration exceeds all the preceding ones, we immediately arrive at a contradiction.)

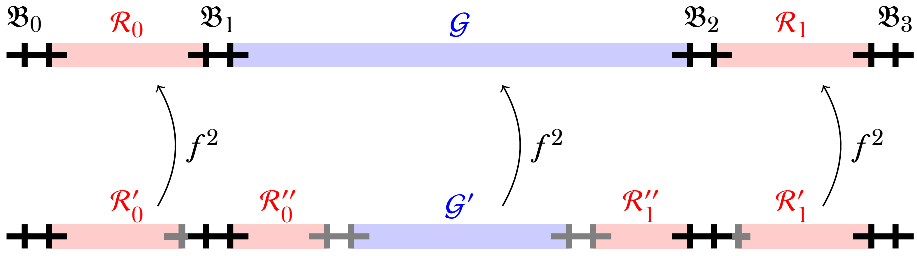

FIGURE 4. — Pull-off in the Feigenbaum combinatorics for non-periodic rectangles. Either most of the horizontal degeneration is within periodic rectangles $\mathcal{R}_0, \mathcal{R}_1$ such that a $(1-\delta)$-portion of each $\mathcal{R}_i$ overflows its periodic lifts $\mathcal{R}'_i$. Or we have $\mathcal{W}^{\text{ver}}_{\text{loc}}(\mathfrak{B}_j) \succeq_\delta \mathcal{W}_{\text{loc}}(\mathfrak{B}_j)$.

1.3.2. *Satellite case.* — Let us now assume that f is an infinitely renormalizable map with the period-doubling combinatorics; i.e., f is hybrid equivalent to a map in the period-doubling Feigenbaum combinatorial class. The general bounded case is similar.

Let $\mathfrak{B} := \mathfrak{K}^1_0 \cup \mathfrak{K}^1_1$ be the bouquet consisting of two level 1 little Julia sets of f. Instead of $\mathcal{W}(f)$, we will measure

$$\mathcal{W}_\bullet(f) := \mathcal{W}(V \setminus \mathfrak{B}).$$

In the dynamical plane of $f = f_0$, consider the periodic cycle of little bouquets $\mathfrak{B}_0, \mathfrak{B}_1, \dots, \mathfrak{B}_{p-1}$ (where $p = 2^n$) associated with $f_n = \mathcal{R}^n f$. We enumerate these bouquets linearly from left-to-right. Consider the thick-thin decomposition of $V' := V \setminus \bigcup_i \mathfrak{B}_i$. The following properties are established similarly as in the primitive case considered above:

- Estimates (1.1) and (1.2), where little Julia sets $\mathfrak{K}_i$ are replaced with bouquets $\mathfrak{B}_i$;
- alignment of the horizontal rectangles with the Hubbard tree of f (after several restrictions of the domain).

However, unlike in the primitive case, Section 1.3.1, f may have periodic horizontal rectangles $\mathcal{R}_k$ aligned with the Hubbard tree between $\mathfrak{B}_{2k}$ and $\mathfrak{B}_{2k+1}$ as shown on Figure 4. This leads us to the following alternative: either most of the total degeneration is within the $\mathcal{R}_k$ or a substantial part of the total degeneration is vertical (i.e. (1.4) holds). More precisely, choose a small $\delta > 0$. We have:

(I) either there is a periodic rectangle $\mathcal{R}_k$ connecting $\mathfrak{B}_{2k}, \mathfrak{B}_{2k+1}$ such that a $(1-\delta)$ part of $\mathcal{R}_k$ overflows its iterated lift $\mathcal{R}'_k$ under $f^{2^{n-1}}$;

(II) or else we have:

$$\mathcal{W}^{\text{ver}}_{\text{loc}}(\mathfrak{B}_j) \asymp_\delta \mathcal{W}_{\text{loc}}(\mathfrak{B}_j) \quad \text{for every } \mathfrak{B}_j$$

If (II) holds, then assuming $p \gg_\delta 1$ and $\mathcal{W}_\bullet(f_n) \gg_{p,\delta} 1$ and repeating the argument of (1.5), we obtain

(II′) $\mathcal{W}_\bullet(f) > 2\mathcal{W}_\bullet(f_n)$;

the Record Argument leads to a contradiction in case (II′).

The dichotomy "Case (I) vs Case (II′)" is stated as Theorem 4.1. This is a refinement of [18].

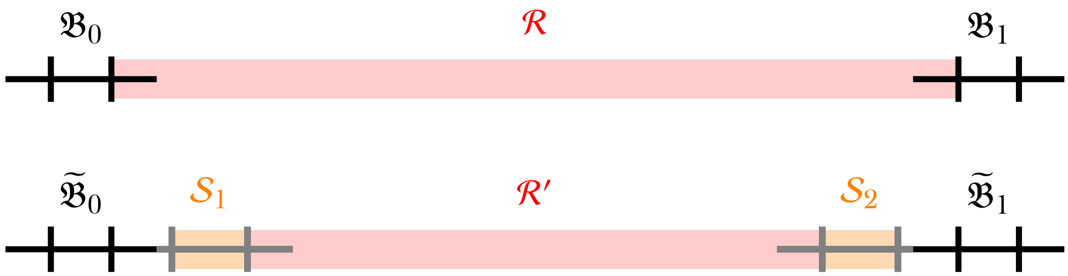

FIGURE 5. — Pull-off in the Feigenbaum combinatorics for periodic rectangles. Assume that a $(1-\delta)$ part of a periodic rectangle $\mathcal{R}$ overflows its iterated lift $\mathcal{R}'$. Since $\mathcal{R}$ also overflows $\mathcal{S}_1, \mathcal{S}_2$ (that are disjoint from $\mathcal{R}'$), we obtain from the Grötzsch inequality that $\mathcal{W}(\mathcal{S}_i) > C_\delta \mathcal{W}(\mathcal{R})$ with $C_\delta \to \infty$ as $\delta \to 0$.

Consider now Case (I). As Figure 5 illustrates, bouquets $\mathfrak{B}_0, \mathfrak{B}_1$ grow under lifting, thus the "combinatorial distance" between $\widetilde{\mathfrak{B}}_0, \widetilde{\mathfrak{B}}_1$ shrinks. This allows us to detect laminations $\mathcal{S}_i$ (the "differences" between $\mathcal{R} = \mathcal{R}_k$ and its lift $\mathcal{R}'$) that are much wider than $\mathcal{W}(\mathcal{R}) \geq \mathcal{W}_\bullet(f_n)$; i.e. $\mathcal{W}(\mathcal{S}_i) \geq C_\delta \mathcal{W}(\mathcal{R})$ with $C_\delta \to \infty$ as $\delta \to 0$. There are two possibilities: either a substantial part of $\mathcal{S}_i$ hits preperiodic bouquets associated with f_{n+1} or a substantial part of $\mathcal{S}_i$ creates a "wave" (see Section 5) above such a preperiodic bouquet. In the latter case, the Wave Lemma 5.1 implies that $\mathcal{W}_\bullet(f_{n-1}) \gg \mathcal{W}_\bullet(f_n)$ and the Record Argument is applicable.

In the former case, we obtain $\mathcal{W}_\bullet(f_{n+1}) \geq \mathcal{W}(\mathcal{S}_i) \geq C_\delta \mathcal{W}_\bullet(f_n)$; i.e.:

$$\mathcal{W}_\bullet(f_{n+1}) \geq C'_\delta \mathcal{W}_\bullet(f_n) \quad \text{with} \quad C'_\delta \longrightarrow \infty \quad \text{as} \quad \delta \longrightarrow 0.$$

For a sufficiently small δ, this eventually contradicts the Teichmüller contraction within hybrid classes [31] (stated as Proposition 2.7):

$$\mathcal{W}_\bullet(f_n) = O(\Delta^n) \quad \text{for some } \Delta > 1.$$

Finally, we will convert in Theorem 7.2 bounds for $\mathcal{W}_\bullet(f_m)$ into bounds for $\mathcal{W}(f_m)$.

Remark **1.2**. — In the paper, we will be measuring the degeneration around "bushes", Section 2.3, which are Hubbard continua enhanced with the periodic cycle of level one little Julia sets. An alternative would be to keep the Julia sets for primitive renormalization and use the satellite flowers in the satellite case. We believe that it is also possible to work with Hubbard continua; however, the Wave Lemma 5.1 in Section 5 and the transition toward classical QL-bounds will be more subtle.

2. QL renormalization, bushes, and invariant arc diagrams

The reader can find most of the needed background material in the book [23]. Let us briefly outline the context of this section with emphasis on the notation.

2.1. *Outline*. — Basics aspects of quadratic-like maps are summarized in Section 2.2. We will usually denote by

- $f: X \to Y$ a ql map,
- $\mathfrak{K}_f$ its (filled) Julia set,
- $\mathfrak{K}_i^{[n]} \subset \mathfrak{K}_f$ its level n little Julia sets,
- $\boldsymbol{\mathfrak{K}}^{[n]} := \bigcup_i \mathfrak{K}_i^{[n]}$ the union of periodic level n little Julia sets, see (2.6),
- $\mathfrak{T}_f \subset \mathfrak{K}$ the Hubbard continuum of f, see Section 2.2.2,
- $\mathfrak{B}_f := \mathfrak{T}_f \cup \boldsymbol{\mathfrak{K}}^{[1]}$ the *bush* of f, see (2.8),
- $\mathfrak{B}_f^{(m)} := f^{-m}(\mathfrak{B}_f)$ the little bush of height m,
- $\mathfrak{B}_i^{[n]} \equiv \mathfrak{B}_i^{[n],(0)} \subset \mathfrak{B}_i^{[n],(m)} \subset \mathfrak{K}_i^{[n]}$ the associated little bushes,
- $\boldsymbol{\mathfrak{B}}^{[n]} := \bigcup_i \mathfrak{B}_i^{[n]} \subset \boldsymbol{\mathfrak{K}}^{[n]}$ the union of periodic level n little bushes,
- $\boldsymbol{\mathfrak{B}}^{[n],(m)} := f^{-m}(\boldsymbol{\mathfrak{B}}^{[n]})$ the union of level n and height m little periodic and preperiodic bushes,
- $\boldsymbol{\mathfrak{B}}_{\mathrm{per}}^{[n],(m)} := \boldsymbol{\mathfrak{B}}^{[n],(m)} \cap \boldsymbol{\mathfrak{K}}^{[n]}$ the union of periodic level n and height m little bushes.

If $\mathfrak{B}_f = \boldsymbol{\mathfrak{K}}^{[1]}$, i.e., $\mathfrak{T}_f \subset \boldsymbol{\mathfrak{K}}^{[1]}$, then we will also refer to $\mathfrak{B}_f$ as the *satellite flower* of f. Preperiodic little Julia sets and bushes are defined accordingly, Section 2.3.1. We will usually assume that

(O) $\mathfrak{K}_0^{[n]}$ contains the critical point of f.

Periodic little Julia sets will be usually labeled dynamically:

$$f\left(\mathfrak{K}_i^{[n]}\right) = \mathfrak{K}_{i+1}^{[n]}, \quad i,\ i+1 \in \mathbb{Z}/\mathfrak{p}_n \simeq \{0, 1, \dots, \mathfrak{p}_n - 1\}.$$

However, in the satellite combinatorics, we will often label the $\mathfrak{K}_i^{[n]}$ with respect to their cyclic order in the satellite flower; see, for instance, Lemma 2.3. Preperiodic little Julia sets $\mathfrak{K}_j^{[n]}$ will be labeled with some subindex. The labeling of little bushes is naturally inherited from the labeling of little Julia sets: $\mathfrak{B}_i^{[n],(m)} \subset \mathfrak{K}_i^{[n]}$. We will always indicate when the cyclic order is used. And we will also mention when a preperiodic object appears.

The Teichmüller contraction of ql renormalization will be established in Section 2.5.

In Lemma 2.2, we will show that invariant horizontal arc diagrams are aligned with the Hubbard dendrite. Figures 7 and 8 demonstrate new subtleties arising in the satellite case. In particular, there are genuinely periodic arcs, described in Lemma 2.3; they encode almost periodic rectangles (see Section 3.5.8 and Figure 4) that will have a separate treatment in Section 6.

2.2. *Quadratic-like maps*. — A *quadratic-like map* $f: X \to Y$, which will also be abbreviated as a *ql map*, is a holomorphic double branched covering between two Jordan disks $X \Subset Y \subset \mathbb{C}$. It has a single critical point that we usually put at the origin 0. The annulus $A = Y \setminus \overline{X}$ is called the *fundamental annulus* of $f : X \to Y$. We let $\operatorname{mod} f := \operatorname{mod} A$. The *filled Julia set* $\mathfrak{K}_f$ is the set of non-escaping points:

$$\mathfrak{K} \equiv \mathfrak{K}_f \equiv \mathfrak{K}(f) := \{z : f^n z \in X, \ n = 0, 1, 2, \dots\}.$$

Its boundary is called the *Julia set* $\mathfrak{J} \equiv \mathfrak{J}(f)$. The (filled) Julia set is either connected or Cantor, depending on whether the critical point is non-escaping (i.e., $0 \in \mathfrak{K}(f)$) or otherwise. In this paper, filled Julia sets will play a bigger role than the Julia sets, and we will often skip the adjective "filled" when it is clear (e.g., from the notation) what we mean.

We say that two quadratic-like maps with connected Julia sets represent the same *germ* if they have the same filled Julia set and coincide in some neighborhood of it. For a ql germ f, we define

(2.1) $$\operatorname{mod} f := \sup \operatorname{mod}(Y' \setminus X'),$$

where the supremum is taken over all $f : X' \to Y'$ representing the ql germ of f.

Two quadratic-like maps $f : X \to Y$ and $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ are called *hybrid conjugate* if they are conjugate by a quasiconformal map $h : (Y, X) \to (\widetilde{Y}, \widetilde{X})$ such that $\bar{\partial} h = 0$ a.e. on $\mathfrak{K}$.

A simplest example of a quadratic-like map is provided by a quadratic polynomial $f_c : z \mapsto z^2 + c$ restricted to a disk $Y = \mathbb{D}_R$ of sufficiently big radius. The Douady and Hubbard *Straightening Theorem* asserts that any quadratic-like map f is hybrid conjugate to some restricted quadratic polynomial f_c. Moreover, if J is connected then the parameter $c \in \mathcal{M}$ is unique.

As for quadratic polynomials, two fixed points of a quadratic-like map with connected Julia set have a different dynamical meaning. One of them, called β, is the landing point of a proper arc $\gamma \subset X \setminus K(f)$ such that $f(\gamma) \supset \gamma$. It is either repelling or parabolic with multiplier one. The other fixed point, called α, is non-repelling or a *cut-point* of the Julia set.

2.2.1. *Combinatorial models.* — A quadratic polynomial f is called *periodically hyperbolic* (resp., *repelling*) if it does not have neutral (resp., non-repelling) cycles. Note that in the periodically repelling case, $\operatorname{int} \mathfrak{K} = \emptyset$, so $\mathfrak{J} = \mathfrak{K}$. The filled Julia set of a periodically hyperbolic map admits a locally connected *combinatorial model* $\mathfrak{K}^{\mathrm{com}} = \mathfrak{K}^{\mathrm{com}}(f)$ obtained by taking the quotient of the unit disk by the associated rational lamination (see [23, Section 32.1.3]). The combinatorial model is endowed with the induced dynamics $f_{\mathrm{com}} : \mathfrak{K}^{\mathrm{com}} \to \mathfrak{K}^{\mathrm{com}}$ which is semi-conjugate to f by a natural projection $\pi : \mathfrak{K} \to \mathfrak{K}^{\mathrm{com}}$ whose *fibers* on the Julia set are "combinatorial classes" (sets of points inseparable by preperiodic points).

2.2.2. *Hubbard continua.* — In the superattracting case (when the critical point is periodic) the *Hubbard tree* $\mathfrak{T}_f$ is defined as the smallest tree in $\mathfrak{K}$ containing the postcritical set such that $\mathfrak{T}_f$ intersects the components of $\operatorname{int} \mathfrak{K}$ along internal rays. It is invariant under f and is marked with the orbit of 0. If $\mathfrak{K}$ is periodically repelling and locally connected, we can define the *Hubbard dendrite* $\mathfrak{T}_f$ as the smallest connected closed forward-invariant subset of $\mathfrak{K}$ containing the postcritical set.

For a general periodically repelling map, the (combinatorial) Hubbard dendrite $\mathfrak{T}^{\mathrm{com}}_f \subset \mathfrak{K}^{\mathrm{com}}$ is defined for its combinatorial model $f\colon \mathfrak{K}^{\mathrm{com}} \circlearrowleft$. Lifting $\mathfrak{T}^{\mathrm{com}}_f$ via $\pi\colon \mathfrak{K} \to \mathfrak{K}^{\mathrm{com}}$, we obtain the *Hubbard continuum* $\mathfrak{T}_f$.

For a periodically repelling f and $x, y \in \mathfrak{J}_f = \mathfrak{K}_f$, we define a *geodesic* continuum

$$[x,y] := \pi^{-1}[\pi(x), \pi(y)], \tag{2.2}$$

where $[\pi(x), \pi(y)]$ is a unique arc in $\mathfrak{J}^{\mathrm{com}}$ connecting $\pi(x), \pi(y)$.

2.2.3. *QL-renormalization.* — A quadratic-like map $f : X \to Y$ is called *DH renormalizable* (after Douady and Hubbard) if there is a quadratic-like restriction

$$f_1 = f_{1,0} = \mathcal{R}f = f^{p_1} : X_{1,0} \longrightarrow Y_{1,0} \tag{2.3}$$

with connected Julia set $\mathfrak{K}^{[1]}_0$ such that the little Julia sets

$$\mathfrak{K}^{[1]}_i := f^i\left(\mathfrak{K}^{[1]}_0\right), \quad k = 0, \dots, p_1 - 1, \tag{2.4}$$

are either pairwise disjoint or else some touch at their β-fixed points. In the former case the renormalization is called *primitive*, while in the latter it is called *satellite*.

Note that there are many ql maps $f_1\colon X'_{1,0} \to Y'_{1,0}$ that satisfy the above requirements (if f is renormalizable). However, all of them represent the same *ql germ*.

The map f_1 in (2.3) is called a *pre-renormalization* of f. If it is considered up to a linear rescaling, it is called the *renormalization* of f. (In what follows, we will often skip the prefix "-pre" as long as it does not lead to a confusion.)

For every $\mathfrak{K}^{[1]}_i$, there are $Y_{1,i} \Supset X_{1,i} \Supset \mathfrak{K}^{[1]}_i$ such that

$$f_1 = f_{1,i} = \mathcal{R}f = f^{p_1} : X_{1,i} \longrightarrow Y_{1,i} \tag{2.5}$$

is a ql map with non-escaping set $\mathfrak{K}^{[1]}_i = \mathfrak{K}_{f_{1,i}}$. All maps (2.5) represent conformally conjugate germs. We assume that $\mathfrak{K}^{[1]}_0$ contains the critical point of f; see (O).

2.2.4. *Little copies of* $\mathcal{M}$. — The sets $\mathfrak{K}^{[1]}_i$ in (2.4) are referred to as the *little (filled) Julia sets*. Their positions in the big Julia set $\mathfrak{K}_f$ determine the renormalization *combinatorics*. By the Douady–Hubbard Straightening Theorem [8], the set of parameters c for which the quadratic polynomial $f_c\colon z \mapsto z^2 + c$ is renormalizable with a given combinatorics forms a *little Mandelbrot copy* $\mathcal{M}_1 \subset \mathcal{M}$ (see [23, Theorem 43.1]). The renormalization combinatorics can be formally encoded by the Hubbard tree $\mathfrak{T}_\circ$ of the superattracting center $c_\circ$ of $\mathcal{M}_1$.

Each little copy $\mathcal{M}_1$ can be canonically mapped onto the whole Mandelbrot set $\mathcal{M}$ by the *straightening homeomorphism* $\chi_1 : \mathcal{M}_1 \to \mathcal{M}$.

A little Mandelbrot copy $\mathcal{M}_1$ is called *primitive* or *satellite* depending on the type of the corresponding renormalization. They can be easily distinguished as any satellite copy is attached to some hyperbolic component of $\operatorname{int}\mathcal{M}$ and does not have the cusp at its root point.

2.2.5. *Infinitely renormalizable maps.* — The notions of an infinitely DH renormalizable map f with renormalization periods p_n, and its renormalizations $f_n = \mathcal{R}^n f$, are defined naturally.

By default, we assume that p_n is the *smallest renormalization period after* p_{n-1}. We will denote by $\mathfrak{K}_i^{[n]}, i \in \{0, 1, \dots, p_n - 1\}$ the level n little Julia sets of f enumerated dynamically so that $\mathfrak{K}_0^{[n]}$ contains the critical point of f. We will write

$$\mathbf{\mathfrak{K}}^{[n]} := \bigcup_i \mathfrak{K}_i^{[n]} \tag{2.6}$$

The ratios $q_n := p_n / p_{n-1}$ are called *relative periods*. One says that such a map has *bounded combinatorics of type* $\bar{p}$ if the relative periods are bounded by $\bar{p}$. In this case, the map f is called *Feigenbaum of type* $\bar{p}$. We say that a Feigenbaum map is *primitive/satellite* if all its renormalizations are such. A Feigenbaum map has a priori bounds if

$$\operatorname{mod} \mathcal{R}^n f \geq \epsilon > 0 \tag{2.7}$$

We say that the family $\mathcal{F}_{\bar{p}}$ of Feigenbaum maps of type $\bar{p}$ have *beau bounds* if there exists $\mu > 0$ depending only on $\bar{p}$ such that for any $\nu > 0$ there exists $n_0 = n_0(\bar{p}, \nu)$ such that for any $f \in \mathcal{F}_{\bar{p}}$ with $\operatorname{mod} f \geq \nu$ we have

$$\operatorname{mod} \mathcal{R}^n f \geq \mu \quad \text{for all} \quad n \geq n_0.$$

2.3. *Bushes.* — Consider a DH renormalizable quadratic-like map $f : X \to Y$. The *bush* of f is

$$\mathfrak{B}_f \equiv \mathfrak{B}(f) := \mathfrak{T}_f \cup \mathbf{\mathfrak{K}}^{[1]}, \tag{2.8}$$

where $\mathfrak{T}_f$ is the Hubbard continuum and $\mathbf{\mathfrak{K}}^{[1]}$ is the periodic cycle of level one little Julia sets (2.6). For $m \geq 0$, we define the *bush of height* m to be

$$\mathfrak{B}^{(m)} = \mathfrak{B}_f^{(m)} := f^{-m}(\mathfrak{B}_f). \tag{2.9}$$

2.3.1. *Little bushes.* — Suppose that f is at least $n + 1$ times DH renormalizable and let $f_n = \mathcal{R}^n f$ be its n^{th} renormalization of f. Then f_n has a well defined bush $\mathfrak{B}(f_n)$. Consider the periodic cycle of little level n filled Julia sets $\mathfrak{K}_i^{[n]}$ associated with f_n in the dynamical plane of f. Let $f_{n,i}$ be the n^{th} prerenormalization around $\mathfrak{K}_i^{[n]}$, compare (2.5). Then $\mathfrak{K}_i^{[n]}$ contains the little bush $\mathfrak{B}_i^{[n]} \equiv \mathfrak{B}(f_{n,i}) \simeq \mathfrak{B}(f_n)$ as well as $\mathfrak{B}_i^{[n],(m)} \equiv \mathfrak{B}^{(m)}(f_{n,i}) \simeq \mathfrak{B}^{(m)}(f_n)$. Note that $\mathfrak{B}_i^{[n],(m)}$ is the unique periodic lift of $\mathfrak{B}_{i+m}^{[n]}$ under f^m.

We write

$$\mathbf{\mathfrak{B}}^{[n]} := \bigcup_i \mathfrak{B}_i^{[n]}, \quad \mathbf{\mathfrak{B}}^{[n],(m)} := f^{-m}(\mathbf{\mathfrak{B}}^{[n]}), \quad \mathbf{\mathfrak{B}}_{\text{per}}^{[n],(m)} := \mathbf{\mathfrak{B}}^{[n],(m)} \cap \mathbf{\mathfrak{K}}^{[n]}.$$

Observe that $\mathbf{\mathfrak{B}}_{\text{per}}^{[n],(m)}$ is the union of periodic little bushes of height m. A *preperiodic bush* $\mathfrak{B}_a^{[n],(m)}$ is a non-periodic connected component of $\mathbf{\mathfrak{B}}^{[n],(m)}$. The *preperiod* of $\mathfrak{B}_a^{[n],(m)}$ is the smallest $s \leq m$ such that $f^s(\mathfrak{B}_a^{[n],(m)})$ is periodic.

Every periodic or preperiodic little bush $\mathfrak{B}_a^{[n],(m)}$ is within a unique little periodic or preperiodic filled Julia set $\mathfrak{K}_a^{[n]}$ associated with f_n. The $\mathfrak{B}_a^{[n],(m)}$ exhaust $\mathfrak{K}_a^{[n]}$:

$$\mathfrak{K}_a^{[n]} = \overline{\bigcup_m \mathfrak{B}_a^{[n],(m)}}\ .$$

By construction, the $\mathfrak{B}_a^{[n],(m)}$ are pairwise disjoint but the $\mathfrak{K}_a^{[n]}$ may touch each other in the satellite case.

2.3.2. *Superattracting model.* — Consider a ql map $f : X \to Y$ and assume it is $n+1$ DH renormalizable. Then f is hybrid equivalent to z^2+c, where c is in a level $n+1$ little copy $\mathcal{M}_i^{[n+1]} \subset \mathcal{M}$. A *superattracting* model for f of level $n+1$ is any ql map $f_\circ : X_\circ \to Y_\circ$ hybrid equivalent to the center of $\mathcal{M}_i^{[n+1]}$.

It is well-known (follows, for instance, from the lamination theory) that the Hubbard continua $\mathfrak{T}_f$ and $\mathfrak{T}_\circ \equiv \mathfrak{T}_{f_\circ}$ are *combinatorially equivalent* up to $\boldsymbol{\mathfrak{B}}^{[n]}$: there is a bijection between:

- components of $\mathfrak{T}_f \setminus f^{-m}(\boldsymbol{\mathfrak{B}}_f^{[n]})$ and components of $\mathfrak{T}_\circ \setminus f_\circ^{-m}(\boldsymbol{\mathfrak{B}}_\circ^{[n]})$ for all $m \geq 0$; and
- components of $\mathfrak{T}_f \cap f^{-m}(\boldsymbol{\mathfrak{B}}_f^{[n]})$ and of $\mathfrak{T}_\circ \cap f_\circ^{-m}(\boldsymbol{\mathfrak{B}}_\circ^{[n]})$ for all $m \geq 0$

that respects the adjacency, natural embedding, and dynamical relations between respective components. In other words, the above components define equivalent Markov partitions for $f, f_\circ$.

2.3.3. *Satellite combinatorics.* — Suppose that f is at least 3 times renormalizable and the first renormalization of f is satellite. Then all $\mathfrak{K}_i^{[1]}$ are organized in the *satellite flower* around the α fixed point of f:

$$\mathfrak{B}_f = \boldsymbol{\mathfrak{K}}_f^{[1]} \quad \text{because} \quad \boldsymbol{\mathfrak{K}}_f^{[1]} \supset \mathfrak{T}_f,$$

where $\mathfrak{T}_f$ is the Hubbard continuum of f. For $s \geq 0$ we write $\mathfrak{T}_f^{(s)} = f^{-s}(\mathfrak{T}_f)$. As before, the p_k are the periods of level k Julia sets.

Lemma **2.1** *(Satellite flower).* — *For a satellite f as above, let $\mathfrak{B}_a^{[1]}, \mathfrak{B}_b^{[1]}$ be two of its level 1 periodic bushes. Then the continuum $\mathfrak{T}_f^{(6p_2)}$ contains geodesic continua ℓ_1 and ℓ_2 (as in Section 2.2.2) connecting preperiodic lifts $\mathfrak{B}', \mathfrak{B}''$ and $\mathfrak{B}''', \mathfrak{B}^{iv}$ of $\mathfrak{B}_b^{[1]}$ of preperiods $\leq 6p_2$ such that, see Figure 6:*

- *ℓ_1 and ℓ_2 separate $\mathfrak{B}_a^{[1]}$ from α (within $\mathfrak{K}_f$),*
- *the geodesic continuum $\ell \subset \mathfrak{T}_f$ connecting ℓ_1, ℓ_2 intersects little preperiodic bushes $\mathfrak{B}_{i_1}^{[2]}, \mathfrak{B}_{i_2}^{[2]}$ of level 2 with preperiods less than $6p_1$;*
- *$\mathfrak{B}_{i_1}^{[2]}, \mathfrak{B}_{i_2}^{[2]}$ are disjoint from $\ell_1 \cup \ell_2$.*

We remark that $\ell, \mathfrak{B}_{i_1}^{[2]}, \mathfrak{B}_{i_2}^{[2]}$ are within $\mathfrak{B}_a^{[1],(6p_2)}$.

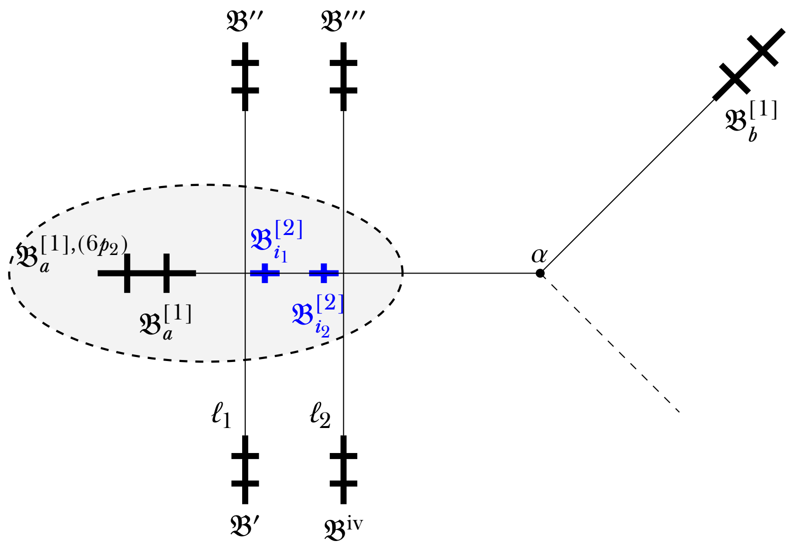

FIGURE 6. — Illustration to Lemma 2.1.

Proof. — Let $\mathfrak{B}_\gamma^{[2]} \subset \mathfrak{B}_a^{[1]}$ be the level 2 periodic bush closest to α; i.e. $(\mathfrak{T}_f \cap \mathfrak{B}_a^{[1]}) \setminus \mathfrak{B}_\gamma^{[2]}$ and α are in different components of $\mathfrak{T}_f \setminus \mathfrak{B}_\gamma^{[2]}$. Let the $\mathfrak{B}_{i(t)}^{[2]} \subset \mathfrak{B}_a^{[1],(tp_1)}$ be the lifts of $\mathfrak{B}_\gamma^{[2]}$ under f^{tp_1} towards α; i.e. each $\mathfrak{B}_{i(t)}^{[2]}$ separates α from $\mathfrak{B}_a^{[1],(tp_1-p_1)}$.

Since f^{6p_2} has critical points in $\mathfrak{B}_{i(1)}^{[2]}$ and in $\mathfrak{B}_{i(4)}^{[2]}$, we can select ℓ_1 and ℓ_2 passing through these critical points such that ℓ_1, ℓ_2 connect preperiodic lifts $\mathfrak{B}', \mathfrak{B}''$ and $\mathfrak{B}''', \mathfrak{B}^{\text{iv}}$ of $\mathfrak{B}_b^{[1]}$. Then ℓ_1, ℓ_2 separate $\mathfrak{B}_a^{[1]}$ from α, and $\ell_1 \cup \ell_2$ separate $\mathfrak{B}_{i(2)}^{[2]}, \mathfrak{B}_{i(3)}^{[2]}$ from α and $\mathfrak{B}_a^{[1]}$; i.e. we can take $\mathfrak{B}_{i_1}^{[2]}, \mathfrak{B}_{i_2}^{[2]}$ to be $\mathfrak{B}_{i(2)}^{[2]}, \mathfrak{B}_{i(3)}^{[2]}$. □

2.4. *Invariant arc diagrams.* — In this subsection, we will discuss invariant up to homotopy arc diagrams of ql maps. Arc diagrams endowed with weights will naturally appear from the thin-thick decompositions of the dynamical planes of $\psi^\bullet$-ql maps, see Sections 3.3, 3.5.

2.4.1. *Arc diagrams.* — Consider a hyperbolic open Riemann surface S of finite type without cusps. We endow S with its ideal boundary $\partial^i S$. This naturally makes $V := \overline{S} \equiv S \cup \partial^i S$ a compact surface.

A *path* (closed or open) ℓ in S is an embedded (closed or open) interval $\ell\colon I \to S$.

An open path $\gamma\colon (0,1) \to S$ is *proper* if it extends to $\gamma\colon [0,1] \to V \equiv S \cup \partial^i S$ with $\gamma\{0,1\} \subset \partial^i S$. Two proper paths γ_0, γ_1 in S are homotopic if there is a homotopy γ_t among proper paths. Similarly, *proper curves* and their homotopy are defined.

An *arc* on S is a class of properly homotopic paths, $\alpha = [\gamma]$. A curve γ is *trivial* if it can be represented in an arbitrary small neighborhood of $\partial^i S$. Two different arcs are *non-crossing* if they can be represented by non-crossing paths.

An *arc diagram* (AD) is a family of non-trivial pairwise non-crossing arcs $A = \{\alpha_i\}$. A *weighted arc diagram* (WAD) $\mathcal{A} = \sum_{\alpha_i \in A} w_i \alpha_i,\ w_i \in \mathbb{R}_{>0}$ is an arc diagram endowed with positive weights.

2.4.2. *Arc diagrams of ql maps.* — Consider a ql map $f: X \to Y$ and assume that it is $n+1$ times DH renormalizable.

An arc diagram $A = \{\alpha_i\}$ on $X \setminus \mathfrak{B}^{[n]}$ is *horizontal* if every α_i connects components of $\mathfrak{B}^{[n]}$. A horizontal arc diagram $A = \{\alpha_i = [\gamma_i]\}$ is called *invariant* if every α_i can be represented up to a proper homotopy in $X \setminus \mathfrak{B}^{[n]}$ in an arbitrary small neighborhood of $f^{-1}(\mathfrak{B}^{[n]} \cup \Gamma)$, where $\Gamma = \bigcup_i \gamma_i$. In other words, every $[\gamma] \in A$ can be presented as a concatenation

$$\gamma = \ell_0 \# \gamma_1 \# \ell_1 \# \gamma_2 \# \dots \# \gamma_s \# \ell_s, \tag{2.10}$$

where:

- $[\gamma_j] \in f^*(A)$ are proper arcs in $X \setminus \mathfrak{B}^{[n],(1)} = f^{-1}(Y \setminus \mathfrak{B}^{[n]})$; and
- every component of $\ell_i \setminus \mathfrak{B}^{[n],(1)}$ is trivial in $X \setminus \mathfrak{B}^{[n],(1)}$ (with respect to a proper homotopy, see Section 2.4.1).

(In other words, the ℓ_i are contractible into $\mathfrak{B}^{[n],(1)}$.) Here $f^*(A)$ is the pullback of A. Figure 7 gives an example of an invariant arc diagram.

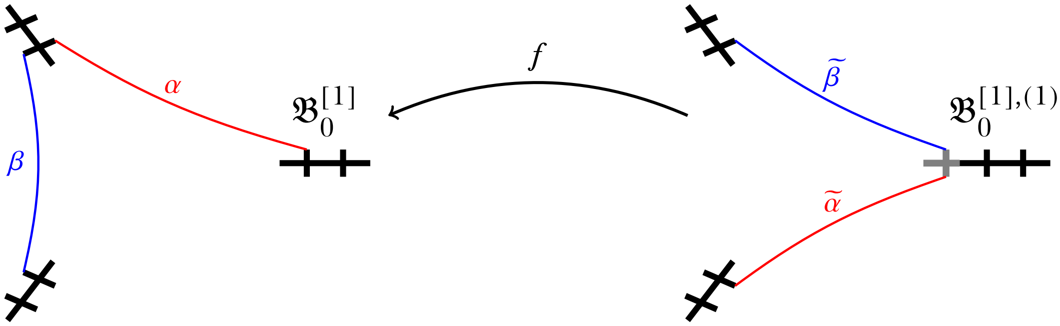

Figure 7. — The arc diagram $\{[\alpha], [\beta]\}$ is invariant. Here $\widetilde{\alpha}, \widetilde{\beta}$ are lifts of α, β, the map f is the Rabbit map tuned with the Feigenbaum map, peripheral lifts of α, β and preperiodic bushes are omitted on the right side. The arc diagram is invariant because α can be properly homotoped into $\widetilde{\beta}$ union $P := \mathfrak{B}_0^{[1],(1)} \setminus \mathfrak{B}_0^{[1]}$ while β can be properly homotoped into $\widetilde{\beta}, P, \widetilde{\alpha}$.

Lemma **2.2** *(Alignment with $\mathfrak{T}_f$, following* [18, Section 4]*).* — *If $A = \{\alpha_i\}$ is an invariant horizontal AD on $X \setminus \mathfrak{B}^{[n]}$, where $f: X \to Y$ is a ql map, then A is* aligned with the Hubbard continuum $\mathfrak{T}_f$: *every α_i can be represented in an arbitrary small neighborhood of a geodesic continuum $T_i \subset \mathfrak{T}_f \setminus \mathfrak{B}^{[n]}$ connecting components of $\mathfrak{B}^{[n]}$.*

Proof. — Write $\alpha_i = [\gamma_i]$ and consider $Y' := Y \setminus (\mathfrak{B}^{[n]} \cup \Gamma)$, where $\Gamma = \bigcup_i \gamma_i$. Then one of the little bushes $\mathfrak{B}_i^{[n]}$ is accessible from the outermost component of $\partial Y'$. Therefore, we can select a proper path $\ell \subset Y'$ connecting ∂Y and $\mathfrak{B}^{[n]}$. For every such a path ℓ, its *legal pullback* $\widetilde{\ell}$ is any of its lift connecting $\mathfrak{B}^{[n]}$ and ∂X and concatenated by a path in $Y \setminus X$ so that $\widetilde{\ell} \subset Y \setminus \mathfrak{B}^{[n]}$ is a proper path connecting ∂Y and $\mathfrak{B}^n$. Since A is invariant, $\widetilde{\ell}$ can be again represented in Y' up to a proper homotopy in $Y \setminus \mathfrak{B}^{[n]}$.

It is well-known that iterated pullbacks of ℓ can realize all periodic rays landing at $\mathfrak{B}^{[n]}$ up to a proper homotopy in $Y \setminus \mathfrak{B}^{[n]}$. (It is sufficient to verify the property for the superattracting model, Section 2.3.2.) Therefore, A is aligned with $\mathfrak{T}_f$. □

2.4.3. *Genuinely periodic arcs.* — Let $A = \{\alpha_i\}$ be a horizontal invariant AD on $Y \setminus \mathfrak{B}^{[n]}$ of $f : X \to Y$. For every $m \geq 1$, we can present $[\gamma] \in A$ as a concatenation

$$\gamma = \ell_0 \# \gamma_1 \# \ell_1 \# \gamma_2 \# \dots \# \gamma_s \# \ell_s \tag{2.11}$$

(compare with (2.10)), where:

- $[\gamma_j] \in (f^m)^*(A)$ are proper arcs in $f^{-m}(Y) \setminus \mathfrak{B}^{[n],(m)} = f^{-m}(Y \setminus \mathfrak{B}^{[n]})$; and
- every component of $\ell_i \setminus \mathfrak{B}^{[n],(m)}$ is trivial in $X^m \setminus \mathfrak{B}^{[n],(m)}$ (with respect to a proper homotopy, see Section 2.4.1).

We call (2.11) the *decomposition of* γ *rel* $\mathfrak{B}^{[n],(m)}$. We define the *expansivity number*

$$\mathrm{EN}_A(f^m, [\gamma]) := \min\{s \mid s \text{ is in (2.11)}\},$$

where the minimum is taken over all possible paths γ as above.

We call an arc $\alpha \in A$ a *genuinely periodic* if $\mathrm{EN}_A(f^m, \alpha) = 1$ for all $m \geq 1$.

Lemma **2.3** *(Expansivity of $f | [\mathfrak{T}_f \setminus \mathfrak{B}^{[n]}]$). — Consider an invariant AD A and an arc $\gamma \in A$ aligned with a proper geodesic continuum T_γ of $\mathfrak{T}_f \setminus \mathfrak{B}^{[n]}$. Then*

- *γ is genuinely periodic if and only if T_γ connects two neighboring bushes $\mathfrak{B}_a^{[n]}, \mathfrak{B}_{a+1}^{[n]}$ (with respect to the cyclic order) of a periodic satellite flower $\mathfrak{B}_j^{[n-1]} \supset \mathfrak{B}_a^{[n]}, \mathfrak{B}_{a+1}^{[n]}$ of level $n-1$. In particular, the n^{th} renormalization of f is satellite.*
- *if γ is not genuinely periodic, then*

$$\mathrm{EN}_A(f^{p_n}, \gamma) \geq 2. \tag{2.12}$$

See Figure 8 for illustration.

Proof. — Clearly, if T_γ is in a satellite flower $\mathfrak{B}_j^{[n-1]}$ and connects two neighboring bushes $\mathfrak{B}_a^{[n]}$, $\mathfrak{B}_{a+1}^{[n]}$, then γ is genuinely periodic. It is also clear that (2.12) holds unless T_γ is in a satellite flower $\mathfrak{B}_j^{[n-1]}$. (It can be easily proven for a superattracting model, Section 2.3.2.)

Assume T_γ is in a satellite flower $\mathfrak{B}_j^{[n-1]}$ but connects two non-neighboring bushes $\mathfrak{B}_a^{[n]}$, $\mathfrak{B}_b^{[n]}$, $|a-b| > 1$. Then a certain lift T' of T_γ under f^m with $m = kp_{n-1} < p_n$ cross-intersects T_γ. It follows that $(f^m)^*A$ has an arc aligned with T' but has no arc aligned with $T_\gamma \setminus (\mathfrak{B}_a^{[n],(m)} \cup \mathfrak{B}_b^{[n],(m)})$. This implies that $\mathrm{EN}_A(f^m, \gamma) \geq 2$. □

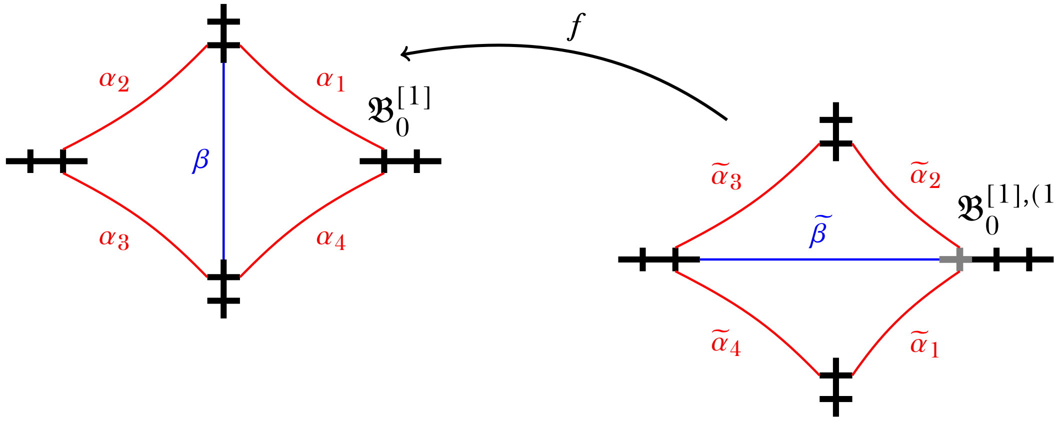

FIGURE 8. — An invariant arc diagram $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta\}$ is depicted for the $1/4$ Rabbit map tuned with the Feigenbaum map. Here, $\mathrm{EN}_A(f, \alpha_i) = 1$ while $\mathrm{EN}_A(f, \beta) = 2$ because β must travel through $\widetilde{\alpha}_2$ and $\widetilde{\alpha}_1$ (β can not cross-intersect its lift $\widetilde{\beta}$). Note that if we replace in A the arc β with its orthogonal arc $\beta^\perp$ connecting $\mathfrak{B}_0^{[1]}$ and $\mathfrak{B}_2^{[1]}$, then the new arc diagram A^{new} will not be invariant: $\widetilde{\beta^\perp} = \beta$ will completely block $\beta^\perp$.

2.4.4. *Arcs inside and outside of satellite flowers.* — Consider an invariant AD A on $X \setminus \mathfrak{B}^{[n]}$. Consider an arc $\alpha \in A$ aligned with a geodesic continuum $T_\alpha \subset \mathfrak{T}_f \setminus \mathfrak{B}^{[n]}$. We say that α is in $\mathfrak{B}_c^{[n-1]}$ if $T_\alpha \subset \mathfrak{B}_c^{[n-1]}$.

Lemma **2.4.** — *Consider an invariant arc diagram A on $X \setminus \mathfrak{B}^{[n]}$ and an arc $\alpha \in A$. Assume that α is not in any satellite flower $\mathfrak{B}_c^{[n-1]}$ of level $n-1$. Then there are two arcs*

$$\alpha_1, \alpha_2 \in \left(f^{2p_n}\right)^*(A) \quad \textit{with} \quad f^{2p_n}(\alpha_1) = f^{2p_n}(\alpha_2) = \alpha_{\mathrm{new}} \in A$$

such that α overflows α_1, α_2 in the following sense: any decomposition (2.11) *of $\alpha = [\gamma]$ rel $\mathfrak{B}^{[n],(2p_n)}$ contains two paths γ_a, γ_b representing $\alpha_1 = [\gamma_a]$ and $\alpha_2 = [\gamma_b]$.*

Proof. — Since T_α is not in any satellite flower $\mathfrak{B}_c^{[n-1]}$, there is a strictly preperiodic component $\mathfrak{B}_d^{[n],(p_n)}$ of $\mathfrak{B}^{[n],(p_n)}$ intersecting T_α. There are two components $T_1, T_2 \subset T_\alpha \setminus \mathfrak{B}^{[n],(2p_n)}$ adjacent to $\mathfrak{B}_d^{[n],(2p_n)}$. Since there is an $m \le 2p_n$ such that $f^m(\mathfrak{B}_d^{[n],(2p_n)})$ contains the critical value, T_1, T_2 have a common injective image of generation $m \le 2p_n$. There must be two arcs $\alpha_1, \alpha_2 \in (f^{2p_n})^*(A)$ aligned with T_1, T_2. □

Assume that the n^{th} renormalization is satellite and consider a satellite flower $\mathfrak{B}_c^{[n-1]}$. Let A be an invariant AD as above. Let $A_c = A(\mathfrak{B}_c^{[n-1]}) \subset A$ be the AD consisting of all arcs from A that are in $\mathfrak{B}_c^{[n-1]}$. Similarly, $A_c^{(p_n)} \subset (f^{p_n})^*(A)$ be the AD consisting of all arcs from $(f^{p_n})^*(A)$ that are in $\mathfrak{B}_c^{[n-1]}$. Then

$$(f^{p_n})^* \colon A_c^{p_n} \xrightarrow{1:1} A_c \tag{2.13}$$

is a bijection and every arc $\alpha \in A_c$ is homotopic to its unique lift $\widetilde{\alpha} \in A_c^{p_n}$, $(f^{p_n})^*(\widetilde{\alpha}) = \alpha$ in the following sense: $T_\alpha \setminus \mathfrak{B}^{[n],(p_n)} = T_{\widetilde{\alpha}}$.

2.4.5. *Inhomogeneous configuration.* — The discussion here will be only used in Section 7. Consider a ql map $f: X \to Y$ with a cycle of bushes $\mathfrak{B}^{[1]}$. Write $\Upsilon := \mathfrak{B}^{[1]} \cup \mathfrak{K}_0^{[1]}$. Note that f does not permute components of Υ.

An AD $A = \{\alpha_i\}$ on $X \setminus \Upsilon$ is *horizontal* if every α_i connects components of Υ. A horizontal AD $A = \{\alpha_i = [\gamma_i]\}$ is f^{p_1}*-invariant* if every α_i can be represented up to a proper homotopy in $X \setminus \Upsilon$ in an arbitrary small neighborhood of $f^{-p_1}(\Upsilon \cup \Gamma)$, where $\Gamma = \bigcup_i \gamma_i$. By replacing f with its superattracting model (Section 2.3.2, compare also with Lemma 2.2), we obtain:

Lemma **2.5** *(Alignment with $\mathfrak{T}_f$). — If $A = \{\alpha_i\}$ is an f^{p_1}-invariant AD on $Y \setminus \Upsilon$ as above, then A is* aligned with the Hubbard continuum $\mathfrak{T}_f$*: every α_i can be represented in an arbitrary small neighborhood of a geodesic continuum T_i of $\mathfrak{T}_f \setminus \Upsilon$ connecting components of Υ.*

For $m \geq 1$ and $\alpha \in A$, the *expansivity number* $\mathrm{EN}_A(f^{mp_1}, \alpha)$ is defined in the same way as in Section 2.4.3; i.e., it is the smallest number of arcs in $f^{-p_1 m}(A)$ overflown by α. An arc $\alpha \in A$ is *genuinely periodic* if $\mathrm{EN}_A(f^{p_1 m}, \alpha) = 1$ for all $m \geq 1$. By the same argument as in Lemma 2.3, we have:

Lemma **2.6** *(Expansivity of $f \mid [\mathfrak{T}_f \setminus \Upsilon]$). — Consider an f^{p_1}-invariant AD A on $Y \setminus \Upsilon$ and an arc $\gamma \in A$ aligned with a proper geodesic continuum T_γ of $\mathfrak{T}_f \setminus \Upsilon$. If the first renormalization of f is primitive, then*

$$\mathrm{EN}_A(f^{p_1}, \gamma) \geq 2. \tag{2.14}$$

2.5. *Teichmüller contraction.* — The Teichmüller contraction comes from the observation that the restriction of a qc conjugacy to deeper renormalization levels can only decrease the dilatation. Therefore, the renormalization orbits (for bounded-type Feigenbaum families) can escape to infinity with at most linear rate. This fact can be traced back to Sullivan [31], where structures of the Teichmüller spaces (reminiscent to the Thurston's machinery) were brought into the Renormalization Theory. The proposition below is stated for quadratic maps; the Straightening Theorem easily extends it quadratic-like maps.

Proposition **2.7.** — *For every combinatorial bound $\bar{p}$, there is a constant $\Delta = \Delta_{\bar{p}} > 1$ such that the following holds. Let $f_c(z) = z^2 + c$ be an infinitely renormalizable quadratic polynomial of bounded type $\bar{p}$, see Section 2.2.5. Then $g_n = \mathcal{R}^n f_c$ has ql prerenormalization $g_n = f_c^{p_n}: X_n \to Y_n$, $X_n \Subset Y_n$ such that*

(I) $\mathcal{W}(Y_n \setminus X_n) = O(\Delta^n)$; and
(II) $Y_n \setminus \mathfrak{K}_{g_n}$ is disjoint from $\mathfrak{B}_{f_c}^{[n]}$.

Proof. — Let $\mathcal{M}_i, i \in I$ be the (finite) set of all maximal $\mathcal{M}$-copies in $\mathcal{M}$ of period $\leq \bar{p}$, and let

$$R\colon \bigcup_{i\in I} \mathcal{M}_i \longrightarrow \mathcal{M}$$

be the ql straightening map. Let us remove from every satellite $\mathcal{M}_i$ a small open neighborhood of its cusp such that the remaining $\mathcal{M}_i^\circ$ contains all the preimages of the $\mathcal{M}_j$ under $R\colon \mathcal{M}_i \to \mathcal{M}$. For a primitive copy $\mathcal{M}_i$, set $\mathcal{M}_i^\circ := \mathcal{M}_i$. By compactness, there is a $\Delta > 1$ such that every f_c with $c \in \bigcup_i \mathcal{M}_i^\circ$ has a ql restriction $g_1 = f_c^{p_1}\colon X_{1,c} \to Y_{1,c}$ around its maximal little Julia set containing 0 such that g_1 satisfies (II) and such that g_1 is hybrid conjugate via $h_c\colon Y_{1,1} \to Y_{R(c)}$ with dilatation $\leq \Delta$ to a ql restriction $f_{R(c)}\colon X_{R(c)} \to Y_{R(c)}$. Moreover, we can select the $Y_{1,c}$ and Y_c so that $Y_c \Supset Y_{1,c}$.

Wright $c_n := R^n(c)$. Then

$$\bar{h} := \left(h_{c_{n-1}} \circ \cdots \circ h_{c_1} \circ h_c\right)^{-1}$$

is a hybrid conjugacy from $f_{c_n}\colon X_{c_n} \to Y_{c_n}$ to a ql restriction $g_n = f_c^{p_1\dots p_n}\colon X_{n,c} \to Y_{n,c}$ satisfying (II). Since the dilatation of $\bar{h}$ is $\leq \Delta^n$, Property (I) follows. □

Remark **2.8.** — Property (II) implies that the prerenormalization $g_n\colon X_n \to Y_n$ is unbranched: $\mathfrak{P}(f_c) \cap Y_n \subset \mathfrak{K}(g_n)$, compare with [28]. Moreover, by induction, Y_n is disjoint from

$$\Upsilon := \bigcup_{m\leq n} \left(\mathfrak{B}_f^{[m]} \setminus \mathfrak{B}_{f,0}^{[m]}\right).$$

This implies that the quadratic-like prerenormalizations $g_n\colon X_n \to Y_n$ can be univalently lifted to the dynamical plane of any $\psi^\bullet$ renormalization

$$\mathcal{R}^{n_1\bullet} \circ \mathcal{R}^{n_2\bullet} \circ \cdots \circ \mathcal{R}^{n_s\bullet}(f_c)$$

of f_c, see Section 3.4.3.

3. $\psi^\bullet$-ql renormalization and near-degenerate regime

In this section, we will introduce $\psi^\bullet$-ql renormalization and discuss tools to detect its degeneration; see [1, 9, 18, 22, 23] for details.

Given a compact subset $K \Subset S$, we denote by $\mathcal{F}(S, K)$ the family of non-trivial proper curves in $S \setminus K$ connecting ∂K to ∂S. We write

$$\mathcal{W}(S,K) = \mathcal{W}\big(\mathcal{F}(S,K)\big). \tag{3.1}$$

3.1. *Outline.* — Consider a ql map $f\colon X \to Y$ and let $\mathfrak{B}^{[n]}$ be its level n cycle of little bushes.

Around every periodic $\mathfrak{K}_i^{[n]}$ there is an associated ql prerenormalization

$$f_{n,i} = f^{p_n}\colon X_{n,i} \longrightarrow Y_{n,i}. \tag{3.2}$$

In Section 3.4, we will define the $\psi^\bullet$-ql renormalization

$$F_{n,i} = (f_{n,i},\ \iota_{n,i})\colon U_{n,i} \rightrightarrows V_{n,i} \tag{3.3}$$

of $f\colon X \to Y$ by *extending* (3.2) *along all curves* in $Y \setminus \mathfrak{B}^{[n]}$, see Section 3.4.3. This is similar to the ψ-ql renormalization [18], where the extension is performed along all curves in $Y \setminus \mathfrak{K}^{[n]}$ under the assumption that the $\mathfrak{K}_i^{[n]}$ are pairwise disjoint (the primitive case); see Section 3.4.1 and Remark 3.5.

The correspondence $F_{n,i}$ in (3.3) is called a $\psi^\bullet$-ql map. It consists of a degree 2 branched covering $f_{n,i}$ and an immersion $\iota_{n,i}$, see definitions in Section 3.4.2. We define the Julia set and bush of $F_{n,i}$ to be that of the ql map $f_{n,i}$ (3.2):

$$\mathfrak{K}_{F_{n,i}} := \mathfrak{K}_{f_{n,i}} \quad \text{and} \quad \mathfrak{B}_{F_{n,i}} := \mathfrak{B}_{f_{n,i}}.$$

Similarly, $\mathfrak{K}_F$ and $\mathfrak{B}_F$ are defined for any $\psi^\bullet$-ql map, see (3.14).

A $\psi^\bullet$-ql renormalization can be naturally defined for a $\psi^\bullet$-ql map; i.e., $\psi^\bullet$-ql renormalization can be iterated. The Sup-Chain Rule for renormalization domains is stated in Section 3.4.4.

Just like a ql map $f\colon X \to Y$ can be restricted to $f\colon X^{k+1} \to X^k$, where $X^{k+1} = f^{-k}(X)$, a $\psi^\bullet$-ql map $F = (f, \iota)\colon U \rightrightarrows V$ can be restricted to

$$F = (f, \iota)\colon U^{k+1} \rightrightarrows U^k$$

by considering the fiber product, Section 3.4.6. We have a natural $2^k : 1$ branched covering $f^k\colon U^k \to V$ and an immersion $\iota^k\colon U^k \to V$.

The *width* of a $\psi^\bullet$-ql map $F\colon U \rightrightarrows V$ is

$$\mathcal{W}_\bullet(F) := \mathcal{W}(V \setminus \mathfrak{B}_F).$$

We denote by $\mathcal{A}^{[n]} \equiv \mathcal{A}_F^{[n]}$ the weighted arc diagram (WAD) of $V \setminus \mathfrak{B}^{[n]}$: it is a formal sum of weighted arcs representing rectangles in the thick-thin decomposition of $V \setminus \mathfrak{B}^{[n]}$; see Section 3.3 and Figure 9. The *horizontal part* $\mathcal{A}_{\text{hor}}^{[n]}$ of $\mathcal{A}^{[n]}$ consists of weighted arcs connecting components of $\mathfrak{B}^{[n]}$. The *vertical part* $\mathcal{A}_{\text{ver}}^{[n]}$ of $\mathcal{A}^{[n]}$ consists of weighted arcs connecting ∂V and components of $\mathfrak{B}^{[n]}$. The *local WAD* $\mathcal{A}_i^{[n]}$ consists of arcs in $\mathcal{A}^{[n]}$ adjacent to $\mathfrak{B}_i^{[n]}$, where the weight of self-arcs to $\mathfrak{B}_i^{[n]}$ is doubled. More generally, $\mathcal{A}^{[n],(m),k}$ is the WAD of $U^k \setminus \mathfrak{B}^{[n],(m)}$, and the WADs $\mathcal{A}_{\text{hor}}^{[n],(m),k}, \mathcal{A}_i^{[n],(m),k}, \dots$ are defined accordingly, see Section 3.5.2. We have

$$\mathcal{W}(F_{n,i}) = \mathcal{W}\left(\mathcal{A}_i^{[n]}\right) + O_{p_n}(1).$$

By *identifying up to homotopy* $U^k \setminus \mathfrak{B}^{[n],(m)}$ and $U^{k+1} \setminus \mathfrak{B}^{[n],(m)}$, we obtain $\mathcal{A}_{\text{hor}}^{[n],(m),k+1} \le \mathcal{A}_{\text{hor}}^{[n],(m),k}$, see Section 3.5.6. Since the complexity of $A_{\text{hor}}^{[n],k} \equiv \text{AD}[\mathcal{A}_{\text{hor}}^{[n],k}]$ decreases, the $A_{\text{hor}}^{[n],k}$ are essentially invariant for $k \ge 3p_n$. Then either most of $\mathcal{A}_{\text{hor}}^{[n]}$ is in $\mathcal{A}_{\text{hor}}^{[n],k}$ or a substantial part of $\mathcal{A}_{\text{hor}}^{[n]}$ is in $\mathcal{A}_{\text{ver}}^{[n],k}$ – this is a key dichotomy in Section 4; see also the dichotomy "(I) vs (II)" in Section 1.3.2.

3.2. *Rectangles.* — A standard *Euclidean rectangle* is a rectangle $E_x := [0,x] \times [0,1] \subset \mathbb{C}$, where:

- $(0,0), (x,0), (x,1), (0,1)$ are four vertices of E_x,
- $\partial^h E_x = [0,x] \times \{0,1\}$ is the horizontal boundary of E_x,
- $\partial^{h,0} E_x := [0,x] \times \{0\}$ is the *base* of E_x,
- $\partial^{h,1} E_x := [0,x] \times \{1\}$ is the *roof* of E_x,
- $\partial^v E_x = \{0,x\} \times [0,1]$ is the *vertical* boundary of E_x,
- $\mathcal{F}(E_x) := \{\{t\} \times [0,1] \mid t \in [0,x]\}$ is the *vertical foliation* of E_x,
- $\mathcal{F}^{\text{full}}(E_x) := \{\gamma\colon [0,1] \to E_x \mid \gamma(0) \in \partial^{h,0} E_x,\ \gamma(1) \in \partial^{h,1} E_x\}$ is the *full family of curves* in E_x;
- $\mathcal{W}(E_x) = \mathcal{W}(\mathcal{F}(E_x)) = \mathcal{W}(\mathcal{F}^{\text{full}}(E_x)) = x$ is the *width* of E_x,
- $\operatorname{mod}(E_x) = 1/\mathcal{W}(E_x) = 1/x$ the extremal length of E_x.

By a *(topological) rectangle* in a Riemann surface we mean a closed Jordan disk $\mathcal{R}$ together with a conformal map $h\colon \mathcal{R} \to E_x$ into the standard rectangle E_x. The vertical foliation $\mathcal{F}(\mathcal{R})$, the full family $\mathcal{F}^{\text{full}}(\mathcal{R})$, the base $\partial^{h,0}\mathcal{R}$, the roof $\partial^{h,1}\mathcal{R}$, the vertices of $\mathcal{R}$, and other objects are defined by pulling back the corresponding objects of E_x. Equivalently, a rectangle $\mathcal{R}$ is a closed Jordan disk together with four marked vertices on $\partial\mathcal{R}$ and a chosen base between two vertices.

A *genuine subrectangle* of E_x is any rectangle of the form $E' = [x_1, x_2] \times [0,1]$, where $0 \le x_1 < x_2 \le x$; it is identified with the standard Euclidean rectangle $[0, x_2 - x_1] \times [0,1]$ via $z \mapsto z - x_1$. A genuine subrectangle of a topological rectangle is defined accordingly.

A *subrectangle* of a rectangle $\mathcal{R}$ is a topological rectangle $\mathcal{R}_2 \subset \mathcal{R}$ such that $\partial^{h,0}\mathcal{R}_2 \subset \mathcal{R}$ and $\partial^{h,1}\mathcal{R}_2 \subset \mathcal{R}$. By monotonicity: $\mathcal{W}(\mathcal{R}_2) \le \mathcal{W}(\mathcal{R})$.

Assume that $\mathcal{W}(E_x) > 2$. The *left and right* 1*-buffers* of E_x are defined

$$B_1^\ell := [0,1] \times [0,1] \quad \text{and} \quad B_1^\rho := [x-1, x] \times [0,1]$$

respectively. We say that the rectangle

$$E_x^{\text{new}} := [1, x-1] \times [0,1] = \overline{E_x \setminus \left(B_1^\ell \cup B_1^\rho\right)}$$

is obtained from E_x by *removing* 1*-buffers*. If $\mathcal{W}(E_x) \le 2$, then we set $E_x^{\text{new}} := \emptyset$. Similarly, buffers of any width are defined.

3.2.1. *Monotonicity and Grötzsch inequality.* — We say a family of curves $\mathcal{S}$ *overflows* a family $\mathcal{G}$ if every curve $\gamma \in \mathcal{S}$ contains a subcurve $\gamma' \in \mathcal{G}$. We also say that:

- a family of curves $\mathcal{F}$ *overflows* a rectangle $\mathcal{R}$ if $\mathcal{F}$ overflows $\mathcal{F}^{\text{full}}(\mathcal{R})$;
- a rectangle $\mathcal{R}_1$ overflows another rectangle $\mathcal{R}_2$ if $\mathcal{F}(\mathcal{R}_1)$ overflows $\mathcal{F}^{\text{full}}(\mathcal{R}_2)$.

If $\mathcal{F}$ overflows a family or a rectangle $\mathcal{G}$, then $\mathcal{G}$ is wider than $\mathcal{F}$:

$$\mathcal{W}(\mathcal{F}) \le \mathcal{W}(\mathcal{G}). \tag{3.4}$$

If $\mathcal{F}$ overflows both $\mathcal{G}_1, \mathcal{G}_2$, and $\mathcal{G}_1, \mathcal{G}_2$ are disjointly supported, then the *Grötzsch* inequality states:

$$\mathcal{W}(\mathcal{F}) \le \mathcal{W}(\mathcal{G}_1) \oplus \mathcal{W}(\mathcal{G}_2), \tag{3.5}$$

where $x \oplus y = (x^{-1} + y^{-1})^{-1}$ is the harmonic sum.

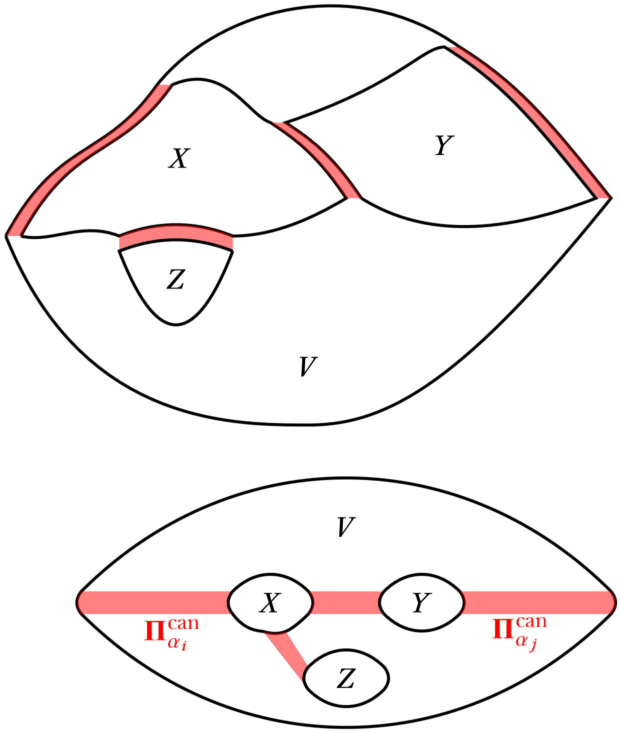

FIGURE 9. — Top: the thin-thick decomposition of $V \setminus (X \cup Y \cup Z)$. As we are interested in homotopy properties of wide rectangles, they will usually be depicted as narrow rectangles (bottom) and denoted by $\mathbf{\Pi}^{\text{can}}_{\alpha_i}$, where α_i is an arc representing the homotopy class of the rectangle $\mathbf{\Pi}^{\text{can}}_{\alpha_i}$. The Weighted Arc Diagram (WAD) is the formal sum $\mathcal{A} = \sum_i W(\mathbf{\Pi}^{\text{can}}_{\alpha_i})\alpha_i$, while the Arc Diagram (AD) is the associated set of curves $A = \{\alpha_i\} \equiv \text{AD}[\mathcal{A}]$. Thin annuli of the thick-thin decomposition are disregarded in the WAD.

3.3. *Weighted arc diagrams.* — Let us recall the notion of the Weighted Arc Diagram (WAD) describing wide rectangles in the thick-thin decomposition of a Riemann surface with boundary, see Figure 9 for illustration and brief summary. We also recall that WAD were abstractly defined in Section 2.4.1.

3.3.1. *Proper rectangles.* — A *proper rectangle* R in an open hyperbolic surface S is a rectangle in its ideal completion $\overline{S} = S \cup \partial^i S$ such that $\partial^h R \subset \partial^i S$. We naturally view vertical curves of R as proper (and open) paths in S. We say that R *connects* boundary components $J_0, J_1 \subset \partial^i S$ if $\partial^{h,0} R \subset J_0$, $\partial^{h,1} R \subset J_1$.

If $S = U \setminus K$, where U is a topological disk and $K \Subset U$ is a compact subset with finitely many connected components, then $\partial^i S$ has one *outer* component $\partial^i U$; all remaining *inner* components are parameterized by the components of K. If J_0, J_1 represent the boundaries of components K_0, K_1 of K, then we say that the above R connects K_0, K_1.

3.3.2. *Universal cover.* — For compact Riemann surface V as in Section 2.4.1, write its non-empty boundary as $\partial V = \bigsqcup_k J_k$. We assume that V has negative Euler characteristic $\chi(V) < 0$. We set $\pi : \mathbb{D} \to \operatorname{int} V$ to be the universal covering, and we let $\Lambda \subset \mathbb{T} = \partial\mathbb{D}$ be the limit set for the group Δ of deck transformations of π. Then π extends continuously to $\mathbb{T} \setminus \Lambda$, and $\pi : \overline{\mathbb{D}} \setminus \Lambda \to V$ is the universal covering of V. Moreover, π restricted to any component of $\mathbb{T} \setminus \Lambda$ gives us a universal covering of some component J_k of ∂V. Such a component of $\mathbb{T} \setminus \Lambda$ will be denoted $\mathbf{J}_k$ (usually we will need only one component for each k, so we will not use an extra label for it). The stabilizer of $\mathbf{J}_k$ in Δ is a cyclic group generated by a hyperbolic Möbius transformation τ_k with fixed points in $\partial\mathbf{J}_k$.

3.3.3. *Covering annuli and local weights.* — We let

$$\mathbb{A}(V, J_k) := \mathbb{D}/\langle\tau_k\rangle \tag{3.6}$$

be the *covering annulus* of V corresponding to J_k. We call its width

$$\mathcal{W}(J_k) := \mathcal{W}\big[\mathbb{A}(V, J_k)\big]$$

the *local weight* of J_k. We let

$$\mathcal{W}(V) = \sum_k \mathcal{W}(J_k)$$

be the *total weight* of V.

3.3.4. *Canonical arc diagram.* — Recall that an arc α is a proper path in V up to proper homotopy. Any arc α connects some components J_k and J_i of ∂V and lifts to an arc $\boldsymbol{\alpha}$ in $\mathbb{D}$ connecting some intervals $\mathbf{J}_k$ and $\mathbf{J}_i$. We let $\mathcal{W}(\boldsymbol{\alpha}) = \mathcal{W}_{\overline{\mathbb{D}}}(\mathbf{J}_k, \mathbf{J}_i)$ be the width of all curves in $\mathbb{D}$ connecting $\mathbf{J}_k$ and $\mathbf{J}_i$. Then $\mathcal{W}(\boldsymbol{\alpha})$ is also the width of the rectangle $\boldsymbol{\Pi}_{\boldsymbol{\alpha}} = \overline{\mathbb{D}}$ with horizontal sides $\mathbf{J}_k$ and $\mathbf{J}_i$. It is independent of the lift used.

If $\mathcal{W}(\boldsymbol{\alpha}) > 2$, then by removing from $\boldsymbol{\Pi}_{\boldsymbol{\alpha}}$ the square buffers we obtain a rectangle $\boldsymbol{\Pi}^{\text{can}}_{\boldsymbol{\alpha}}$. In this case we let

$$\mathcal{W}^{\text{can}}(\alpha) = \mathcal{W}^{\text{can}}(\boldsymbol{\alpha}) = \mathcal{W}\big(\boldsymbol{\Pi}^{\text{can}}_{\boldsymbol{\alpha}}\big) = \mathcal{W}(\boldsymbol{\alpha}) - 2.$$

By construction, the Δ-orbit of $\boldsymbol{\Pi}^{\text{can}}_{\boldsymbol{\alpha}}$ consists of pairwise disjoint rectangles. Therefore, $\boldsymbol{\Pi}^{\text{can}}_{\boldsymbol{\alpha}}$ projects injectively onto a rectangle $\boldsymbol{\Pi}^{\text{can}}_{\alpha}$ in V. Moreover, different $\boldsymbol{\Pi}^{\text{can}}_{\boldsymbol{\alpha}_i}$ project to pairwise disjoint $\boldsymbol{\Pi}^{\text{can}}_{\alpha_i}$.

If $\mathcal{W}(\boldsymbol{\alpha}) \le 2$ we let $\mathcal{W}^{\text{can}}(\alpha) = 0$ and $\boldsymbol{\Pi}^{\text{can}}(\alpha) := \emptyset$. Arcs with positive weight form the *canonical weighted arc diagram (WAD) on V:*

$$\mathcal{A}_V \equiv \text{WAD}(V) := \sum_{\mathcal{W}^{\text{can}}(\alpha) > 0} \mathcal{W}^{\text{can}}(\alpha)\alpha. \tag{3.7}$$

Every $\boldsymbol{\Pi}^{\text{can}}(\alpha)$ supports the canonical vertical foliation $\mathcal{F}(\boldsymbol{\Pi}^{\text{can}}(\alpha)$. Altogether these foliations form the *canonical lamination* of V:

$$\mathcal{F}(\mathcal{A}_V) := \bigsqcup_{\alpha \in \mathcal{A}_V} \mathcal{F}(\boldsymbol{\Pi}^{\text{can}}(\alpha)), \tag{3.8}$$

We set

$$\mathcal{W}(\mathcal{A}_V) := \sum_{\alpha\in\mathcal{A}_V} \mathcal{W}(\alpha) = \mathcal{W}\big(\mathcal{F}(\mathcal{A}_V)\big). \tag{3.9}$$

The canonical *arc diagram* of V is the set of arcs in $\mathcal{A}_V$

$$A_V := \{\alpha \mid \mathcal{W}^{\mathrm{can}}(\alpha) > 0\} \equiv \mathrm{AD}[\mathcal{A}_V].$$

3.3.5. *Local WADs and the Thin-Thick Decomposition.* — For a boundary component J_k of ∂V, its *local WAD* $\mathcal{A}_{J_k}$ consists of weighted arcs of $\mathcal{A}_V$ adjacent to J_k such that the weights of self-arcs adjacent to J_k are doubled. We have:

$$\sum_{J_k} \mathcal{A}_{J_k} = 2\mathcal{A}_V.$$

See [23, Section 7.6] for a reference of the following fact (originated in [19]).

Theorem **3.1** *(Thin-Thick Decomposition). — For any compact Riemann surface V with boundary $\partial V = \bigsqcup_k J_k$, we have:*

$$\mathcal{W}(J_k) - C \le \mathcal{W}(\mathcal{A}_{J_k}) \ \le \mathcal{W}(J_k) \quad \textit{for every } \ J_k$$

and

$$\sum_{J_k} \mathcal{W}(J_k) - C \le 2\mathcal{W}(\mathcal{A}_V) \le \sum_{J_k} \mathcal{W}(J_k) = \mathcal{W}(V),$$

where C depends only on the topological type of V.

3.3.6. *Sub-diagrams and removing buffers.* — Let $\mathcal{A} \equiv \mathcal{A}_V$ be the canonical WAD of V as in Section 3.3.4, and let $B \subset A$ be a subset of arcs from $A \equiv \mathrm{AD}(\mathcal{A})$. Then the *WAD induced by* B is the formal sum $\mathcal{B} := \sum_{\alpha\in B} \mathcal{W}^{\mathrm{can}}(\alpha)\alpha$, compare to (3.7). The weight $W(\mathcal{B})$ of $\mathcal{B}$ and its canonical lamination $\mathcal{F}(\mathcal{B})$ is defined as for $\mathcal{A}$, see (3.8) and (3.9). In Section 3.5.2, we will specify the vertical and horizontal parts of the WAD associated with $\psi^\bullet$-ql maps.

For $B, \mathcal{B}$ as above and $c > 0$, we define

$$\mathcal{B} - c := \sum_{\substack{\alpha\in B\\ \mathcal{W}(\alpha)>c}} (\mathcal{W}^{\mathrm{can}}(\alpha) - c)\alpha, \quad \text{and} \quad B - c := \mathrm{AD}(\mathcal{B} - c);$$

i.e., we remove the c-weight from each arc. For a rectangle $\mathbf{\Pi}^{\mathrm{can}}(\alpha)$ as in Section 3.3.4, we denote by $\mathbf{\Pi}^{\mathrm{can}}(\alpha) - c$ the rectangle obtained from $\mathbf{\Pi}^{\mathrm{can}}(\alpha)$ by removing the $c/2$-buffers on each side. We call

$$\mathcal{F}(\mathcal{B} - c) := \bigsqcup_{\alpha\in\mathcal{B}-c} \mathcal{F}(\mathbf{\Pi}^{\mathrm{can}}(\alpha) - c)$$

the *canonical lamination of* $\mathcal{B} - c$. We also denote by

$$\mathcal{F}^{\mathrm{full}}(\mathcal{B} - c) := \bigsqcup_{\alpha\in\mathcal{B}-c} \mathcal{F}^{\mathrm{full}}(\mathbf{\Pi}^{\mathrm{can}}(\alpha) - c)$$

the full family of vertical curves within respective rectangles, see Section 3.2.

3.3.7. *Transformation rules.* —

Lemma **3.2.** — *Let $i : S' \to S$ be a holomorphic map between two compact Riemann surfaces with boundary. Then*

(i) *If a boundary component J' of S' is mapped with degree d to a boundary component J of S then $\mathcal{W}(J') \le d \cdot \mathcal{W}(J)$;*

(ii) *If an arc α' of S' is mapped to an arc α of S then $\mathcal{W}(\alpha') \le \mathcal{W}(\alpha)$.*

Lemma **3.3.** — *Let $\mathbf{i} : \mathbf{\Pi}' \to \mathbf{\Pi}$ be a holomorphic map between two rectangles which extends to homeomorphisms between the respective horizontal sides $\boldsymbol{I}' \to \boldsymbol{I}$ and $\mathbf{J}' \to \mathbf{J}$. Then*

$$\mathcal{W}(\mathbf{\Pi}') \le \mathcal{W}(\mathbf{\Pi}).$$

3.3.8. *WAD under covering.* — Assume that $f : U \to V$ as a finite-degree covering between surfaces with boundaries, where $\chi(V) < 0$. Then, see [18, Lemma 3.3]

$$\mathcal{A}_U = f^*\mathcal{A}_V, \quad \text{where} \quad f^*\mathcal{A}_V = \sum_{\alpha \in \mathcal{A}_V} W^{\text{can}}(\alpha) f^{-1}(\alpha) \tag{3.10}$$

and $f^{-1}(\alpha)$ is the set of preimages of α.

3.4. *$\psi^\bullet$-ql renormalization.* — As in Section 2.2.3, consider a ql map $f : X \to Y$ and its ql prerenormalization

$$f_{n,0} = f^{p_n} : X_n \longrightarrow Y_n. \tag{3.11}$$

3.4.1. *Motivation.* — A major technical issue is that there is no natural choice of Y_n in (3.11) so that $\mathcal{W}(Y_n \setminus \mathfrak{K}_n)$ is optimal and satisfies, in particular, (1.2). To handle this problem, a ψ-ql renormalization was introduced in [18]: assuming that (3.11) is primitive and $Y_n \cap \boldsymbol{\mathfrak{K}}^{[n]} = \mathfrak{K}_0^{[n]}$, *extend $f_n : X_n \to Y_n$ along all curves in $Y_n \setminus \boldsymbol{\mathfrak{K}}^{[n]}$*; the result is a correspondence $F_n = (f_n, \iota_n) : U_n \rightrightarrows V_n$, where

$$V_n \coloneqq \mathbb{A}\left(Y \setminus \boldsymbol{\mathfrak{K}}^{[n]},\ \partial\mathfrak{K}_0^{[n]}\right) \bigcup_{\partial\mathfrak{K}_0^{[n]}} \mathfrak{K}_0^{[n]}$$

is the covering annulus (3.6) of $Y \setminus \boldsymbol{\mathfrak{K}}^{[n]}$ rel $\partial\mathfrak{K}_0^{[n]}$ glued with $\mathfrak{K}_0^{[n]}$ and

$$U_n \coloneqq \mathbb{A}\left(X \setminus f^{-1}(\boldsymbol{\mathfrak{K}}^{[n]}),\ \partial\mathfrak{K}_0^{[n]}\right) \bigcup_{\partial\mathfrak{K}_0^{[n]}} \mathfrak{K}_0^{[n]}$$

is the covering annulus of $X \setminus f^{-1}(\boldsymbol{\mathfrak{K}}^{[n]})$ rel $\partial\mathfrak{K}_0^{[n]}$ glued with $\mathfrak{K}_0^{[n]}$. The correspondence F_n is called a *ψ-ql map* and it is independent of the choice of Y_n in (3.11). A ψ-ql renormalization can be naturally iterated. Moreover, ψ-ql bounds can be converted into ql bounds, see Lemma 3.7.

In this section, we will introduce $\psi^\bullet$-renormalization by replacing $\boldsymbol{\mathfrak{K}}^{[n]}$ with the periodic cycle $\boldsymbol{\mathfrak{B}}^{[n]}$ of little bushes so that it is also applicable for satellite combinatorics.

Remark **3.4.** — The viewpoint that various classes of self-correspondences $(g,h)\colon A \rightrightarrows B$ form interesting dynamical systems was popularized by Sullivan and later by A. Epstein in the context of deformation spaces. In the 2000s, it became apparent that self-correspondences give a natural framework to study some of the classical dynamical systems [17, 18, 30]. See also [3, 33] for more recent developments.

3.4.2. *Definitions.* — A *pseudo•-quadratic-like map* ("$\psi^\bullet$-ql map") is a pair of holomorphic maps

$$F = (f,\iota)\colon\ (U,\mathfrak{B}') \rightrightarrows (V,\mathfrak{B}) \tag{3.12}$$

between two conformal disks U, V with the following properties:

(I) $f\colon U \to V$ is a double branched covering (we usually normalize it so that its critical point is located at 0);

(II) $\iota\colon U \to V$ is an immersion;

(III) $\mathfrak{B}$ and $\mathfrak{B}'$ are hulls (compact connected full sets) such that

$$\iota^{-1}(\mathfrak{B}) = \mathfrak{B}' \subset f^{-1}(\mathfrak{B});$$

(IV) there exist neighborhoods $X' \supset \mathfrak{B}'$ and $X \supset \mathfrak{B}$ with the following property: $\iota : X' \to X$ is a conformal isomorphism such that

$$f_X \coloneqq f\circ(\iota \mid X')^{-1} : X \longrightarrow f(X') =: Y \tag{3.13}$$

is a quadratic-like map with connected filled Julia set

$$\mathfrak{K}_F \coloneqq \mathfrak{K}(f_X : X \longrightarrow Y), \tag{3.14}$$

and such that $\mathfrak{B} \equiv \mathfrak{B}_F \equiv \mathfrak{B}(f_X)$ is the bush of f_X.

Since ι is a conformal isomorphism in a neighborhood of $\mathfrak{K}_F$, we will below identify

$$\mathfrak{K}_F \simeq (\iota \mid X)^{-1}(\mathfrak{K}_F) \quad \text{and hence} \quad \mathfrak{B} \simeq \mathfrak{B}' \equiv \mathfrak{B}_F,$$

and write

$$F\colon (U,\mathfrak{B}) \rightrightarrows (V,\mathfrak{B}) \quad \text{or} \quad F\colon U \rightrightarrows V.$$

Let us say that a subset $\Omega \subset \mathfrak{K}_F$ is *ι-proper* if

$$\iota^{-1}(\Omega) = \Omega.$$

(III) implies that $\mathfrak{B}_F$ is ι-proper. If $\mathfrak{K}_F$ is ι-proper, then F is ψ-ql map [18].

3.4.3. *$\psi^\bullet$-ql renormalization.* — Consider a $\psi^\bullet$-ql map F from (**3.12**), and assume that f_X (see (**3.13**)) is $n+1$ DH renormalizable. Let $f_{n,i} = f^{p_n}\colon X_{n,i} \to Y_{n,i}$ be a ql prerenormalization of $f_X\colon X \to Y$. The associated $\psi^\bullet$-ql renormalization is the extension of $f_{n,i}\colon X_{n,i} \to Y_{n,i}$ along all curves in $V \setminus \boldsymbol{\mathfrak{B}}_n$, compare with Section 3.4.1; the result is a $\psi^\bullet$-ql map $F_{n,i} = (f_{n,i}, \iota_{n,i})\colon U_{n,i} \rightrightarrows V_{n,i}$, where

$$V_{n,i} \coloneqq \mathbb{A}\Big(V \setminus \boldsymbol{\mathfrak{B}}^{[n]},\ \partial\mathfrak{B}_i^{[n]}\Big) \bigcup_{\partial\mathfrak{B}_i^{[n]}} \mathfrak{B}_i^{[n]}$$

is the covering annulus (3.6) of $V \setminus \mathfrak{B}^{[n]}$ rel $\partial\mathfrak{B}_i^{[n]}$ glued with $\mathfrak{B}_i^{[n]}$ and

$$U_{n,i} := \mathbb{A}\Big(U \setminus f_{n,i}^{-1}(\mathfrak{B}^{[n]}),\ \partial\mathfrak{B}_i^{[n](1)}\Big) \bigcup_{\partial\mathfrak{B}_i^{[n](1)}} \mathfrak{B}_i^{[n](1)}$$

is the covering annulus of $U \setminus f^{-1}(\mathfrak{B}^{[n]})$ rel $\partial\mathfrak{B}_i^{[n](1)}$ glued with $\mathfrak{B}_i^{[n](1)}$.

We will write suppress index "0" for $F_{n,0}$:

$$F_{n,0} \equiv F_n = (f_n, \iota_n) \colon U_n \rightrightarrows V_n, \quad \text{where} \quad U_n \equiv U_{n,0},\ V_n \equiv V_{n,0}.$$

We write

$$\mathcal{R}^{n\bullet}(F) := F_n \equiv F_{n,0}. \tag{3.15}$$

Remark **3.5.** — The theory works similarly if the bush $\mathfrak{B}$ is replaced with any connected forward invariant set Υ satisfying $\mathfrak{T} \subset \Upsilon \subset \mathfrak{K}$, where $\mathfrak{T}$ is the Hubbard continuum. Namely, let us say that a map

$$F = (f, \iota) \colon (U, \Upsilon') \rightrightarrows (V, \Upsilon) \tag{3.16}$$

is ψ^Υ-*ql* if it satisfies (I)–(IV) from Section 3.4.2 so that $\mathfrak{B}' \simeq \mathfrak{B} = \mathfrak{B}(f_X)$ are replaced with

$$\Upsilon' = \iota^{-1}(\Upsilon) \simeq \Upsilon = \Upsilon(f_X), \quad \text{i.e.,}\ \ \Upsilon \text{ is } \iota\text{-proper.}$$

For example, one can take $\Upsilon = \mathfrak{T}$ or $\Upsilon = \mathfrak{T} \cup \mathfrak{B}^{[m]}$. To define "$\psi^\Upsilon$-renormalization", one should extend a ql prerenormalization $f_{n,i} \colon X_{n,i} \to Y_{n,i}$ along all curves in $V \setminus \bigcup_j \Upsilon_{f_{n,j}}$. Moreover, $\Upsilon_{f_{n,j}}$ can often be enlarged, see Section 3.4.5.

3.4.4. *Sup-Chain Rule for renormalization domains.* — Consider a $\psi^\bullet$-ql map F as in (3.12). Since

(1) $\mathcal{R}^{n_1\bullet} \circ \mathcal{R}^{n_2\bullet}(F)$, see (3.15), is the extension of $f_{n_1+n_2,0}$ along curves in $V \setminus \boldsymbol{\Upsilon}$ (subject to certain restrictions illustrated on Figure 10), where

$$\boldsymbol{\Upsilon} := \mathfrak{B}^{[n_1+n_2]} \cup \Big(\mathfrak{B}^{[n_2]} \setminus \mathfrak{B}_0^{[n_2]}\Big),$$

(2) $\mathcal{R}^{n_1+n_2\bullet}(F)$ is the extension of $f_{n_1+n_2,0}$ along all curves in $V \setminus \mathfrak{B}^{[n_1+n_2]}$,

(3) and every connected component of $\mathfrak{B}^{[n_1+n_2]}$ is within a connected component of $\boldsymbol{\Upsilon}$ (compare with Figure 10), we obtain the natural embedding of dynamical systems (respecting all the maps)

$$\mathcal{R}^{n_1\bullet} \circ \mathcal{R}^{n_2\bullet}(F) \ \subset\ \mathcal{R}^{n_1+n_2\bullet}(F). \tag{3.17}$$

3.4.5. *Unbalanced renormalization from $\psi^\bullet$ to ψ-ql maps.* — Consider a $\psi^\bullet$-ql map F from (3.12), and assume that f_X (see (3.13)) is twice DH renormalizable. Let $f_{1,0} \colon X_{1,0} \to Y_{1,0}$ be a ql prerenormalization of f_X.

By Item (III) of the definition of F, the little Julia set $\mathfrak{K}_0^{[1]} \equiv \mathfrak{K}(f_{1,0})$ is ι-proper. Therefore, we can extend $f_{1,0} \colon X_{1,0} \to Y_{1,0}$ along all curves in $\boldsymbol{\Upsilon} := \mathfrak{K}_0^{[1]} \cup \mathfrak{B}^{[1]}$ and

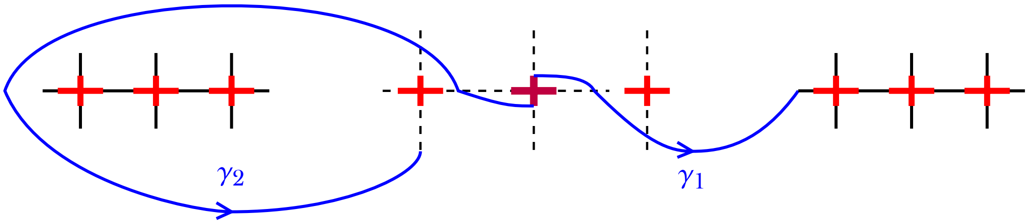

FIGURE 10. — Illustration to Sup-Chain Rule (3.17), where $n_1 = n_2 = 1$. Level one bushes $\mathfrak{B}^{[1]}$ are depicted black and dashed black (the central bush $\mathfrak{B}_0^{[1]}$), while level two bushes $\mathfrak{B}^{[2]}$ are depicted red and purple (the central bush $\mathfrak{B}_0^{[2]}$). The map $\mathcal{R}^\bullet \circ \mathcal{R}^\bullet(F)$ is obtained from F by extending the ql prerenormalization $f_{2,0}$ (defined near $\mathfrak{B}_0^{[2]}$, purple) along some paths γ_i in $V \setminus \Upsilon$ (where Υ consists of non-dashed sets). The curve γ_1 terminates at Υ (i.e., no extension beyond). The curve γ_2 terminates at $\mathfrak{B}_0^{[1]}$ because it makes a non-trivial loop around $\mathfrak{B}^{[1]} \setminus \mathfrak{B}_0^{[1]}$. Since $\mathcal{R}^{2\bullet}F$ is obtained by extending $f_{2,0}$ along *all* paths in $V \setminus \mathfrak{B}^{[2]}$ (where $\mathfrak{B}^{[2]}$ is red and purple), we obtain the Sup-Chain Rule $\mathcal{R}^\bullet \circ \mathcal{R}^\bullet(F) \subset \mathcal{R}^{2\bullet}F$.

obtain ψ-ql map

(3.18) $$F_{\Upsilon,1} = (f_{1,0}, \iota_{1,0})\colon U_{\Upsilon,1} \longrightarrow V_{\Upsilon,1}.$$

We recall from Section 2.4.5 that the set Υ is inhomogeneous: f_X does not permute components of Υ.

3.4.6. *Restrictions.* — A $\psi^\bullet$-ql map F as in (3.12) has the natural *restriction* (also known as the *pullback*, *fiber product*, *graph*) denoted by

$$F = (f, \iota)\colon U^2 \rightrightarrows U^1 = U, \quad \text{where} \quad U^2 = \{(x,y) \in U \times U \mid f(x) = \iota(y)\},$$

where $f, \iota\colon U^2 \to U$ are component-wise projections; see [18, Section 2.2.2]. Note that $F\colon U^2 \rightrightarrows U^1$ is also a $\psi^\bullet$-ql map. Repeating the construction, we obtain the sequence

$$F\colon U^k \rightrightarrows U^{k-1}, \quad n \geq 1, \quad U^0 = V, \quad U^1 = U,$$

together with induced iterations denoted by

(3.19) $$F^k = (f^k, \iota^k)\colon U^k \longrightarrow V.$$

Since $\iota|U^{k+1}$ is the lift of $\iota|U$ under the covering f^k, we have:

Lemma **3.6.** — *The set $f^{-k}(\mathfrak{B}_F)$ is ι-proper for $F\colon U^{k+1} \rightrightarrows U^k$.*

3.4.7. *Sup-Chain rule for iterations.* — Consider a ql map $g\colon A \to B$. Assume that a ψ-ql map $F\colon U \rightrightarrows V$ is obtained from $g\colon A \to B$ using finitely many $\psi^\bullet$-ql renormalizations followed by the unbalanced renormalization from Section 3.4.5. Then F has a ql restriction $f_X\colon X \to Y$ as in (3.13) that is realized in the dynamical plane of g:

$$f_X \simeq f_{X,g}, \quad \text{where} \quad f_{X,g} = g^p\colon X_g \longrightarrow Y_g.$$

Consider the filled Julia set $\mathfrak{K}_i^{[n]} \equiv \mathfrak{K}(f_{X,g}) \subset B$ of $f_{X,g} = g^p : X_g \to Y_g$. Let $G\colon N \rightrightarrows M$ be the ψ-ql map obtained from $g\colon A \to B$ by extending $f_{X,g} = g^p\colon X_g \to Y_g$ along all paths in $\Upsilon := \mathfrak{K}_i^{[n]} \cup \mathfrak{B}^{[n]}$; compare with Section 3.4.5 where the case $(n,i) = (1,0)$ is defined. By construction, we have a covering map

$$\rho\colon M \setminus \mathfrak{K}_G \longrightarrow B \setminus \Upsilon$$

so that ρ induces an isomorphism $\rho\colon \partial\mathfrak{K}_G \xrightarrow{\simeq} \partial\mathfrak{K}_i^{[n]}$. The Sup-Chain Rule (Section 3.4.4) implies by induction that $F \subset G$. Therefore, we have the induced partial covering:

$$\rho'\colon V \setminus \mathfrak{K}_F \hookrightarrow M \setminus \mathfrak{K}_G \xrightarrow{\rho} B \setminus \Upsilon \tag{3.20}$$

so that $\rho'\colon \partial\mathfrak{K}_F \xrightarrow{\simeq} \partial\mathfrak{K}_i^{[n]}$. Let γ be the core hyperbolic geodesic of the annulus $V \setminus \mathfrak{K}_F$. Then γ_F can be described as the set points x such that the Brownian motion starting at x has equal probabilities of hitting each of the boundary components of the annulus $V \setminus \mathfrak{K}_F$. Therefore, the image of the subdisk $V_\gamma \Subset V$ bounded by γ under $V \setminus \mathfrak{K}_F \hookrightarrow M \setminus \mathfrak{K}_G$ is within the subdisk $M_\beta \Subset M$ bounded by the core hyperbolic geodesic β of the annulus $M \setminus \mathfrak{K}_G$. Since ρ is a covering with $\rho\colon \partial\mathfrak{K}_G \xrightarrow{\simeq} \partial\mathfrak{K}_i^{[n]}$, we have $\rho\colon \beta \xrightarrow{1:1} \rho(\beta)$ and $\rho(\beta)$ is a hyperbolic geodesic of $B \setminus \Upsilon$. Therefore, $V\gamma \setminus \mathfrak{K}_F$ descends univalently to the dynamical plane of $g\colon A \to B$ via the partial covering (3.20).

3.5. *Width and WADs of $\psi^\bullet$ maps.* — For a $\psi^\bullet$-ql map $F\colon U \rightrightarrows V$ we write

$$\mathcal{W}_\bullet(F) := \mathcal{W}(V \setminus \mathfrak{B}_F).$$

If F is ψ-ql map (i.e., $\mathfrak{K}_F$ is ι-proper), then we can also measure

$$\mathcal{W}(F) := \mathcal{W}(V \setminus \mathfrak{K}_F).$$

By monotonicity,

$$\mathcal{W}\big(U^k \setminus \mathfrak{B}\big) \le \mathcal{W}\big(U^k \setminus f^{-k}(\mathfrak{B})\big) = 2^k \mathcal{W}_\bullet(F).$$

The Sup-Chain Rule (3.17) implies that

$$\mathcal{W}[\mathcal{R}^{n_1\bullet} \circ \mathcal{R}^{n_2\bullet}(F)] \ge \mathcal{W}\big[\mathcal{R}^{n_1+n_2\bullet}(F)\big]. \tag{3.21}$$

3.5.1. *Compactness of ψ and $\psi^\bullet$-ql maps.* — If $\mathcal{W}(F)$ is bounded for ψ-ql map F, then F also has regular ql bounds:

Lemma **3.7** *([18, Lemma 2.4]).* — *There is a positive function $\mu(K)$ with the following property. If F is a ψ-ql map with $\mathcal{W}(F) \le K$, then the quadratic-like map $f_X\colon X \to Y$ in (3.13) can be selected so that*

$$\operatorname{mod}(Y \setminus X) \ge \mu(K). \tag{3.22}$$

Remark **3.8.** — Assume that ψ-ql map $F\colon U \rightrightarrows V$ is obtained from a ql map $g\colon A \to B$ using finitely many $\psi^\bullet$-renormalizations followed by the unbalanced renormalization from Section 3.4.5. As in Section 3.4.7, let γ be the core hyperbolic geodesic of the annulus $V \setminus \mathfrak{K}_F$. In Lemma 3.7 we can assume (by replacing X, Y with $f_X^{-1}(X), X$) that the ql restriction $f_X\colon X \to Y$ is within the subdisk $V_\gamma \Subset V$ bounded by γ. It is shown in Section 3.4.7 that $V_\gamma \setminus \mathfrak{K}_F$ (and hence V_γ) descents univalently to the dynamical plane of g via the partial covering (3.20). Therefore, the ql map $f_X : X \to Y$ descents univalently to the dynamical plane of g.

It is shown in the proof of [18, Lemma 2.4] that ι is an embedding in a neighborhood of $\mathfrak{K}_F$; this argument is applicable to $\psi^\bullet$ maps:

Lemma **3.9.** — *There is a positive function $\mu_\bullet(K)$ with the following property. If F is a $\psi_\bullet$-ql map with $\mathcal{W}_\bullet(F) \le K$, then $\mathfrak{B}_F$ has a neighborhood Ω such that ι is injective on Ω and*

$$\operatorname{mod}(\Omega \setminus \mathfrak{B}_F) \ge \mu_\bullet(K).$$

3.5.2. *WADs of F.* — Assume that $\mathbf{B}^{[n]}$ is well-defined, i.e., F is $n+1$ renormalizable. Let us consider the canonical WAD of $U^k \setminus \mathbf{B}^{[n],(m)}$, $k \ge m$:

$$\mathcal{A}^{(m),k} \equiv \mathcal{A}^{[n],(m),k} := \operatorname{WAD}\big(U^k \setminus \mathbf{B}^{[n],(m)}\big).$$

We say (compare with Section 1.3.1) that an arc $\alpha \in \mathcal{A}^{[n],(m),k}$ is:

- *vertical:* if it connects the outer boundary $\partial^{\text{out}}(U^k \setminus \mathbf{B}^{[n],(m)}) := \partial U^k$ of $U^k \setminus \mathbf{B}^{[n],(m)}$ to one of its inner boundary components;
- *horizontal:* if it connects two inner (i.e., non-outer) boundary components of $U^k \setminus \mathbf{B}^{[n],(m)}$.

We set (see Section 3.3.6):

- $\mathcal{A}^{(m),k}_{\text{ver}}$ to be *vertical part* of $\mathcal{A}^{(m),k}$: it consists of arcs connecting ∂U^k and one of the components of $\partial \mathbf{B}^{[n],(m)}$;
- $\mathcal{A}^{(m),k}_{\text{hor}}$ to be *horizontal part* of $\mathcal{A}^{(m),k}$: it consists of arcs connecting components of $\partial \mathbf{B}^{[n],(m)}$;
- $\mathcal{A}^{(m),k}_i$ to be the *local WAD* for $\mathfrak{B}^{[n],(m),k}_i$, see Section 3.3.5: $\mathcal{A}^{(m),k}_i$ consisting of arcs adjacent to $\mathfrak{B}^{[n],(m),k}_i$ such that the weights of self-arcs adjacent to $\mathfrak{B}^{[n],(m),k}_i$ are doubled;
- $\mathcal{A}^{(m),k}_{i,\text{ver}}, \mathcal{A}^{(m),k}_{i,\text{hor}}$ to be the local parts of $\mathcal{A}^{(m),k}_{\text{ver}}$, $\mathcal{A}^{(m),k}_{\text{hor}}$.

We will usually suppress upper zero induces:

$$\mathcal{A}_i = \mathcal{A}^{(0),0}_i, \quad \mathcal{A}^{(m)}_i = \mathcal{A}^{(m),0}_i, \quad \mathcal{A}^k_i = \mathcal{A}^{(0),k}_i, \quad \dots$$

The following lemma provides a reverse to (3.21) estimate.

Lemma **3.10**. — *Assume that in the dynamical plane of* $F = \mathcal{R}^{n_0\bullet}G\colon U \rightrightarrows V$, *there is a non-trivial horizontal lamination of curves* $\mathcal{L} \subset V \setminus \mathfrak{B}^{[n]}$ *emerging from* $\mathfrak{B}^{[n]}_0$ *and landing at* $\mathfrak{B}^{[n]}$. *Then*

$$\mathcal{W}\big(\mathcal{R}^{n_0+n\bullet}G\big) \geq \mathcal{W}(\mathcal{L}).$$

Proof. — By definition, see Section 3.4.2, the $\mathcal{L}$ lifts univalently to a horizontal lamination $\widetilde{\mathcal{L}}$ in the dynamical plane of G emerging from $\mathfrak{B}^{n_0+n}_{G,0}$ and landing at $\mathfrak{B}^{[n_0+n]}_G$. After that $\widetilde{\mathcal{L}}$ lifts univalently to a vertical lamination in the dynamical plane of $\mathcal{R}^{n_0+n\bullet}G$. We obtain that $\mathcal{W}(\mathcal{R}^{n_0+n\bullet}G) \geq \mathcal{W}(\widetilde{\mathcal{L}}) \geq \mathcal{W}(\mathcal{L})$. □

3.5.3. *WADs and* $\mathcal{W}_\bullet(F_{n,i})$. — It follows from Theorem 3.1 that

$$\mathcal{W}_\bullet(F_{n,i}) = \mathcal{W}(\mathcal{A}_i) + O_{p_n}(1). \tag{3.23}$$

As with ψ-ql renormalization (compare with [27, Theorem 9.3]), we have

$$\frac{1}{2}\mathcal{W}_\bullet(F_{n,0}) \leq \mathcal{W}_\bullet(F_{n,i}) \leq \mathcal{W}_\bullet(F_{n,0}) \quad \text{for all } i. \tag{3.24}$$

Indeed, let $\gamma_i \subset V \setminus \mathfrak{B}^{[n]}$ be the peripheral hyperbolic geodesic around $\mathfrak{B}^{[n]}_i$. Then the hyperoblic length of $|\gamma_i|_{V\setminus\mathfrak{B}^{[n]}}$ is proportional to $\mathcal{W}_\bullet(F_{n,i})$. Let γ^1_i be the lift of γ_i under $f\colon U^1 \setminus \mathfrak{B}^{[n],(1)} \to V \setminus \mathfrak{B}^{[n]}$ so that γ^1_i is the peripheral hyperbolic geodesic around $\mathfrak{B}^{[n],(1)}_{i-1}$, where the subscript is mod p_n. Counting the degree of $f\colon \gamma^1_i \to \gamma_i$, we obtain

- $|\gamma^1_i|_{U^1\setminus\mathfrak{B}^{[n],(1)}} = |\gamma_i|_{V\setminus\mathfrak{B}^{[n]}}$ if $i \neq 1$; and
- $\frac{1}{2}|\gamma^1_1|_{U^1\setminus\mathfrak{B}^{[n],(1)}} = |\gamma_1|_{V\setminus\mathfrak{B}^{[n]}}$.

Finally, $|\gamma^1_i|_{U^1\setminus\mathfrak{B}^{[n],(1)}} \geq |\gamma_{i-1}|_{V\setminus\mathfrak{B}^{[n]}}$; this implies (3.24).

3.5.4. *Covering relation: from* $U^k \setminus \mathfrak{B}^{(m)}$ *to* $U^{k+s} \setminus \mathfrak{B}^{(m+s)}$. — It follows from Section 3.3.8 that the WADs change naturally under the covering:

$$\mathcal{A}^{(m+s),k+s}_{\text{hor}} = (f^s)^*\mathcal{A}^{(m),k}_{\text{hor}}, \quad \mathcal{A}^{(m+s),k+s}_{i-s} = (f^s)^*\mathcal{A}^{(m),k}_i, \tag{3.25}$$

and similar for other WAD such as $\mathcal{A}^{(m),k}, \mathcal{A}^{(m),k}_{\text{ver}}, \dots$.

3.5.5. *Restriction by* ι^s*: from* $U^k \setminus \mathfrak{B}^{(m)}$ *to* $U^{k+s} \setminus \mathfrak{B}^{(m)}$. — Assume that $m \leq k$. By Lemma 3.6, the set $\mathfrak{B}^{[n],(m)} \subset U^k$ is ι-proper. Let γ be a proper curve in $U^k \setminus \mathfrak{B}^{[n],(m)}$ emerging from $\mathfrak{B}^{[n],(m)}_a$. Applying ι^{-s} along γ, we construct, see (3.26) below, its *restriction* $(\iota^*)^s[\gamma]$ which is a proper curve in $U^{k+s} \setminus \mathfrak{B}^{[n],(m)}$. Moreover,

- if γ is vertical, then so is $(\iota^*)^s[\gamma]$,
- if γ is horizontal, then $(\iota^*)^s[\gamma]$ is either horizontal or vertical.

If $(\iota^*)^s[\gamma]$ is horizontal, then we will also call $(\iota^*)^s[\gamma]$ the *lift* of γ under ι^s.

More precisely, write $\gamma\colon (0,1) \to U^k \setminus \mathbf{B}^{[n],(m)}$. For every $s > 0$, there is a $t_s \in (0,1]$ such that ι^{-s} extends along $\gamma \mid (0, t_s)$ and the resulting curve

$$(\iota^*)^s[\gamma] \coloneqq \iota^{-s} \circ \gamma\colon (0, t_s) \longrightarrow U^{k+s} \tag{3.26}$$

is a proper curve in $U^{k+s} \setminus \mathbf{B}^{[n],(m)}$ (because $\mathbf{B}^{[n],(m)} \subset U^k$ is ι-proper). Moreover,

- if $t_m = 1$, then $(\iota^*)^n\gamma$ is vertical if and only if γ is vertical,
- if $t_m < 1$, then $(\iota^*)^n\gamma$ is vertical.

The following lemma implies that curves in a horizontal rectangle that restrict under $(\iota^s)^*$ to vertical curves form buffers of the rectangle.

Lemma **3.11**. — *Assume that two disjoint horizontal paths $\gamma_1, \gamma_2 \subset U^k \setminus \mathbf{B}^{(m)}$ representing the same arc $\alpha = [\gamma_1] = [\gamma_2]$ lift under $(\iota^s)^*$ to horizontal paths in $U^{k+s} \setminus \mathbf{B}^{(m)}$. Let $R \subset U^k \setminus \mathbf{B}^{(m)}$ be the proper rectangle between γ_1 and γ_2 such that all vertical curves in R also represent α. Then all curves in $\mathcal{F}^{\text{full}}(R)$ lift under $(\iota^s)^*$ to homotopic horizontal curves in $U^{k+s} \setminus \mathbf{B}^{(m)}$.*

Proof. — The curve $\gamma \in \mathcal{F}^{\text{full}}(R)$ becomes vertical under the restriction by $(\iota^s)^*$ if and only if $\gamma(t)$ hits $\iota^s(\partial U^{k+s})$. This is impossible because $\gamma_1(t), \gamma_2(t)$ do not hit $\iota^s(\partial U^{k+s})$. □

3.5.6. *Monotonicity of the $\mathcal{A}^k_{\text{hor}}$.* — Since $\iota\colon U^{k+1} \to U^k$ is an embedding in a neighborhood of $\mathfrak{K}_F$, we *identify up to homotopy* $U^k \setminus \mathbf{B}^{[n],(m)}$ and $U^s \setminus \mathbf{B}^{[n],(m)}$ for all $k < s$. In particular, horizontal arcs α in $U^k \setminus \mathbf{B}^{[n],(m)}$ are naturally viewed as horizontal arcs in $U^s \setminus \mathbf{B}^{[n],(m)}$ by realizing the α in a small neighborhood of $\mathfrak{K}_F$ where ι^{s-k} is an embedding.

It follows essentially from Lemma 3.3 (by lifting ι to the universal coverings, see [18, Section 3.5] for reference) that for $k \geq m$

$$\mathcal{A}^{[n],(m),k}_{\text{hor}} \geq \mathcal{A}^{[n],(m),k+1}_{\text{hor}} \quad \text{and} \quad \mathcal{A}^{[n],(m),k}_{i,\text{hor}} \geq \mathcal{A}^{[n],(m),k+1}_{i,\text{hor}}. \tag{3.27}$$

3.5.7. *Domination: from $U^k \setminus \mathbf{B}^{(m)}$ to $U^k \setminus \mathbf{B}^{(m+1)}$ (following* [18, Section 3.6]*).* — Assume that $\gamma \subset U^k \setminus \mathbf{B}^{[n],(m)}$ is a proper horizontal curve. Then it has the decomposition

$$\gamma = \ell_0 \# \gamma_1 \# \ell_1 \# \gamma_2 \# \dots \# \gamma_s \# \ell_s \tag{3.28}$$

(compare with Section 2.4.3) such that:

- $\gamma_j \subset U^k \setminus \mathbf{B}^{[n],(m+1)}$ are proper horizontal curves; and
- every component of $\ell_i \setminus \mathbf{B}^{[n],(m+1)}$ is trivial in $U^k \setminus \mathbf{B}^{[n],(m+1)}$ (with respect to a proper homotopy, see Section 2.4.1).

We call s the *expansivity number* of γ in rel $\mathbf{B}^{(m+1)}$.

It follows from Decomposition (3.28) that there is a combinatorial constant $C \equiv C_{p_n,m}$ (essentially, the maximal number of possible arcs rel $\mathbf{B}^{[n],(m+1)}$) such that for all $M > 1$

$$\mathcal{A}^{[n],(m),k}_{\text{hor}} - MC \quad \text{is } \textit{dominated} \text{ by} \quad \mathcal{A}^{[n],(m+1),k}_{\text{hor}} - M \tag{3.29}$$

in the following sense, see Section 3.3.6 for notation. For every arc α in $\mathcal{A}^{[n],(m),k}_{\text{hor}} - MC$, there is a vertical sublamination

$$\mathcal{L}_\alpha \subset \mathcal{F}(\alpha) \equiv \mathbf{\Pi}^{\text{can}}(\alpha) \quad \text{with} \quad \mathcal{W}(\mathcal{L}_\alpha) \geq \mathcal{W}(\alpha) - MC$$

such that Decomposition 3.28 of every $\gamma \in \mathcal{L}_\alpha$ has the additional property that

$$\gamma_i \in \mathcal{F}^{\text{full}}\left(\mathcal{A}^{[n],(m+1),k}_{\text{hor}} - M\right). \tag{3.30}$$

The lamination $\mathcal{L}_a$ is constructed by removing all γ from $\mathcal{F}(\alpha)$ that do not satisfy (3.30) (the width of removed curves is bounded by MC).

3.5.8. *Almost periodic rectangles.* — Consider a $\psi^\bullet$-ql map $F\colon U \rightrightarrows V$ as in (3.12). We say that a proper rectangle R in $V \setminus \mathfrak{B}^{[1]}$ is almost periodic if most of the width of R overflows its iterative lift; more precisely:

Definition **3.12.** — *For twice renormalizable a $\psi^\bullet$-ql map $F\colon U \rightrightarrows V$, a proper rectangle $R \subset V \setminus \mathfrak{B}^{[1]}$ connecting $\mathfrak{B}^{[1]}_a, \mathfrak{B}^{[1]}_{a+1}$ is called δ-*almost periodic *if R represents a genuinely periodic arc (of some AD, see Section 2.4.3) and R has a vertical sublamination $\mathcal{G} \subset \mathcal{F}(R)$ with $\mathcal{W}(\mathcal{G}) \geq (1-\delta)\mathcal{W}(R)$ such that Conditions (I), (II) stated below hold for all $s \leq 10\bar{p}$, where $\bar{p}$ is the combinatorial bound from Section 2.2.5.*

(I) Under the immersion, $\mathcal{G}$ lifts to the lamination

$$\mathcal{G}^s := (\iota^{s p_1})^*(\mathcal{G}) \subset U^{s p_1} \setminus \mathfrak{B}^{[1]}$$

still connecting $\mathfrak{B}^{[1]}_a, \mathfrak{B}^{[1]}_{a+1}$.

Let $R^{(s)} \subset U^{s p_1} \setminus \mathfrak{B}^{[1],(s p_1)}$ be the periodic lift of R under $f^{s p_1}$; i.e., $R^{(s)}$ connects $\mathfrak{B}^{[1],(s p_1)}_a$, $\mathfrak{B}^{[1],(s p_1)}_{a+1}$.

(II) The lamination $\mathcal{G}^s$ overflows $R^{(s)}$ as follows: every curve γ in $\mathcal{G}^s$ is the concatenation $\gamma = \ell_a \# \gamma' \# \ell_{a+1}$ such that
- *$\gamma' \in \mathcal{F}^{\text{full}}(R^{(s)})$, see Section 3.2; and*
- *every component of $\ell_a \setminus \mathfrak{B}^{[1],(s p_1)}_a$ and every component of $\ell_{a+1} \setminus \mathfrak{B}^{[1],(s p_1)}_{a+1}$ is trivial in $U^{s p_1} \setminus \mathfrak{B}^{[1],(s p_1)}$ with respect to a proper homotopy, see Section 2.4.1.*

By Lemma 2.3, the first renormalization of F is satellite and $\mathfrak{B}^{[1]}_a, \mathfrak{B}^{[1]}_{a+1}$ are neighboring bushes with respect to the cyclic order.

Remark **3.13.** — An almost periodic rectangle R between little bushes $\mathfrak{B}^{[n]}_a, \mathfrak{B}^{[n]}_{a+1}$ is defined in the same way as in the case $n = 1$. Such a rectangle R can be lifted to the dynamical plane of $F_{n-1,c}$ (via the covering map representing the $\psi^\bullet$-renormalization, see Section 3.4.3) and its lift will be an almost periodic rectangle between level 1 little bushes as in Definition 3.12.

4. Pull-off for non-periodic rectangles

In this section we will establish the following theorem that refines a result from [18].

Theorem **4.1**. — *For every bound $\bar{p}$ on renormalization periods as in Section 2.2.5, every small $\delta > 0$, and every $n \gg_{\bar{p},\delta} 1$, the following holds.*

Let $F = (f, \iota)\colon U \rightrightarrows V$ be a $\psi^\bullet$-ql map $n+1$ times renormalizable of type $\bar{p}$, and let $F_{n,i} = (f_{n,i}, \iota_{n,i})\colon U_{n,i} \rightrightarrows V_{n,i}$ be its n^{th} $\psi^\bullet$-renormalization, see Section 3.4.3. If

$$\mathcal{W}_\bullet(F_{n,i}) = K \gg_{\bar{p},\delta,n} 1,$$

then

- *(P) either $\mathcal{W}_\bullet(F) \geq 2K$;*
- *(S) or the n^{th} renormalization of F is satellite and the $(n-1)^{th}$ $\psi^\bullet$-ql renormalization F_{n-1} has a δ-almost periodic rectangle R with $\mathcal{W}(R) \geq K/20$ (see Definition 3.12) between two neighboring bushes in its satellite flower.*

If the n^{th} renormalization of F is primitive, then F_{n-1} has no periodic rectangles; i.e., Case (P) holds.

Proof. — (See also Section 1.3.1.) Consider the periodic cycle $\mathfrak{B}^{[n]}$ of level n little bushes in the dynamical plane of $F\colon U \rightrightarrows V$. Recall from (3.19) that $F^k\colon U^k \rightrightarrows V$ represents the k^{th} iteration of F.

The degeneration of $U^k \setminus \mathfrak{B}^{[n]}$ is described by the WAD $\mathcal{A}^k \equiv \mathcal{A}^{[n],k}$; let us consider its horizontal and vertical parts $\mathcal{A}^k_{\text{hor}}, \mathcal{A}^k_{\text{ver}}$, see details in Section 4.1. Since the $U^k \setminus \mathfrak{B}^{[n]}$ decrease, so are the arc diagrams representing $\mathcal{A}^k_{\text{hor}}$; i.e., the $A^k_{\text{hor}} \equiv \text{AD}(\mathcal{A}^k_{\text{hor}})$ stabilize for $k \leq 3p_n$. And since $f^{-1}(U^k \setminus \mathfrak{B}^{[n]}) \subset U^{k+1} \setminus \mathfrak{B}^{[n]}$, the A^k_{hor} stabilize at an invariant AD; i.e., A^k_{hor} is aligned with the Hubbard tree of the superattracting model.

Consider the local part $\mathcal{A}^k_i$ of $\mathcal{A}^k$ around $\mathfrak{B}^n_i$. Write $q := (10\bar{p}+3)p_n$. We remark that since q is linear in p_n, the q^{th} iterate has a bounded degree on small bushes and the application of the Covering Lemma below will lead to estimates independent of n.

If for all i we have

$$\mathcal{W}(\mathcal{A}^q_i) - \mathcal{W}(\mathcal{A}^0_i) \leq \delta_1 K, \quad \delta_1 \ll \delta, \tag{4.1}$$

then most of the degeneration in $\mathcal{A}^0_i$ is represented by genuinely periodic arcs Section 2.4.3; this leads to Case (S), see Section 4.3.

If (4.1) does not hold some i, then $\mathcal{W}(\mathcal{A}^q_{i,\text{ver}}) \geq \delta_1 K$ and applying the Covering Lemma we obtain

$$\mathcal{W}\left(V \setminus \mathfrak{B}^{[n]}_j\right) \succeq_{\delta_1,\bar{p}} K \quad \text{for all } j.$$

Applying the Quasi-Additivity Law and using $p_n \gg_{\delta_1,\bar{p}} 1$, we obtain

$$\mathcal{W}(F) \geq_{\delta_1,\bar{p}} \sum_j \mathcal{W}\left(V \setminus \mathfrak{B}_j^{[n]}\right) \geq_{\delta_1,\bar{p}} p_n K \gg 2K,$$

see Section 4.2; this is Case (P). □

Below we will make a more technical exposition.

4.1. *Alignment of WAD with the Hubbard continuum.* — Following Section 2.3.1, let

$$\mathfrak{B} \equiv \mathfrak{B}^{[n],(0)} \quad \text{and} \quad \mathfrak{B}^{(m)} \equiv \mathfrak{B}^{[n],(m)} = f_X^{-m}(\mathfrak{B})$$

be the periodic cycle of level-n bushes and its preimage, where $f_X = [f : X \to Y]$ is a ql restriction of f, see (3.13) in Section 3.4.2. The similar suppression of indices is applied to WAD from Section 3.5.2.

Recall from Section 3.5.4 and Section 3.5.6 that we have

$$\mathcal{A}_{\text{hor}}^{(m+1),k+1} = f^*\left(\mathcal{A}_{\text{hor}}^{(m),k}\right) \quad \text{and} \quad \mathcal{A}_{\text{hor}}^{(m),k+1} \leq \mathcal{A}_{\text{hor}}^{(m),k}. \tag{4.2}$$

Let us choose a sufficiently big threshold $C \gg_{p_n} 1$ so that (3.29) is applicable for $m = 0$. By (4.2), the ADs forming $\mathcal{A}_{\text{hor}}^k - C^k$ decrease, hence for a certain

$$\mathrm{t} \leq \left[\text{maximal number of horizontal arcs on } V \setminus \mathfrak{B}_n\right] \leq 3p_n \tag{4.3}$$

the arc diagram

$$H = \mathrm{AD}(\mathcal{H}), \quad \text{where} \quad \mathcal{H} := \mathcal{A}_{\text{hor}}^{\mathrm{t}} - C^{\mathrm{t}} \tag{4.4}$$

coincides with the AD of $\mathcal{A}_{\text{hor}}^{\mathrm{t}+1} - C^{\mathrm{t}+1}$.

Since the WAD $\mathcal{A}_{\text{hor}}^{\mathrm{t}+1} - C^{\mathrm{t}+1}$ on $U^{\mathrm{t}+1} \setminus \mathfrak{B}$ is dominated by the WAD $f^*(\mathcal{H}) = f^*(\mathcal{A}_{\text{hor}}^{\mathrm{t}} - C^{\mathrm{t}})$ on $U^{\mathrm{t}+1} \setminus \mathfrak{B}^{(1)}$, see (3.29) (where $M = C^{\mathrm{t}}$) in Section 3.5.7, we obtain that H is an invariant AD, and therefore it is aligned with the Hubbard continuum by Lemma 2.2.

4.2. *Case (P).* — Consider local WAD $\mathcal{A}_i^k$ as in Section 3.5.2, we have by Section 3.5.3

$$\mathcal{W}(\mathcal{A}_i^k) \geq \mathcal{W}(\mathcal{A}_i^0) = \mathcal{W}_\bullet(F_{n,i}) - O_{p_n}(1) \geq K/2 - O_{p_n}(1). \tag{4.5}$$

Write $q := (10\bar{p} + 3)p_n$ and fix a sufficiently small $\delta_1 > 0$. Assume:

(P_{loc}) there is a κ such that $\mathcal{W}(\mathcal{A}_{\kappa,\text{ver}}^q) \geq \delta_1 K$.

Since all local weights are comparable (Section 3.5.3), we have the Collar Assumption:

$$\mathcal{W}\left(V \setminus \bigcup_{j \neq i} \mathfrak{B}_j, \quad \mathfrak{B}_i\right) \asymp K \quad \text{for all } i. \tag{4.6}$$

Since $\mathcal{W}(\mathcal{A}_{\kappa,\text{ver}}^{q+j}) \geq \mathcal{W}(\mathcal{A}_{\kappa,\text{ver}}^q) \geq \delta_1 K$ and (4.6) holds, we obtain from the [22, Covering Lemma] that

$$\mathcal{W}(V, \mathfrak{B}_{\kappa+i}) \geq_{\delta_1,\bar{p}} \mathcal{W}(U^{q+i}, \mathfrak{B}_\kappa) \geq \mathcal{W}\left(\mathcal{A}_{\kappa,\text{ver}}^{q+i}\right) - O_{p_n}(1) \geq_{\delta_1,\bar{p}} K$$

for all $i \in \{0, 1, \dots, p_n - 1\}$. Therefore,

$$\sum_{j=0}^{p_n} \mathcal{W}(V, \mathfrak{B}_j) \succeq_{\delta_1, \bar{p}} \ p_n K.$$

Applying [22, Quasi Additivity Law with separation] together with (4.6) and using $p_n \gg_{\delta_1, \bar{p}} 1$, we obtain

$$\mathcal{W}(V \setminus \mathfrak{B}) \succeq_{\delta_1, \bar{p}} \sum_{j=0}^{p_n} \mathcal{W}(V, \mathfrak{B}_j) \succeq_{\delta_1, \bar{p}} \ p_n K \geq 2K.$$

We conclude that

$$\mathcal{W}_\bullet(F) = \mathcal{W}(V,\ \mathfrak{B}_F) \geq \mathcal{W}(V,\ \mathfrak{B}) \geq 2K.$$

This establishes Case (P) from ($\mathrm{P}_{\mathrm{loc}}$).

4.3. *Case (S).* — Let us now assume that ($\mathrm{P}_{\mathrm{loc}}$) does not hold: for all i we have $\mathcal{W}(\mathcal{A}^q_{i,\mathrm{ver}}) \leq \delta_1 K$. This implies that most paths in the canonical lamination $\mathcal{F}(\mathcal{A}_i)$ of $\mathcal{A}_i$ are horizontal and they restrict to horizontal paths under the immersions $\iota^s \colon (U^s, \mathfrak{B}) \to (V, \mathfrak{B})$ for $s \leq q$. We will use notations of Section 2.4.4.

Let $\mathcal{H}$ be the WAD from (4.4) on $(U^{\mathfrak{t}}, \mathfrak{B})$. Consider an arc e in H with maximal $\mathcal{W}(e)$. Since the number of arcs in H is bounded by $3p_n$ and $\mathcal{W}(\mathcal{H}) \geq p_n K/4 - \delta_1 p_n K$ by (3.24), we obtain that $\mathcal{W}(e) \geq K/14$. We claim that e is within a periodic satellite flower $\mathfrak{B}^{[n-1]}_c$. In particular, the n^{th} renormalization of f is satellite.

Assume converse: e is not in any satellite flower $\mathfrak{B}^{[n-1]}_c$. By Lemma 2.4, e overflows arcs $e_1, e_2 \in (f^{2p_n})^*(H)$ with $f^{2p_n}(e_1) = f^{2p_n}(e_2) = e_{\text{new}} \in H$. By Lemma 3.11, $(\iota^{2p_n})^*[\mathcal{F}(e - \delta_1 K)]$ is a horizontal family of curves in $U^{\mathfrak{t}+2p_n} \setminus \mathfrak{B}$; after removing $O_{p_n}(1)$ curves, this family is dominated by $(f^{2p_n})^*(H)$. Therefore, by the Series Law:

$$\max\big(\mathcal{W}(e_1),\ \mathcal{W}(e_2)\big) \geq 2\mathcal{W}(e) - 2.1\ \delta_1 K.$$

Applying f^{2p_1}, we obtain the contradiction: $\mathcal{W}(e_{\text{new}}) > \mathcal{W}(e)$.

Consider now a satellite flower $\mathfrak{B}^{[n-1]}_c$. As in Section 2.4.4, we denote by $\mathcal{H}_c$ the WAD consisting of arcs of $\mathcal{H}$ that are aligned with $\mathfrak{B}^{[n-1]}_c$.

Write $q' = 10\bar{p}p_n$. Let $\mathcal{H}^{(q')}_c$ be the WAD consisting of arcs of $(f^{q'})^*\mathcal{H}$ that are aligned with $\mathfrak{B}^{[n-1]}_c$. By Lemma 3.11, the restriction $(\iota^{q'})^*[\mathcal{H}_c - \delta_1 K]$ consists of horizontal curves. Since $H = \mathrm{AD}(\mathcal{H})$ is aligned with the Hubbard dendrite, $(\iota^{q'})^*[\mathcal{H}_c - \delta_1 K]$ is, after removing $O_{p_n}(1)$ curves, dominated by $\mathcal{H}^{(q')}_c$, see Section 3.5.7. And since $(f^{q'})_* \colon \mathcal{H}^{q'}_c \to \mathcal{H}_c$ is a bijection, as most $\bar{p}\delta_1 K + 2C^2 < \delta_2 K$ curves in $\mathcal{H}_c$ can have expansivity number greater than 2, where $\delta_1 \ll \delta_2 \ll \delta$. This implies that most curves in $\mathcal{F}(\mathcal{H}_c)$ are within rectangles representing genuinely periodic arcs (Section 2.4.3) that connect neighboring level n bushes of $\mathfrak{B}^{[n-1]}_c$, see Lemma 2.3. This also demonstrates that all the $\mathcal{H}_c$ have comparable width – the difference of their weights are bounded by $\delta_2 K$.

Since $\mathcal{H}$ has an edge e with $\mathcal{W}(e) \geq K/14$, the map $F\colon U^{\mathfrak{t}+1} \rightrightarrows U^{\mathfrak{t}}$ has a $\delta/2$-almost periodic rectangle R between neighboring level n bushes of $\mathfrak{B}_0^{[n-1]}$ with $\mathcal{W}(R) \geq K/15$ and satisfying Remark 3.13.

4.3.1. *Pushing forward R into $V \setminus \mathfrak{B}^n$ and then into the dynamical plane of F_{n-1}.* — Since $\mathcal{W}(\mathcal{A}^q_{i,\mathrm{ver}}) \leq \delta_1 K$, curves in $\mathcal{F}(\mathcal{A}_{i,\mathrm{hor}} - 2\delta_1 K)$ restrict under $(\iota^{\mathfrak{t}})^*$ to horizontal curves in $(U^{\mathfrak{t}}, \mathfrak{B})$. Since $\mathcal{A}^{\mathfrak{t}}_i \leq \mathcal{A}_i$, see (3.27), we obtain from Lemma 3.11 that

$$\mathcal{F}(\mathcal{A}^{\mathfrak{t}}_{i,\mathrm{hor}} - 2\delta_1 K - 2) \subset (\iota^{\mathfrak{t}})^*\Big[\mathcal{F}^{\mathrm{full}}(\mathcal{A}_{i,\mathrm{hor}} - 2\delta_1 K)\Big].$$

Therefore, $\mathcal{F}(\mathcal{A}^{\mathfrak{t}}_{i,\mathrm{hor}} - 2\delta_1 K - 2)$ can be univalently pushed forward under $(\iota^{\mathfrak{t}})_*$. By removing $\delta_1 K$ buffers from R, we push forward R into the dynamical plane of $F\colon U \rightrightarrows V$ and then push forward R into the dynamical plane of F_{n-1}, see Remark 3.13. This establishes Case (S) of the theorem.

5. Waves

Given a compact connected filled set $X \subset \mathbb{C}$, we denote by $\partial^c X$ its Carathéodory boundary; i.e., the set of prime ends of $\widehat{\mathbb{C}} \setminus X$. A Riemann map identifies $\partial^c \widetilde{X}$ with the unit circle $\mathbb{T}$. For a compact connected subset $Y \subset X$, we denote $\partial^c_X Y \subset \partial^c X$ the set of prime ends of X accumulating at Y. A *side of Y rel X* is a connected component of $\partial^c_X Y$ viewed as a subset of $\mathbb{T} \simeq \partial^c X$.

Let $F = (f, \iota)\colon U \rightrightarrows V$ be a $\psi^\bullet$-ql map. Assume that

- $(\mathfrak{K}_k)_k$ is a forward invariant collection of periodic and preperiodic little filled Julia sets of f of the same level;
- $T \supset \bigcup_k \mathfrak{K}_k$ is forward invariant compact connected filled subset of the Julia set of f;
- T is ι-proper;
- every $\mathfrak{K}_k$ has finitely many sides in T.

A relevant example for us is $T = \mathfrak{B}^{(m)}_F$. Let us denote by M the total number of sides of all $\mathfrak{K}_k$.

Consider a side $\mathfrak{K}^\iota_k$ of $\mathfrak{K}_k$ in T. A *wave $\mathcal{S}$ above $\mathfrak{K}^\iota_k$* is a lamination of proper paths in $U \setminus T$ such that every curve $\gamma \in \mathcal{S}$ starts and ends at $\partial^c X \setminus \mathfrak{K}^\iota_k$ and goes above $\mathfrak{K}^\iota_k$: the bounded component O of $U \setminus (\gamma \cup T)$ contains $\mathfrak{K}^\iota_k$ on its Carathéodory boundary (i.e., prime ends of $\mathfrak{K}^\iota_k$ are accessible from O).

Lemma **5.1** *(Wave Lemma).* — *Let $\mathcal{S}$ be a wave as above. Then $\mathcal{W}(U \setminus T) \geq_M \mathcal{W}(\mathcal{S})$.*

Proof. — We will use the following fact:

Lemma **5.2.** — *Suppose $f : A \to B$ is a degree m covering between closed annuli. Suppose $J \subset \partial A$ is an interval. Let $\mathcal{S}$ be a wave in A above J. Then $\mathcal{S}$ contains a genuine* *subwave* $\mathcal{S}^{\mathrm{new}}$ *such that* $f \mid \mathcal{S}^{\mathrm{new}}$ *is injective and*

$$\mathcal{W}(\mathcal{S}^{\mathrm{new}}) \geq \mathcal{W}(\mathcal{S}) - O(\ln m).$$

In particular, if $f \mid J$ is not injective, then $\mathcal{W}(\mathcal{S}) = O(\ln m)$.

Proof. — Consider universal covering maps $X \to A$ and $X \to B$. Their group of deck transformations are isomorphic to $m\mathbb{Z}$ and $\mathbb{Z}$ respectively. Let $\mathcal{S}_k$, $k \in m\mathbb{Z}$ be the lifts of $\mathcal{S}$ under $X \to A$; all these lifts are disjoint and permuted by $m\mathbb{Z}$. Let $\mathcal{S}_k$, $k \in \mathbb{Z}$ be the orbit of $\mathcal{S}$ under $\mathbb{Z}$. We *claim* that, by removing $O(\ln m)$ outermost curves from $\mathcal{S}$, we obtain $\mathcal{S}^{\mathrm{new}}$ and the new $\mathcal{S}_k^{\mathrm{new}}, k \in \mathbb{Z}$ so that

$$\mathcal{S}_0^{\mathrm{new}}, \mathcal{S}_1^{\mathrm{new}}, \dots, \mathcal{S}_{m-1}^{\mathrm{new}}$$

are pairwise disjoint. Then, by the claim, all $\mathcal{S}_k^{\mathrm{new}}, k \in \mathbb{Z}$ are pairwise disjoint, i.e.

$$f \colon \mathcal{S}^{\mathrm{new}} \longrightarrow f(\mathcal{S}^{\mathrm{new}}) = \mathcal{S}_0^{\mathrm{new}}/\mathbb{Z}$$

is injective.

Let us verify the claim. Let us denote by A_1 the component of ∂A containing J, by $H \simeq \mathbb{Z}/m$ the group of deck transformations of $f \colon A \to B$. Then H acts on A_1. We can decompose $\mathcal{S} = \mathcal{S}^0 \sqcup \mathcal{S}^1 \sqcup \dots \mathcal{S}^\tau$ into possibly empty pairwise-disjoint laminations such that

- curves in $\mathcal{S}^0$ start and end at an interval $I^0 \subset A_1$ that is a fundamental interval for the action $H \curvearrowright A_1$;
- curves in $\mathcal{S}^t$, $t > 0$ start at an interval $I_-^t \subset A_1$ and end at an interval $I_+^t \subset A_1$ such that one of the intervals I_-^t, I_+^t is within a union of $2^t - 1$ fundamental intervals for the action $H \curvearrowright A_1$ and the interval $J^t \subset A_1$, $J^t \cap J \neq \emptyset$ between I_-^t, I_+^t is the union of exactly $2^t - 1$ fundamental intervals of the action $H \curvearrowright A_1$;
- $\tau = O(\ln m)$.

Then $\mathcal{W}(\mathcal{S}^t) = O(1)$ for $t > 0$ because, after removing 1-buffers, $\mathcal{S}^t$ crosses its shift under H, see [9, Shift Argument in Section A.3].

Replacing $\mathcal{S}$ with $\mathcal{S}^0$, we obtain that curves in the $\mathcal{S}_k^0$ start and end at pairwise disjoint intervals. Removing an extra $O(1)$ outermost curves, we obtain that the new $\mathcal{S}_l^{\mathrm{new}}$ are pairwise disjoint. □

Let us assume first that $\mathfrak{R}_k$ is periodic. Since the number of sides is M and since the period of small Julia sets is bounded by M, we can fix an iteration $f^n \mid T$, where n is bounded in terms of M (for instance, $n \leq M^3$) such that $f^n \mid \mathfrak{R}_k^\iota$ covers $\partial_T^c \mathfrak{R}_k$ at least twice. Consider the associated iteration

$$(f^n, \iota^n) \colon U^n \rightrightarrows V, \quad \text{see (3.19)}.$$

Under the immersion ι^n, either $\frac{1}{3}\mathcal{W}(\mathcal{S})$ part of $\mathcal{S}$ lifts to a vertical family of $U^n \setminus T$ or $\frac{2}{3}\mathcal{W}(\mathcal{S})$ part of $\mathcal{S}$ lifts univalently into the lamination $\mathcal{S}'$. In the former case, Lemma 5.1 is proven:

$$\mathcal{W}(V \setminus T) \geq \frac{1}{2^n}\mathcal{W}(U^n \setminus T) \geq \frac{1}{2^n 3}\mathcal{W}(\mathcal{S}). \tag{5.1}$$

Assume the latter case. Let $\widetilde{T}$ be the full preimage of T under f^n. Then $\mathfrak{K}^\iota_k$ splits into finitely many sides $X_1, X_2, \dots, X_t$ of $\mathfrak{K}_k$ in $\widetilde{T}$. Every X_i maps univalently to the side $f^n(X)$ of $\mathfrak{K}_k$ in T under f^n. Since $f^n \mid \mathfrak{K}^\iota_k$ covers $\partial^c_T \mathfrak{K}_k$ at least twice, we can choose two sides X_a, X_b that map onto $\mathfrak{K}^\iota_k$. Every curve $\gamma \in \mathcal{S}'$ has first shortest subcurves γ_a, γ_b (which may coincide) in $U^n \setminus \widetilde{T}$ above X_a, X_b respectively. By Lemma 5.2, the width of the family of curves γ with $\gamma_a = \gamma_b$ is $O_M(1)$.

Write $K := \mathcal{W}(\mathcal{S})$. Let $\mathcal{F}'$ be the family of all γ in $\mathcal{S}$ with $\gamma_a \neq \gamma_b$. Let $\mathcal{S}_a, \mathcal{S}_b$ be family of curves consisting of γ_a and γ_b with γ in $\mathcal{S}'$. Since $\mathcal{S}'$ overflows $\mathcal{S}_a$ and then $\mathcal{S}_b$, either $\mathcal{W}(\mathcal{S}_a)$ or $\mathcal{W}(\mathcal{S}_b)$ is at least $4/3K - O_M(1)$. After removing $O_M(1)$ buffers and applying Lemma 5.2, the waves $\mathcal{S}_a, \mathcal{S}_b$ map univalently by f^n onto waves above $\mathfrak{K}^\iota_k$. We obtain a wave with width $\geq \frac{4}{3}K - O_M(1)$ and the whole argument can be repeated with a bigger wave leading eventually to a contradiction.

Assume now that $\mathfrak{K}_k$ is strictly preperiodic. Let f^n be the smallest iteration so that $f^n(\mathfrak{K}_k)$ is periodic. As above, either the $1/3$ part of $\mathcal{S}$ lifts to a vertical family under ι^n or the $2/3$ part of $\mathcal{S}$ lifts univalently. In the former case, (5.1) concludes the proof of Lemma 5.1. In the latter case, we construct the family $\mathcal{S}_a$ as above and then we pushforward $\mathcal{S}_a$ under f^n. The result will be a wave above a side of a periodic Julia set. This reduces the preperiodic to the periodic case. □

6. Pull-off for periodic rectangles

Consider a ψ-ql like map $F = (f, \iota) \colon U \rightrightarrows V$. We assume that the first renormalization of F is satellite.

Theorem **6.1**. — *Fix a combinatorial bounds $\bar{p}$ on the renormalization period (Section 2.2.5). Then for every sufficiently small $\delta > 0$ there is a $C_\delta = C_{\delta,\bar{p}} > 1$ with*

$$C_\delta \longrightarrow \infty \quad \textit{as} \quad \delta \longrightarrow 0$$

such that the following holds for every $\psi^\bullet$-ql map F, and its $\psi^\bullet$-ql renormalizations $F_1 = \mathcal{R}^\bullet(F)$, $F_2 = \mathcal{R}^{2\bullet}(F)$, see (3.15).

Suppose that R with $\mathcal{W}(R) \gg_{\delta,\bar{p}} 1$ is a δ-almost periodic rectangle (see Definition 3.12) in the dynamical plane of F between bushes $\mathfrak{B}^{[1]}_a$ and $\mathfrak{B}^{[1]}_{a+1}$. Then

$$\text{(6.1)} \qquad \textit{either} \quad \mathcal{W}_\bullet(F_2) \geq C_\delta K \quad \textit{or} \quad \mathcal{W}_\bullet(F) \geq C_\delta K.$$

Moreover, if $F = \mathcal{R}^{n\bullet}(G)$, then we also have the alternative

$$\text{(6.2)} \qquad \textit{either} \quad \mathcal{W}_\bullet\left[\mathcal{R}^{n+2\bullet}(G)\right] \geq C_\delta K \quad \textit{or} \quad \mathcal{W}_\bullet(F) \geq C_\delta K$$

(independent of n).

6.1. *Proof of Theorem 6.1.* — Let $\Pi \subset \mathfrak{B}_F$ be the geodesic continuum between $\mathfrak{B}^{[1]}_a$ and $\mathfrak{B}^{[1]}_{a+1}$. We will use notations from Definition 3.12 such as $\mathcal{G}, \mathcal{G}^s, R^{(s)}$.

6.1.1. *Spiraling Numbers.* — First we will introduce the spiraling parameters rel Π for a curve $\gamma \in R$ shortly summarized in the following remark:

Remark **6.2.** — Recall the pure Mapping Class Group MCG of a 3-punctured sphere $S^2 \setminus \{a, b, \infty\}$ is trivial. By replacing a, b with small open Jordan disks D_a, D_b, we obtain the new group $\mathrm{MCG}(S^2 \setminus (D_a \cup D_b \cup \{\infty\})) \simeq \mathbb{Z}^2$ consisting of Dehn-twists around D_a, D_b, see [12]. This fact has the following implication.

Let O be a small disk-neighborhood of $\mathfrak{B}_a^{[1]} \cup \Pi \cup \mathfrak{B}_{a+1}^{[1]}$. Then $O' := O \setminus (\mathfrak{B}_a^{[1]} \cup \mathfrak{B}_{a+1}^{[1]})$ is topologically a disk with two holes. Therefore, a path (simple curve) $\beta \subset O'$ from $\mathfrak{B}_a^{[1]}$ to $\mathfrak{B}_{a+1}^{[1]}$ has a simple description: it first spirals $\widetilde{t}_a \in \mathbb{Z}$ times around $\mathfrak{B}_a^{[1]}$ and then spirals $\widetilde{t}_{a+1} \in \mathbb{Z}$ times around $\mathfrak{B}_{a+1}^{[1]}$, where Π can be used as a reference "path" of zero spiralings. Below we will ignore the spiraling orientation and introduce the absolute quantity $t_a = |\widetilde{t}_a|$.

Recall from Section 3.3 that $\gamma \in R$ lands at the ideal boundary of $V \setminus \mathbf{\mathfrak{B}}^{[1]}$. Let $\gamma_\Pi \subset V \setminus \mathbf{\mathfrak{B}}^{[1]}$ be a path homotopic rel the endpoints to γ in $V \setminus \mathbf{\mathfrak{B}}^{[1]}$ such that $\gamma_\Pi \cap \Pi$ is the minimal possible number. In other words, γ_Π and Π are in the minimal position – this is well defined because Π has infinitely many cut-points.

Consider the components $\gamma_0, \gamma_1, \dots, \gamma_f, \gamma_{f+1}$ of $\gamma_\Pi \setminus \Pi$. Since γ is properly homotopic into (a small neighborhood of) Π, there is a $t_a \le f$ such that

- $\gamma_t \cup \Pi$ surrounds $\mathfrak{B}_a^{[1]}$ for $t \in \{1, 2, \dots, t_a\}$;
- $\gamma_t \cup \Pi$ does not surround $\mathfrak{B}_a^{[1]}$ for $t \in \{t_a + 1, \dots, f\}$.

(It follows that $\gamma_t \cup \Pi$ surrounds $\mathfrak{B}_{a+1}^{[1]}$ for $t > t_a$, and only $\gamma_{t_a} \cup \Pi$ can surround both $\mathfrak{B}_a^{[1]}$ and $\mathfrak{B}_{a+1}^{[1]}$.)

We say that $t_a = t_a(\gamma)$ is the *spiraling number of* γ *around* $\mathfrak{B}_a^{[1]}$. Similarly, the *spiraling number of* $\gamma \in R^{(s)}$ *around* $\mathfrak{B}_a^{[1],(sp_1)}$ is introduced. Below are some properties of spiraling numbers:

(A) The spiraling numbers of $\gamma_1, \gamma_2 \in R$ differ by at most 1.
(B) If $\gamma_s \in R^{(s)}$ is the lift of $\gamma \in R$ to $R^{(s)}$, then $t_a(\gamma_s) \le \frac{1}{2^s} t_a(\gamma)$.

Indeed, (A) follows from the fact that (the disjoint curves) γ_1, γ_2 can be simultaneously put into the minimal position with Π. And (B) follows from the fact that $f^{sp_1} : \mathfrak{B}_a^{[1],(sp_1)} \to \mathfrak{B}_a^{[1]}$ has degree 2^s.

6.1.2. *The non-spiraling case.* — $t_a(\gamma) \le 4$ for all $\gamma \in R$; see Figure 11 for illustration. Write $q := 10\bar{p}$ and note that $qp_1 > 10p_2$. We continue using the notations of Definition 3.12.

Let us consider the objects introduced in Lemma 2.1. It follows from (B) that curves in $R^{(3)}$ do not spiral around $\mathfrak{B}_a^{[1],(3p_1)}$. Therefore, ℓ_2 separates the base of $R^{(q)}$ from $\mathfrak{B}_a^{[1]}$: the base $\partial^{h,0} R^{(q)}$ and $\mathfrak{B}_a^{[1]}$ are in different components of $\mathfrak{B}_a^{[1],(qp_1)} \setminus \ell_2$.

Let $\mathcal{G}_a$ be the restriction of $\mathcal{G}^q$ between ℓ_1 and ℓ_2 (this lamination consists of the first shortest subarcs $\gamma' \subset \gamma$ between ℓ_1 and ℓ_2 for all $\gamma \in \mathcal{G}^q$). Since $\mathcal{G}^q$ consequently

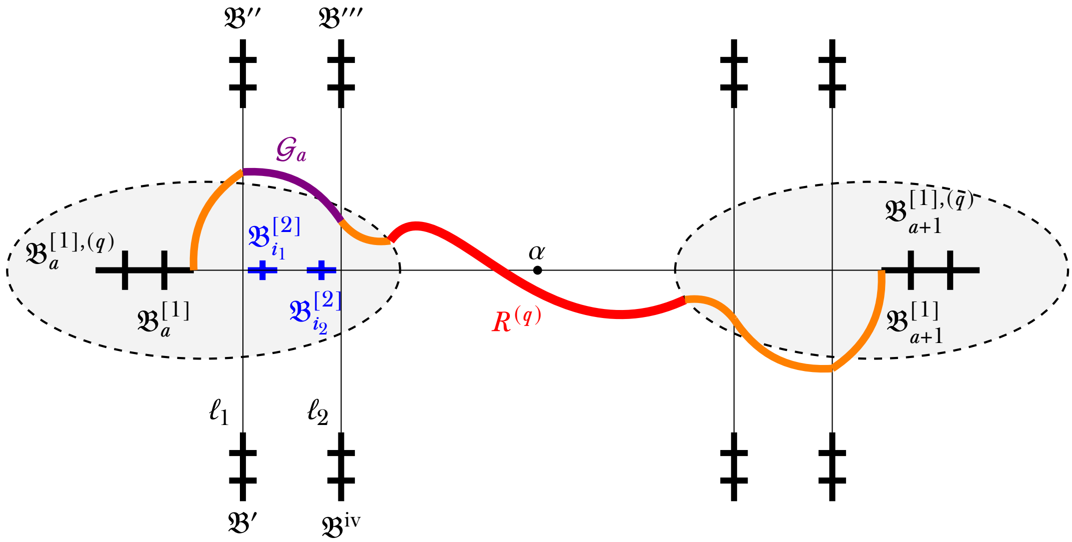

Figure 11. — The non-spiraling case. The rectangle R (orange, violet, red) connects $\mathfrak{B}_a^{[1]}$ and $\mathfrak{B}_{a+1}^{[1]}$. Between $\mathfrak{B}_a^{[1],(q)}$ and $\mathfrak{B}_{a+1}^{[1],(q)}$ most of the rectangle R travels through its lift $R^{(q)}$ (red). Therefore, the part $\mathcal{G}_a$ (violet) of R between ℓ_1 and ℓ_2 is tremendously wide. The bushes $\mathfrak{B}', \mathfrak{B}', \mathfrak{B}''', \mathfrak{B}^{\text{iv}}$ are lifts of $\mathfrak{B}_{a+1}^{[1]}$, see Figure 6.

overflows $\mathcal{G}_a$ and then $R^{(q)}$, we obtain from the Grötzsch inequality that

$$(1-\delta)\mathcal{W}(R) \le \mathcal{W}(\mathcal{G}^q) \le \mathcal{W}\big(R^{(q)}\big) \oplus \mathcal{W}(\mathcal{G}_a) = \mathcal{W}(R) \oplus \mathcal{W}(\mathcal{G}_a), \tag{6.3}$$

where $\mathcal{W}(R^{(q)}) = \mathcal{W}(R)$ because $R^{(q)}$ is a lift of R. Therefore, $\mathcal{W}(\mathcal{G}_a) \ge C_{0,\delta}\mathcal{W}(R) \ge C_{0,\delta}K$, where $C_{0,\delta} \to \infty$ as $\delta \to 0$.

There are two possibilities. If a substantial part of $\mathcal{G}_a$ travels through both $\mathfrak{B}_{i_1}^{[2]}, \mathfrak{B}_{i_2}^{[2]}$, then this substantial part of $\mathcal{G}_a$ restricts to a wide lamination $\mathcal{L}$ between $\mathfrak{B}_{i_1}^{[2]}, \mathfrak{B}_{i_2}^{[2]}$. Pushing $\mathcal{L}$ with respect to f_* and ι^* and using Lemma 3.2 (the number of iterations and the degree is bounded in terms of $\bar{p}$), we obtain a family $\mathcal{F}$ of non-trivial proper curves in $V \setminus \mathfrak{B}^{[2]}$ starting at any periodic bush $\mathfrak{B}_k^{[2]}$ such that $\mathcal{W}(\mathcal{F}) \ge_{\bar{p}} C_{0,\delta}K$. If a substantial part of $\mathcal{F}$ is vertical, then we obtain the second estimate in (6.1), (6.2). If a substantial part of $\mathcal{F}$ is horizontal, then we obtain the first estimate in (6.1); applying Lemma 3.10 to $\mathcal{F}$, we obtain the first estimate in (6.2).

If a substantial part of $\mathcal{G}_a$ omits either $\mathfrak{B}_{i_1}^{[2]}$ or $\mathfrak{B}_{i_2}^{[2]}$, then we obtain a wide wave above one of the sides of $\mathfrak{B}_{i_1}^{[2]}, \mathfrak{B}_{i_2}^{[2]}$; Wave Lemma 5.1 implies that $\mathcal{W}_\bullet(F) \ge C_\delta K$.

6.1.3. *The spiraling case.* — $t_a(\gamma) \ge 4$ for all $\gamma \in R$; see Figure 12 for illustration. It follows from (B) that the spiraling number of any curve in $\mathcal{G}^1$ around $\mathfrak{B}_a^{[1]}$ is strictly less than the spiraling number of any curve in $R^{(1)}$ around $\mathfrak{B}_a^{[1],(1)}$. Therefore, curves in $\mathcal{G}^1$ first spirals around $\mathfrak{B}_a$ before they continue within $\mathcal{F}^{\text{full}}(R^{(1)})$. Applying the Grötzsch inequality as in (6.3), we obtain a wide lamination $\mathcal{G}_a$ with $\mathcal{W}(\mathcal{G}_a) \ge C_{2,\delta}\mathcal{W}(R)$ such that $\mathcal{G}_a$ creates a wave above one of the sides of $\mathfrak{B}_a$. Wave Lemma 5.1 implies that $\mathcal{W}_\bullet(f) \geq C_\delta K$.

Remark **6.3.** — The argument of this section will be repeated in the proof of (7.2) below with $\mathfrak{B}^{[1]}_{a+1}$ being replaced by a little Julia set $\mathfrak{K}^{[1]}_0$. Observe that no specific properties of $\mathfrak{B}^{[1]}_{a+1}$ has been used in this section (only that it is a forward invariant set that is disjoint from $\mathfrak{B}^{[1]}_a$).

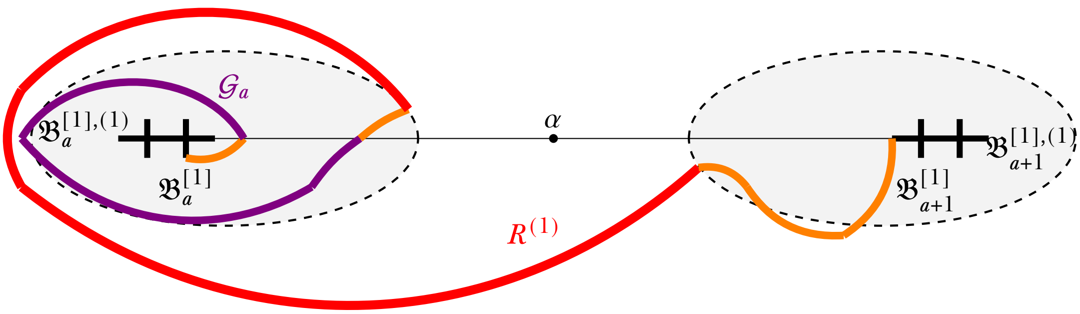

FIGURE 12. — The spiraling case. Since R and its lift $R^{(1)}$ have different spiraling numbers around $\mathfrak{B}^{[1]}_a$ rel Π respectively, (most of) R (namely $\mathcal{G}^1$) spirals first around $\mathfrak{B}^{[1]}_a$ before it continues as $R^{(1)}$ (red). The part $\mathcal{G}_a$ (violet) of R before $R^{(1)}$ is tremendously wide.

7. Conclusions

In this section, we will deduce the main theorems.

Theorem **7.1** *(A priori beau $\psi^\bullet$-bounds).* — *For any combinatorial bound $\bar{p}$, there is an $n > 1$ and $K_{\bar{p}} > 1$ such that the following holds. If F is an infinitely renormalizable $\psi^\bullet$-ql map of bounded type $\bar{p}$, then*

$$\mathcal{W}_\bullet[(\mathcal{R}^{n\bullet})^m(F)] \leq K_{\bar{p}} \quad \text{for} \quad m \gg_{\mathcal{W}_\bullet(F)} 1,$$

where $\mathcal{R}^{n\bullet}$ is a $\psi^\bullet$-ql renormalization (3.15).

Proof. — Let us choose a sufficiently small $\delta > 0$ such that $C_\delta \gg \Delta_{\bar{p}}$, where C_δ is from Theorem 6.1 and $\Delta_{\bar{p}}$ is from Proposition 2.7. We next choose a sufficiently big $n \gg_{\bar{p},\delta} 1$ such that Theorem 4.1 is applicable as follows: if

$$\mathcal{W}_\bullet[(\mathcal{R}^{n\bullet})^m(F)] = K \gg_{\bar{p},\delta,n} 1,$$

then

(I) either $\mathcal{W}_\bullet\big[(\mathcal{R}^{n\bullet})^{(m-1)}(F)\big] \geq 2K$;

(II) or $\mathcal{W}_\bullet\big[\mathcal{R}^{n+1\bullet} \circ (\mathcal{R}^{n\bullet})^{m-1}(F)\big] \geq C_\delta K$,

where (II) follows from Case (S) of Theorem 4.1 combined with Theorem 6.1.

Write $G := (\mathcal{R}^{n\bullet})^{(m-1)}(F)$. Reapplying Theorem 4.1 for G and its renormalization $\mathcal{R}^{n+1\bullet}G$, we obtain the alternative "(I) vs (II′)", where

(II′) $\mathcal{W}_\bullet\big[\mathcal{R}^{n+2\bullet}(G)\big] \geq C_\delta^2 K$.

Repeating the argument, we eventually obtain the alternative "(I) vs (II″)", where

(II″) $\mathcal{W}_\bullet\big[\mathcal{R}^{2n\bullet}(G)\big] \geq C_\delta^n K$.

Therefore, if there are no a priori beau $\psi^\bullet$-bounds, then we obtain

$$\mathcal{W}_\bullet[(\mathcal{R}^{n\bullet})^m(F)] \geq_F \ C_\delta^{n(m-1)} K.$$

This contradicts the Teichmüller contraction: by Proposition 2.7, see also Remark 2.8, the sequence $(\mathcal{R}^{n\bullet})^m(F)$ restricts to ql maps $f_{nm} : X_{nm} \to Y_{nm}$ such that

$$\mathcal{W}(Y_{nm} \setminus X_{nm}) = O_F\Big(\Delta_{\bar{p}}^{nm}\Big),$$

where $\Delta_{\bar{p}} \ll C_\delta$. □

Theorem **7.2** *(A priori beau ql-bounds).* — *For any combinatorial bound $\bar{p}$, there is a $K_{\bar{p}} > 1$ such that the following holds. If $f : U \to V$ is an infinitely renormalizable ql map of bounded type $\bar{p}$, then for $n \gg_{\mathcal{W}(f)} 1$, the map f_n has a ql restriction*

$$f_n \colon X_n \longrightarrow Y_n \quad \textit{with} \quad \mathcal{W}(Y_n \setminus X_n) \leq K_{\bar{p}}.$$

Proof. — Write $F := f$, and consider the sequence $G_m := (\mathcal{R}^{n\bullet})^m(F)$ from Theorem 7.1. We have

$$\mathcal{W}_\bullet(G_m) \leq K_{\bar{p}} \quad \text{for } m \gg_{\mathcal{W}_\bullet(F)} 1 \tag{7.1}$$

for $K_{\bar{p}}$ from Theorem 7.1.

For m satisfying (7.1), consider the dynamical plane of $G_m \colon U \to V$. Let $\mathfrak{B}^{[1]}$ be the cycle of little bushes in the dynamical plane of G_m. We *claim* that there is a space between $\mathfrak{K}_0^{[1]}$ and $\mathfrak{B}^{[1]} \setminus \mathfrak{B}_0^{[1]}$; i.e., that there is a $K_2 > 0$ depending on $K_{\bar{p}}, \bar{p}$ such that

$$\mathcal{W}\Bigg(V \setminus \bigcup_{i\neq 0} \mathfrak{B}_i^{[1]},\ \mathfrak{K}_0^{[1]}\Bigg) \leq K_2. \tag{7.2}$$

Then (7.2) together with Section 3.4.5, Lemma 3.7, and Remark 3.8 will imply that G_m has a ql renormalization around $\mathfrak{K}_0^{[1]}$ with a definite modulus. □

Proof of (7.2). — Assuming converse, the associated degeneration is arbitrary big:

$$K := \mathcal{W}(\mathcal{G}) \gg_{K_{\bar{p}}} 1, \quad \text{where} \quad \mathcal{G} := \mathcal{F}\Bigg(V \setminus \bigcup_{i\neq 0} \mathfrak{B}_i^{[1]},\ \mathfrak{K}_0^{[1]}\Bigg).$$

We will now argue that a substantial part of $\mathcal{G}$ travels through level two little bushes; this will lead to a contradiction to the $\psi^\bullet$-bounds of Theorem 7.2.

Write $\Upsilon := \mathfrak{B}^{[1]} \cup \mathfrak{K}_0^{[1]}$, and consider the horizontal and vertical WAD $\mathcal{A}_{\text{hor}}^{[1],k}$, $\mathcal{A}_{\text{ver}}^{[1],k}$ of $U^k \setminus \Upsilon$. By (7.1),

$$\mathcal{W}\left(\mathcal{A}_{\text{ver}}^{[1],k}\right) = O_{k,\bar{p}}\left(K_{\bar{p}}\right) \quad \text{and hence} \quad \mathcal{W}\left(\mathcal{A}_{\text{hor}}^{[1],k}\right) = K - O_{k,\bar{p}}\left(K_{\bar{p}}\right). \tag{7.3}$$

As in the proof of Theorems 4.1, see Section 4.1, the arc diagrams $\mathcal{A}_{\text{hor}}^{[1],kp_1}$ eventually stabilize:

$$\text{AD}\left(\mathcal{A}_{\text{hor}}^{[1],(t+1)p_1} - C^{t+1}\right) = \text{AD}\left(\mathcal{A}_{\text{hor}}^{[1],tp_1} - C^t\right) \quad \text{for } t \le 3p_1,$$

where $C \gg_{p_1} 1$ is fixed. By Lemma 2.5, $H = \text{AD}(\mathcal{A}_{\text{hor}}^{[1],(t+1)p_1} - C^{t+1})$ is invariant under f^{p_1}. Since most of the horizontal curves restrict to horizontal curves (by (7.3)) the same argument as in Section 4.3 provides a δ-almost periodic rectangle $R \subset V \setminus \Upsilon$, $\mathcal{W}(R) \asymp K$ between $\mathfrak{K}_0^{[1]}$ and some $\mathfrak{B}_a^{[1]}$, where δ is sufficiently small and $\mathfrak{K}_0^{[1]}$ replacing $\mathfrak{B}_{a+1}^{[1]}$ in Definition 3.12. By Lemma 2.6, the first renormalization of G_m is satellite.

We can now repeat the argument of Theorem 6.1, see Remark 6.3, with $\mathfrak{K}_0^{[1]}$ replacing $\mathfrak{B}_{a+1}^{[1]}$ to obtain

$$\text{either} \quad \mathcal{W}_\bullet\left[\mathcal{R}^{2\bullet}(G_m)\right] \succeq_{\bar{p}} K \quad \text{or} \quad \mathcal{W}_\bullet(G_m) \succeq_{\bar{p}} K;$$

both estimates leading to a contradiction with $\psi^\bullet$-bounds of Theorem 7.1. □

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Manuscript received December 4, 2023
Revised version November 5, 2024
Manuscript accepted November 25, 2025