p-ADIC HODGE PARAMETERS IN THE CRYSTABELLINE REPRESENTATIONS OF GL_n

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ABSTRACT

Let K be a finite extension of $\mathbf{Q}_p$, and ρ be an n-dimensional (non-critical generic) crystabelline representation of the absolute Galois group of K of regular Hodge-Tate weights. We associate to ρ an explicit locally $\mathbf{Q}_p$-analytic representation $\pi_1(\rho)$ of $\mathrm{GL}_n(\mathrm{K})$, which encodes some p-adic Hodge parameters of ρ. When $\mathrm{K}=\mathbf{Q}_p$, it encodes the full information hence reciprocally determines ρ. When ρ is associated to p-adic automorphic representations, we show under mild hypotheses that $\pi_1(\rho)$ is a subrepresentation of the $\mathrm{GL}_n(\mathrm{K})$-representation globally associated to ρ.

CONTENTS

1. Introduction

The locally analytic p-adic Langlands program for $\mathrm{GL}_n(\mathbf{Q}_p)$ aims at building a correspondence between n-dimensional p-adic continuous representations of the abosulte Galois group $\mathrm{Gal}_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ and certain locally analytic representations of $\mathrm{GL}_n(\mathbf{Q}_p)$. In particular, it is expected to match the parameters on both sides via the conjectural correspondence.

On the Galois side, the p-adic $\mathrm{Gal}_{\mathbf{Q}_p}$-representations are central objects in the p-adic Hodge theory, and are classified by Fontaine's theory. Among these representations, the de Rham ones are particularly important, as they include those arising from geometry ([39]). The p-adic Langlands program for de Rham representations is expected to be compatible with the classical local Langlands correspondence (e.g. see [19]). More precisely, by Fontaine's theory, for a de Rham representation ρ over a p-adic field E, one can associate an n-dimensional Weil-Deligne representation $\mathtt{r}$, which furthermore corresponds, via the classical local Langlands correspondence, to an irreducible smooth representation $\pi_{\mathrm{sm}}(\mathtt{r})$ of $\mathrm{GL}_n(\mathbf{Q}_p)$ over E. If ρ has regular Hodge-Tate weights $\mathbf{h}=(h_1,\dots,h_n)$, then the locally algebraic representation

$$\pi_{\mathrm{alg}}(\mathtt{r},\mathbf{h}):=\pi_{\mathrm{sm}}(\mathtt{r})\otimes_{\mathrm{E}}\mathrm{L}(\mathbf{h}-\theta)$$

https://doi.org/10.1007/s10240-025-00156-2

is expected to be the locally algebraic subrepresentation of the conjectural locally analytic representation $\pi^?(\rho)$ associated to ρ, where $\theta=(0,-1,\dots,1-n)$ and $\mathrm{L}(\mathbf{h}-\theta)$ is the algebraic representation of $\mathrm{GL}_n(\mathbf{Q}_p)$ of highest weight $\mathbf{h}-\theta$. One can clearly recover $\mathtt{r}$ (up to F-semi-simplification) and $\mathbf{h}$ from the representation of $\pi_{\mathrm{alg}}(\mathtt{r},\mathbf{h})$. However, passing from ρ to $(\mathtt{r},\mathbf{h})$, one loses the information of Hodge filtration of ρ. A fundamental question in the p-adic Langlands program is to find the missing information on Hodge filtration on the automorphic side, say, in the conjectural locally analytic representation $\pi^?(\rho)$. After the pioneer work of Breuil ([9, 11]), the question was settled for $\mathrm{GL}_2(\mathbf{Q}_p)$ by Colmez, establishing the p-adic Langlands correspondence ([28]). It remains quite mysterious for general $\mathrm{GL}_n(\mathbf{Q}_p)$. In this paper, we address the question for (non-critical generic) crystabelline $\mathrm{Gal}_{\mathbf{Q}_p}$-representations ρ, those that become crystalline when restricted to the absolute Galois group of a certain abelian extension of $\mathbf{Q}_p$.

For simplicity, we assume in the introduction that ρ itself is crystalline. Then by Fontaine's theory, ρ is equivalent to the associated filtered φ-module $\mathrm{D}_{\mathrm{cris}}(\rho)$. We assume the φ-action is *generic* (and we simply call such ρ generic), which means the φ-eigenvalues $\underline{\alpha}=(\alpha_i)$ on $\mathrm{D}_{\mathrm{cris}}(\rho)$ are distinct, and $\alpha_i\alpha_j^{-1}\neq p$ for $i\neq j$. In this case, $\mathtt{r}\cong\oplus_{i=1}^n \mathrm{unr}(\alpha_i)$ and we denote $\mathtt{r}$ by $\underline{\alpha}$. The classical local Langlands correspondence in this case is simply given by

$$\pi_{\mathrm{sm}}(\underline{\alpha})\cong\big(\mathrm{Ind}_{\mathrm{B}^-(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)}\mathrm{unr}(\underline{\alpha})\eta\big)^{\infty}$$

where $\mathrm{unr}(\underline{\alpha})=\mathrm{unr}(\alpha_1)\boxtimes\cdots\mathrm{unr}(\alpha_n)$, $\eta=1\boxtimes|\cdot|^1\boxtimes\cdots|\cdot|^{n-1}$ are unramified characters of $\mathrm{T}(\mathbf{Q}_p)$, and B^- is the Borel subgroup of lower triangular matrices. Let $\mathrm{Fil}_{\mathrm{H}}^{\bullet}$ denote the Hodge filtration, which is a complete flag in $\mathrm{D}_{\mathrm{cris}}(\rho)$ as $\mathbf{h}$ is regular. Let $e_i\in\mathrm{D}_{\mathrm{cris}}(\rho)$ be an eigenvector for α_i. Under the basis $\{e_i\}$, $\mathrm{Fil}_{\mathrm{H}}^{\bullet}$ is parametrized by an element in $\mathrm{T}\backslash\mathrm{GL}_n/\mathrm{B}$, which we call the *p-adic Hodge parameter* of ρ. Recall that ρ is called *non-critical* if $\mathrm{Fil}_{\mathrm{H}}^{\bullet}$ is in a relative general position with respect to all the $n!$ φ-stable (complete) flags. When $n=2$, $\mathrm{T}\backslash\mathrm{GL}_2/\mathrm{B}$ is a finite set of cardinality 3. So there are at most 3 isomorphism classes[1] of ρ, distinguished by the relative position of $\mathrm{Fil}_{\mathrm{H}}^{\bullet}$ with the two φ-stable flags. The information is reflected by the extra socle phenomenon on the $\mathrm{GL}_2(\mathbf{Q}_p)$-side. In this context, Breuil formulated a conjecture concerning the locally analytic socle of GL_n, which characterizes the relative positions of $\mathrm{Fil}_{\mathrm{H}}^{\bullet}$ with the φ-stable flags. The conjecture was subsequently proved (under Taylor-Wiles hypotheses) by Breuil-Hellmann-Schraen ([22]). However, a significant difference between the cases $n=2$ and $n\geq 3$ lies in the extra parameters for non-critical ρ (with fixed $(\underline{\alpha},\mathbf{h})$): when $n=2$, the non-critical ρ is unique, whereas for $n\geq 3$, there are additional (new) parameters for non-critical ρ (as $\mathrm{T}\backslash\mathrm{GL}_n/\mathrm{B}$ is now an infinite set). We refer to Example 2.9 for a concrete example of $n=3$.

[1] The étaleness of ρ will imply that some of these classes may not occur. In most general cases, there is typically a unique isomorphism class. But note if we relax the étaleness condition, and consider crystalline (φ,Γ)-modules instead of ρ, all these classes can appear.

In the paper, we reveal these p-adic Hodge parameters on the $\mathrm{GL}_n(\mathbf{Q}_p)$-side. It turns out it is convenient to work with (φ,Γ)-modules over the Robba ring instead of Galois representations. Denote by $\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\mathbf{h})$ the set of isomorphism classes of non-critical crystalline (φ,Γ)-modules overlying $\underline{\alpha}$ of regular Hodge-Tate weights $\mathbf{h}$. Under the basis of φ-eigenvectors $\{e_i\}$ in the precedent paragraph (noting that $\mathrm{D}_{\mathrm{cris}}(\mathrm{D})\cong\oplus_{i=1}^n \mathrm{E}e_i$, as φ-module, for all $\mathrm{D}\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\mathbf{h})$), the set $\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\mathbf{h})$ can be identified with a Zariski open subset of $\mathrm{T}\backslash\mathrm{GL}_n/\mathrm{B}$. For each $\mathrm{D}\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\mathbf{h})$, we associate an explicit locally analytic $\mathrm{GL}_n(\mathbf{Q}_p)$-representation $\pi_1(\mathrm{D})$ (see Theorem 1.3 below for the construction). We have:

Theorem **1.1.** — *(1) (Local correspondence) For* $\mathrm{D}\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\boldsymbol{h})$, $\mathrm{soc}_{\mathrm{GL}_n(\mathbf{Q}_p)}\pi_1(\mathrm{D})\cong\pi_{\mathrm{alg}}(\underline{\alpha},\boldsymbol{h})$*, and* $\pi_1(\mathrm{D})\twoheadrightarrow\pi_{\mathrm{alg}}(\underline{\alpha},\boldsymbol{h})^{\oplus(2^n-\frac{n(n+1)}{2}-1)}$*. Moreover, for* $\mathrm{D}'\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\boldsymbol{h})$, $\pi_1(\mathrm{D})\cong\pi_1(\mathrm{D}')$ *if and only if* $\mathrm{D}'\cong\mathrm{D}$.

(2) (Local-global compatibility) Suppose ρ is automorphic for the setting of [23] (or the setting in Section 4.2.2), and let $\widehat{\pi}(\rho)$ be the unitary Banach representation of $\mathrm{GL}_n(\mathbf{Q}_p)$ *(globally) associated to ρ. Assume* $\mathrm{D}_{\mathrm{rig}}(\rho)\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\boldsymbol{h})$*. Then for* $\mathrm{D}\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\boldsymbol{h})$,

$$\pi_1(\mathrm{D})\hookrightarrow\widehat{\pi}(\rho)^{\mathrm{an}}\ \textit{if and only if}\ \mathrm{D}\cong\mathrm{D}_{\mathrm{rig}}(\rho).$$

In particular, $\widehat{\pi}(\rho)^{\mathrm{an}}$ determines ρ.[2]

The quotient $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})^{\oplus(2^n-\frac{n(n+1)}{2}-1)}$ of $\pi_1(\mathrm{D})$ appears in the "third" layer in its socle filtration. Let $\pi_{\min}(\mathrm{D})$ be the minimal subrepresentation of $\pi_1(\mathrm{D})$ such that the composition $\pi_{\min}(\mathrm{D})\hookrightarrow\pi_1(\mathrm{D})\twoheadrightarrow\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})^{\oplus(2^n-\frac{n(n+1)}{2}-1)}$ is surjective. The representation $\pi_{\min}(\mathrm{D})$ has a much cleaner structure. For example, its socle filtration has only three grades (see Section 3.2). Note that one can replace everywhere $\pi_1(\mathrm{D})$ in the statements by $\pi_{\min}(\mathrm{D})$. The extra locally algebraic constituents in the cosocle of $\pi_1(\mathrm{D})$ were unexpected, not to mention their huge multiplicity. It is one of the reasons why it took a long time to find the Hodge parameters. In fact, the work grows out from the finding of such extra constituents while excluding such constituents for GL_2 in [34]. We remark that the existence of the extra locally algebraic constituent was first proved by Hellmann-Hernandez-Schraen in the split case for $\mathrm{GL}_3(\mathbf{Q}_p)$ ([42]).

For a finite extension K of $\mathbf{Q}_p$, we also construct a locally $\mathbf{Q}_p$-analytic representation $\pi_1(\mathrm{D})$ of $\mathrm{GL}_n(\mathrm{K})$ such that $\mathrm{soc}_{\mathrm{GL}_n(\mathrm{K})}\pi_1(\mathrm{D})\cong\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})$ and $\pi_1(\mathrm{D})\twoheadrightarrow\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})^{\oplus(2^n-\frac{n(n+1)}{2}-1)[\mathrm{K}:\mathbf{Q}_p]}$. The local-global compatibility result still holds. But a major difference is that when $\mathrm{K}\neq\mathbf{Q}_p$, $\pi_1(\mathrm{D})$ just determines the filtered φ^f-module $\mathrm{D}_{\mathrm{cris}}(\mathrm{D})_\sigma$ (where f is the unramified degree of K over $\mathbf{Q}_p$) for each embedding $\sigma:\mathrm{K}\hookrightarrow\mathrm{E}$ rather than D itself. For example, when $n=2$, $\pi_1(\mathrm{D})$ are all isomorphic (for different $\mathrm{D}\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\mathbf{h})$) but there are still extra parameters, see for example [10, Section 3] [34, Conj. 1.7].

We make a few additional remarks on Theorem 1.1.

[2] Note that the information that $\mathrm{D}_{\mathrm{rig}}(\rho)$ is non-critical is determined by $\widehat{\pi}(\rho)^{\mathrm{an}}$ by [12, Thm. 9.3].

Remark **1.2.** — (1) Very little was known about such a local correspondence when $n \geq 3$. We highlight some related results. When $n = 3$, in [14], we showed how to recover the Hodge parameters in the semi-stable non-crystalline case (given by the Fontaine-Mazur $\mathcal{L}$-invariants) in the locally analytic $\mathrm{GL}_3(\mathbf{Q}_p)$-representations and proved a local-global compatibility result in the ordinary case. When the Weil-Deligne representation $\mathtt{r}$ associated to ρ is indecomposable, the (largely open) conjecture on Ext^1 in [13] (see also [15]) suggests a way to recover the p-adic Hodge parameters on the automorphic side. In contrast, the (non-critical) crystalline case was somewhat more mysterious, as such parameters are entirely new for $n \geq 3$. We finally mention that the results for $\mathrm{GL}_3(\mathbf{Q}_p)$ were presented in the note [33] (not intended for publication), which may help readers quickly understand the story.

(2) The phenomenon where the Hodge parameters lie in the extension group of certain locally algebraic representation by certain locally analytic representation traces back to Breuil's initializing work in [9].

(3) Similar results are also obtained in the patched setting. Let Π_∞ be the patched Banach representation over the patched Galois deformation ring R_∞ of [23]. We show that if there is a maximal ideal $\mathfrak{m}_\rho$ of $\mathrm{R}_\infty[1/p]$ associated to ρ such that $\Pi_\infty[\mathfrak{m}_\rho]^{\mathrm{lalg}} \neq 0$, then for $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$, $\pi_1(\mathrm{D}) \hookrightarrow \Pi_\infty[\mathfrak{m}_\rho]$ if and only if $\mathrm{D} \cong \mathrm{D}_{\mathrm{rig}}(\rho)$.

(4) We finally remark the representation $\pi_1(\mathrm{D})$ should still be far from the final complete locally analytic $\mathrm{GL}_n(\mathbf{Q}_p)$-representation associated to D (so we choose not to use the notation $\pi(\mathrm{D})$).

We now give the construction of $\pi_1(\mathrm{D})$. We first look at the Galois side. For each $w \in S_n$, let $\mathrm{Ext}^1_w(\mathrm{D}, \mathrm{D})$ be the extension group of trianguline deformations of D with respect to the refinement $w(\underline{\alpha})$ (see the discussion above (2.12)). Recall there is a natural (weight) map $\kappa_w : \mathrm{Ext}^1_w(\mathrm{D}, \mathrm{D}) \to \mathrm{Hom}(\mathrm{T}(\mathbf{Q}_p), \mathrm{E})$, sending $\widetilde{\mathrm{D}}$ to ψ such that $\widetilde{\mathrm{D}}$ is trianguline with parameter $\mathrm{unr}(w(\underline{\alpha}))z^{\mathbf{h}}(1 + \psi\epsilon)$ (that is a character of $\mathrm{T}(\mathbf{Q}_p)$ over $\mathrm{E}[\epsilon]/\epsilon^2$). The map κ_w is surjective (e.g. see [3, Prop. 2.3.10]). One can show that $\mathrm{Ker}\,\kappa_w$, as a subspace of $\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathrm{D}, \mathrm{D})$, is independent of the choice of w, denoted by $\mathrm{Ext}^1_0(\mathrm{D}, \mathrm{D})$ (cf. Lemma 2.11). For a subspace $\mathrm{Ext}^1_?(\mathrm{D}, \mathrm{D}) \subset \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathrm{D}, \mathrm{D})$ containing $\mathrm{Ext}^1_0(\mathrm{D}, \mathrm{D})$, set

$$\overline{\mathrm{Ext}}^1_?(\mathrm{D}, \mathrm{D}) := \mathrm{Ext}^1_?(\mathrm{D}, \mathrm{D}) / \mathrm{Ext}^1_0(\mathrm{D}, \mathrm{D}).$$

We have hence a bijection

$$\kappa_w : \overline{\mathrm{Ext}}^1_w(\mathrm{D}, \mathrm{D}) \xrightarrow{\sim} \mathrm{Hom}\big(\mathrm{T}(\mathbf{Q}_p), \mathrm{E}\big).$$

By [26], the following "amalgamating" map is surjective (see also [41, 46], noting it is already surjective before quotienting by $\mathrm{Ext}^1_0(\mathrm{D}, \mathrm{D})$ on both sides)

$$\oplus_{w \in S_n} \overline{\mathrm{Ext}}^1_w(\mathrm{D}, \mathrm{D}) \twoheadrightarrow \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D}, \mathrm{D}). \tag{1.1}$$

Remark that here we use that all the refinements of D are non-critical.

Now we look at the $\mathrm{GL}_n(\mathbf{Q}_p)$-side. For each w, consider the locally analytic principal series $\mathrm{PS}(w,\underline{\alpha},\mathbf{h}) := (\mathrm{Ind}_{\mathrm{B}^-(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)} \mathrm{unr}(w(\underline{\alpha}))z^{\mathbf{h}}\varepsilon^{-1}\circ\theta)^{\mathrm{an}}$, where ε denotes the cyclotomic character. The explicit structure of $\mathrm{PS}(w,\underline{\alpha},\mathbf{h})$ is well-understood by Orlik-Strauch ([55]). For example, $\mathrm{soc}_{\mathrm{GL}_n(\mathbf{Q}_p)}\mathrm{PS}(w,\underline{\alpha},\mathbf{h}) \cong \pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})$, which has multiplicity one as irreducible constituent of $\mathrm{PS}(w,\underline{\alpha},\mathbf{h})$. For $w\in \mathrm{S}_n$, consider the composition

$$\begin{aligned}\zeta_w : \mathrm{Hom}\big(\mathrm{T}(\mathbf{Q}_p),\mathrm{E}\big) &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}\big(\mathrm{PS}(w,\underline{\alpha},\mathbf{h}),\mathrm{PS}(w,\underline{\alpha},\mathbf{h})\big)\\ &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}\big(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\mathrm{PS}(w,\underline{\alpha},\mathbf{h})\big),\end{aligned}$$

where the first map sends ψ to $(\mathrm{Ind}_{\mathrm{B}^-(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)} \mathrm{unr}(w(\underline{\alpha}))z^{\mathbf{h}}(\varepsilon^{-1}\circ\theta)(1+\psi\epsilon))^{\mathrm{an}}$, and the second map is the natural pull-back map. Using Schraen's spectral sequence ([58, (4.37)]), one can show that ζ_w is in fact bijective. Now we amalgamate these principal series: let $\pi(\underline{\alpha},\mathbf{h})$ be the unique quotient of the amalgamation $\oplus^{w\in \mathrm{S}_n}_{\pi_{\mathrm{alg}}(\underline{\alpha},\lambda)}\mathrm{PS}(w,\underline{\alpha},\mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\underline{\alpha},\lambda)$ (which was introduced and denoted by $\pi(\mathrm{D})^{\mathrm{fs}}$ in [17, Def. 5.7]). For each $w\in \mathrm{S}_n$, there is a natural injection $\mathrm{PS}(w,\underline{\alpha},\mathbf{h})\hookrightarrow \pi(\underline{\alpha},\mathbf{h})$ which induces an injection

$$\mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}\big(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\mathrm{PS}(w,\underline{\alpha},\mathbf{h})\big)\lhook\joinrel\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}\big(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h})\big).$$

We denote by $\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h}))$ its image. The following "amalgamating" map is also surjective (see Proposition 3.8 (2) and compare with (1.1)):

$$\textbf{(1.2)}\qquad \oplus_{w\in\mathrm{S}_n}\mathrm{Ext}^1_w\big(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h})\big)\twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}\big(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h})\big).$$

The following theorem is crucial in the paper:

Theorem **1.3** *(cf. Theorem 3.21, Theorem 3.34). — (1) For* $\mathrm{D}\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\boldsymbol{h})$*, there is a unique (surjective) map* $t_{\mathrm{D}}:\mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha},\boldsymbol{h}),\pi(\underline{\alpha},\boldsymbol{h}))\twoheadrightarrow \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D},\mathrm{D})$ *such that the following diagram commutes:*

$$\begin{array}{ccc}\oplus_{w\in\mathrm{S}_n}\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D}) & \xrightarrow[\sim]{(\zeta_w\circ\kappa_w)} & \oplus_{w\in\mathrm{S}_n}\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha},\boldsymbol{h}),\pi(\underline{\alpha},\boldsymbol{h}))\\ {\scriptstyle(1.1)}\big\downarrow & & \big\downarrow{\scriptstyle(1.2)}\\ \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D},\mathrm{D}) & \xleftarrow{t_{\mathrm{D}}} & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha},\boldsymbol{h}),\pi(\underline{\alpha},\boldsymbol{h})).\end{array}$$

Moreover, $\dim_{\mathrm{E}}\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D},\mathrm{D})=\frac{n(n+1)}{2}+n$, $\dim_{\mathrm{E}}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha},\boldsymbol{h}),\pi(\underline{\alpha},\boldsymbol{h}))=2^n+n-1$ *hence* $\dim_{\mathrm{E}}\mathrm{Ker}(t_{\mathrm{D}})=2^n-\frac{n(n+1)}{2}-1$.

(2) For $\mathrm{D},\mathrm{D}'\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\boldsymbol{h})$, $\mathrm{Ker}(t_{\mathrm{D}})=\mathrm{Ker}(t_{\mathrm{D}'})$ *if and only if* $\mathrm{D}\cong\mathrm{D}'$.

Remark **1.4.** — Consider the composition

$$(\mathbf{1.3})\qquad \oplus_{w\in S_n}\mathrm{Hom}\big(\mathrm{T}(\mathbf{Q}_p),\mathrm{E}\big)\xrightarrow[\sim]{(\kappa_w^{-1})}\oplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})\xrightarrow{(1.1)}\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D},\mathrm{D}).$$

By Theorem 1.3 (1), the map (1.2) induces an exact sequence

$$0\longrightarrow \mathrm{Ker}(1.2)\longrightarrow (\zeta_w)_{w\in S_n}\big(\mathrm{Ker}(1.3)\big)\longrightarrow \mathrm{Ker}(t_{\mathrm{D}})\longrightarrow 0.$$

As the maps (1.2) and ζ_w's are all independent of $\mathrm{D}\in\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha},\mathbf{h})$, Theorem 1.3 (2) implies that Ker(1.3) also determines D. This fact (purely on Galois side) is of interest on its own right.

The representation $\pi_1(\mathrm{D})$ is then defined to be the (tautological) extension of $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\otimes_{\mathrm{E}}\mathrm{Ker}(t_{\mathrm{D}})\cong\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})^{\oplus(2^n-\frac{n(n+1)}{2}-1)}$ by $\pi(\underline{\alpha},\mathbf{h})$. More precisely, choosing a basis $\{v_i\}$ of $\mathrm{Ker}(t_{\mathrm{D}})$ with $\mathscr{E}(v_i)$ the associated extension of $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})$ by $\pi(\underline{\alpha},\mathbf{h})$, $\pi_1(\mathrm{D})$ is the amalgamated sum of these $\mathscr{E}(v_i)$ along $\pi(\underline{\alpha},\mathbf{h})$, which is clearly independent of the choice of $\{v_i\}$. The structure of $\pi(\underline{\alpha},\mathbf{h})$ is complicated (see for example [17, Section 5.3]). However, the theorem actually holds with $\pi(\underline{\alpha},\mathbf{h})$ replaced by its subrepresentation given by the first two layers in its socle filtration, which has a much easier and cleaner structure, see Theorem 3.21 and Section 3.1.2. The extension of $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\otimes_{\mathrm{E}}\mathrm{Ker}(t_{\mathrm{D}})$ by this subrepresentation actually gives $\pi_{\min}(\mathrm{D})$ in the discussion below Theorem 1.1. Theorem 1.1 (1) is then a direct consequence of Theorem 1.3.

One can deduce from Theorem 1.3 (1):

Corollary **1.5** *(cf. Corollary 3.25). — The map* t_{D} *induces a bijection*

$$(\mathbf{1.4})\qquad t_{\mathrm{D}}:\mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}\big(\pi_{\mathrm{alg}}(\underline{\alpha},\boldsymbol{h}),\pi_1(\mathrm{D})\big)\xrightarrow{\sim}\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D},\mathrm{D}).$$

Before discussing the proof of Theorem 1.3, we first explain the proof of the local-global compatibility (Theorem 1.1 (2)). For this, we will use an alternative formulation of Theorem 1.3 given as follows. Let π^{univ} (resp. π_w^{univ}) be the (universal) extension of $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\otimes_{\mathrm{E}}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathbf{Q}_p)}(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h}))$ (resp. of $\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\otimes_{\mathrm{E}}\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h}))$) by $\pi(\underline{\alpha},\mathbf{h})$ (defined in a similar way as in the discussion below Theorem 1.3). By (1.2), π^{univ} is generated by all the subrepresentations π_w^{univ} for $w\in S_n$. On the Galois side, let R_{D} be the universal deformation ring of deformations of D over Artinian local E-algebras and $\mathfrak{m}$ be its maximal ideal. The quotient $\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D},\mathrm{D})$ corresponds to a local Artinian E-subalgebra A_{D} of $\mathrm{R}_{\mathrm{D}}/\mathfrak{m}^2$, and $\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})$ corresponds to a quotient $\mathrm{A}_{\mathrm{D},w}$ of A_{D}. Using the isomorphism $\zeta_w\circ\kappa_w$, there exists a natural action of $\mathrm{A}_{\mathrm{D},w}$ on π_w^{univ} such that $x\in\mathfrak{m}_{\mathrm{A}_{\mathrm{D},w}}/\mathfrak{m}^2_{\mathrm{A}_{\mathrm{D},w}}\cong\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})^\vee\cong\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h}))^\vee$ acts via

$$\pi_w^{\mathrm{univ}}\twoheadrightarrow\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\otimes_{\mathrm{E}}\mathrm{Ext}^1_w\big(\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h}),\pi(\underline{\alpha},\mathbf{h})\big)\xrightarrow{x}\pi_{\mathrm{alg}}(\underline{\alpha},\mathbf{h})\hookrightarrow\pi_w^{\mathrm{univ}}.$$

The following corollary gives a reformulation of Theorem 1.3 (1).

Corollary **1.6** *(cf. Theorem 3.36, Corollary 3.37). — There exists a unique action of* A_{D} *on* π^{univ} *such that for each* $w \in \mathrm{S}_n$*, the* A_{D}*-action on its subrepresentation* π_w^{univ} *factors through the natural* $\mathrm{A}_{\mathrm{D},w}$*-action. Moreover, we have* $\pi_1(\mathrm{D}) \cong \pi^{\mathrm{univ}}[\mathfrak{m}_{\mathrm{A}_{\mathrm{D}}}]$.

Suppose we are in the patched setting as in Remark 1.2 (4), and let $\mathrm{D} = \mathrm{D}_{\mathrm{rig}}(\rho)$. Let $\mathfrak{a}$ be an ideal of $\mathrm{R}_\infty[1/p]$ with $\mathfrak{a} \supset \mathfrak{m}_\rho^2$ (cf. Remark 1.2 (3)) such that the composition $\mathrm{A}_{\mathrm{D}} \to \mathrm{R}_{\mathrm{D}}/\mathfrak{m}^2 \to \mathrm{R}_\infty[1/p]/\mathfrak{a}$ is an isomorphism (see the discussion below (4.3)). Working with the patched eigenvariety of [21], and using Emerton's adjunction formula [37], we can obtain $\mathrm{A}_{\mathrm{D}} \times \mathrm{GL}_n(\mathbf{Q}_p)$-equivariant injections $\pi_w^{\mathrm{univ}} \hookrightarrow \Pi_\infty[\mathfrak{a}]$ for all $w \in \mathrm{S}_n$, where the A_{D}-action on the right hand side comes from the R_∞-action (noting R_{D} is isomorphic to the universal Galois deformation ring of ρ). These injections "amalgamate" to an $\mathrm{A}_{\mathrm{D}} \times \mathrm{GL}_n(\mathbf{Q}_p)$-equivariant injection

$$\pi^{\mathrm{univ}} \lhook\joinrel\longrightarrow \Pi_\infty[\mathfrak{a}]. \tag{1.5}$$

By Corollary 1.6, it induces an injection $\iota : \pi_1(\mathrm{D}) \cong \pi^{\mathrm{univ}}[\mathfrak{m}_{\mathrm{A}_{\mathrm{D}}}] \hookrightarrow \Pi_\infty[\mathfrak{a} + \mathfrak{m}_{\mathrm{A}_{\mathrm{D}}}] \cong \Pi_\infty[\mathfrak{m}_\rho]$. Now for $\mathrm{D}' \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$, if $\pi_1(\mathrm{D}') \hookrightarrow \Pi_\infty[\mathfrak{m}_\rho]$, one can prove (cf. the proof of Corollary 4.8) that it factors through the injection (1.5), i.e. we have $\pi_1(\mathrm{D}') \hookrightarrow \pi^{\mathrm{univ}} \hookrightarrow \Pi_\infty[\mathfrak{a}]$. As $\mathrm{A}_{\mathrm{D}}(\hookrightarrow \mathrm{R}_{\mathrm{D}}/\mathfrak{m}^2)$ acts on $\Pi_\infty[\mathfrak{m}_\rho]$ hence on its sub $\pi_1(\mathrm{D}')$ via $\mathrm{A}_{\mathrm{D}}/\mathfrak{m}_{\mathrm{A}_{\mathrm{D}}}$ and (1.5) is A_{D}-equivariant, $\pi_1(\mathrm{D}') \hookrightarrow \pi^{\mathrm{univ}}$ has image contained in $\pi^{\mathrm{univ}}[\mathfrak{m}_{\mathrm{A}_{\mathrm{D}}}] \cong \pi_1(\mathrm{D})$. Since $\pi_1(\mathrm{D}')$ and $\pi_1(\mathrm{D})$ have the same irreducible constituents with the same multiplicities, this implies $\pi_1(\mathrm{D}') \xrightarrow{\sim} \pi_1(\mathrm{D})$.

We now discuss the proof of Theorem 1.3. First, the case of $n = 2$ is clear, as now $\#\Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h}) = 1$, t_{D} is bijective, and $\pi_1(\mathrm{D}) \cong \pi(\underline{\alpha}, \mathbf{h})$ (which is the locally analytic $\mathrm{GL}_2(\mathbf{Q}_p)$-representation associated to D, see [29, 48]). For general $n \geq 3$, we use an induction argument. For simplicity, in the rest of the introduction, we restrict to the case of $n = 3$. This case already presents the key arguments. Let D_1 (resp. C_1) be the (unique) non-critical (φ, Γ)-module of rank 2 over $\mathcal{R}_{\mathrm{E}}$ of refinement $\underline{\alpha}^1 := (\alpha_1, \alpha_2)$ and of Hodge-Tate weights $\mathbf{h}^1 := (h_1 > h_2)$ (resp. $\mathbf{h}^2 := (h_2 > h_3)$). Then for any $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$, D admits two filtrations:

$$\begin{aligned}
\mathscr{F} &: 0 \longrightarrow \mathrm{D}_1 \longrightarrow \mathrm{D} \longrightarrow \mathcal{R}_{\mathrm{E}}\big(\mathrm{unr}(\alpha_3) z^{h_3}\big) \longrightarrow 0,\\
\mathscr{G} &: 0 \longrightarrow \mathcal{R}_{\mathrm{E}}\big(\mathrm{unr}(\alpha_3) z^{h_1}\big) \longrightarrow \mathrm{D} \longrightarrow \mathrm{C}_1 \longrightarrow 0.
\end{aligned}$$

Similarly as in (1.1) by considering the paraboline deformations with respect to $\mathscr{F}$ and $\mathscr{G}$, we have a natural map

$$f_{\mathrm{D}} = (f_{\mathscr{F}}, f_{\mathscr{G}}) : \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D}_1, \mathrm{D}_1) \oplus \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{C}_1, \mathrm{C}_1) \longrightarrow \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D}, \mathrm{D}), \tag{1.6}$$

sending $\widetilde{\mathrm{D}}_1$ (resp. $\widetilde{\mathrm{C}}_1$) to a (or any) deformation $\widetilde{\mathrm{D}}$ of D of the form (whose image in $\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{D}, \mathrm{D})$ does not depend on the choice): $0 \to \widetilde{\mathrm{D}}_1 \to \widetilde{\mathrm{D}} \to \mathcal{R}_{\mathrm{E}[\epsilon]/\epsilon^2}(\mathrm{unr}(\alpha_3) z^{h_3}) \to$

0 (resp. $0 \to \mathcal{R}_{E[\epsilon]/\epsilon^2}(\mathrm{unr}(\alpha_3)z^{h_1}) \to \widetilde{D} \to \widetilde{C}_1 \to 0$). The kernel of (1.6) is particularly important for our application. For $(\widetilde{D}_1, \widetilde{C}_1) \in \mathrm{Ker}(1.6)$, let $\widetilde{D}$ be a deformation of D whose image in $\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D, D)$ is equal to $f_{\mathscr{F}}(\widetilde{D}_1) = -f_{\mathscr{G}}(\widetilde{C}_1)$. Then $\widetilde{D}$ admits two different parabolic filtrations (of saturated (φ, Γ)-submodules over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$). We refer to this as a *higher intertwining* property (see Section 2.4). The following theorem is purely on Galois side, and follows from an explicit description of Ker(1.6) together with a reinterpretation of the p-adic Hodge parameters of D given in Section 2.2.

Theorem **1.7** *(cf. Corollary 2.34). — For* $D, D' \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \boldsymbol{h})$, $D \cong D'$ *if and only if* $\mathrm{Ker}(f_D) = \mathrm{Ker}(f_{D'})$.

We move to the automorphic side. Using parabolic inductions, one can show there is a natural map

$$(\mathbf{1.7}) \qquad \zeta : \mathrm{Ext}^1_{\mathrm{GL}_2}\big(\pi_{\mathrm{alg}}\big(\underline{\alpha}^1, \mathbf{h}^1\big), \pi_1(D_1)\big) \oplus \mathrm{Ext}^1_{\mathrm{GL}_2}\big(\pi_{\mathrm{alg}}\big(\underline{\alpha}^1, \mathbf{h}^2\big), \pi_1(C_1)\big) \xrightarrow{(\zeta_{\mathscr{F}}, \zeta_{\mathscr{G}})} \mathrm{Ext}^1_{\mathrm{GL}_3}\big(\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}), \pi(\underline{\alpha}, \mathbf{h})\big).$$

For example, $\zeta_{\mathscr{F}}$ is constructed using $(\mathrm{Ind}_{P^-}^{\mathrm{GL}_3} - \boxtimes\, \mathrm{unr}(\alpha_3)\varepsilon^2)^{\mathrm{an}}$, for $P^- = \begin{pmatrix} \mathrm{GL}_2 & 0 \\ * & \mathrm{GL}_1 \end{pmatrix}$, and $\zeta_{\mathscr{G}}$ uses $\begin{pmatrix} \mathrm{GL}_1 & 0 \\ * & \mathrm{GL}_2 \end{pmatrix}$. Moreover, (1.7) is surjective (roughly because $\pi(\underline{\alpha}, \mathbf{h})$ can be "amalgamated" from the two corresponding parabolic inductions). We refer to Proposition 3.13 for details.

Now a key fact is that for any $D \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$, $\mathrm{Ker}(\zeta)$ is sent to $\mathrm{Ker} f_D$ (cf. (1.6)) via the isomorphism for $n = 2$ (cf. (1.4)):

$$t_{D_1,C_1} : \mathrm{Ext}^1_{\mathrm{GL}_2}\big(\pi_{\mathrm{alg}}\big(\underline{\alpha}^1, \mathbf{h}^1\big), \pi_1(D_1)\big) \oplus \mathrm{Ext}^1_{\mathrm{GL}_2}\big(\pi_{\mathrm{alg}}\big(\underline{\alpha}^1, \mathbf{h}^2\big), \pi_1(C_1)\big) \xrightarrow[\sim]{(t_{D_1}, t_{C_1})} \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D_1, D_1) \oplus \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(C_1, C_1).$$

The map t_D in Theorem 1.3 (1) can now be easily constructed: there is a unique map t_D such that the following diagram commutes

$$\begin{array}{ccc}
\begin{array}{c}\mathrm{Ext}^1_{\mathrm{GL}_2}(\pi_{\mathrm{alg}}(\underline{\alpha}^1,\mathbf{h}^1),\pi_1(D_1)) \\ \oplus \mathrm{Ext}^1_{\mathrm{GL}_2}(\pi_{\mathrm{alg}}(\underline{\alpha}^1,\mathbf{h}^2),\pi_1(C_1))\end{array} & \xrightarrow{\ \zeta\ } & \mathrm{Ext}^1_{\mathrm{GL}_3}(\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h}), \pi(\underline{\alpha}, \mathbf{h})) \\
{\scriptstyle t_{D_1,C_1}} \downarrow \wr & & \downarrow {\scriptstyle t_D} \\
\begin{array}{c}\overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(D_1,D_1) \\ \oplus \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(C_1,C_1)\end{array} & \xrightarrow{\ f_D\ } & \overline{\mathrm{Ext}}^1(D, D).
\end{array}$$

It is not very difficult to check t_D satisfies the properties in Theorem 1.3 (1), and we refer to the proof of Theorem 3.21 for details. Theorem 1.3 (2) is then a consequence of Theorem 1.7, as $\mathrm{Ker}(t_D) = \zeta(t_{D_1,C_1}^{-1}(\mathrm{Ker}(f_D)))$ (noting $t_{D_1,C_1}(\mathrm{Ker}(\zeta)) \subset \mathrm{Ker}(f_{D'})$ for all

$\mathrm{D}' \in \Phi\Gamma_{\mathrm{nc}}(\underline{\alpha}, \mathbf{h})$). Remark the existence of the extra one copy of $\pi_{\mathrm{alg}}(\underline{\alpha}, \mathbf{h})$ in $\pi_1(\mathrm{D})$ (for $n = 3$) then comes from the fact $\dim_{\mathrm{E}} \mathrm{Ker}(f_{\mathrm{D}}) = \dim_{\mathrm{E}} \mathrm{Ker}(1.7) + 1$.

We refer to the context for the more precise and detailed statements. One main difference from what's discussed in the introduction is that we mainly work with $\pi_{\min}(\mathrm{D})$ instead of $\pi_1(\mathrm{D})$ in the introduction, which has a cleaner structure but requires a bit more on Orlik-Strauch representations.

2. Hodge filtration and higher intertwining

2.1. *Notation and preliminaries.* — Let K be a finite extension of $\mathbf{Q}_p$, E be a finite extension of $\mathbf{Q}_p$ containing all the embeddings of K in $\overline{\mathbf{Q}_p}$. Let $\Sigma_{\mathrm{K}} := \{\sigma : \mathrm{K} \hookrightarrow \mathrm{E}\}$, and $d_{\mathrm{K}} := [\mathrm{K} : \mathbf{Q}_p]$. For $\mathbf{k} = (k_\sigma)_{\sigma\in\Sigma_{\mathrm{K}}} \in \mathbf{Z}^{\Sigma_{\mathrm{K}}}$, denote by $z^{\mathbf{k}} := \prod_{\sigma\in\Sigma_{\mathrm{K}}} \sigma(z)^{k_\sigma}$ the ($\mathbf{Q}_p$-)algebraic character of $\mathrm{K}^\times$ of weight $\mathbf{k}$. Let $|\cdot|_{\mathrm{K}} : \mathrm{K}^\times \to \mathrm{E}^\times$ be the unramified character such that $|\varpi_{\mathrm{K}}|_{\mathrm{K}} = p^{-[\mathrm{K}_0:\mathbf{Q}_p]}$ for a uniformizer ϖ_{K} of K, where K_0 is the maximal unramified subextension of K over $\mathbf{Q}_p$. Let $\mathrm{Gal}_{\mathrm{K}}$ be the absolute Galois group of K, and $\varepsilon : \mathrm{Gal}_{\mathrm{K}} \to \mathbf{Z}_p^\times \to \mathrm{E}^\times$ be the cyclotomic character. We normalize the local class field theory by sending a uniformizer to a (lift of the) geometric Frobenius. In this way, we view ε as a character of $\mathrm{K}^\times$, which is equal to $\mathrm{N}_{\mathrm{K}/\mathbf{Q}_p}(\cdot)|\cdot|_{\mathrm{K}}$.

For a locally K-analytic group H (e.g. $\mathrm{H} = \mathrm{K}^\times$), set $\mathrm{Hom}(\mathrm{H}, \mathrm{E})$ to be the E-vector space of locally $\mathbf{Q}_p$-analytic E-valued characters on H, $\mathrm{Hom}_{\mathrm{sm}}(\mathrm{H}, \mathrm{E})$ the subspace of smooth (i.e. locally constant) E-valued characters on H. Let $\mathfrak{h}$ be the Lie algebra of H (over K). For $\chi \in \mathrm{Hom}(\mathrm{H}, \mathrm{E})$, by derivation, it induces a $\mathbf{Q}_p$-*linear* map $\mathfrak{h} \to \mathrm{E}$ hence an E-linear map $d\chi : \mathfrak{h} \otimes_{\mathbf{Q}_p} \mathrm{E} \to \mathrm{E}$. It is clear that $\chi \in \mathrm{Hom}_{\mathrm{sm}}(\mathrm{H}, \mathrm{E})$ if and only if $d\chi = 0$. For $\sigma \in \Sigma_{\mathrm{K}}$, we call χ locally σ-analytic if $d\chi$ factors through $\mathfrak{h} \otimes_{\mathrm{K},\sigma} \mathrm{E} \to \mathrm{E}$. Set $\mathrm{Hom}_\sigma(\mathrm{H}, \mathrm{E}) \subset \mathrm{Hom}(\mathrm{H}, \mathrm{E})$ to be the subspace of locally σ-analytic characters. Note we have $\dim_{\mathrm{E}} \mathrm{Hom}_{\mathrm{sm}}(\mathrm{K}^\times, \mathrm{E}) = 1$, $\dim_{\mathrm{E}} \mathrm{Hom}_\sigma(\mathrm{K}^\times, \mathrm{E}) = 2$ and $\dim_{\mathrm{E}} \mathrm{Hom}(\mathrm{K}^\times, \mathrm{E}) = 1 + d_{\mathrm{K}}$ (e.g. see [30, Section 1.3.1]).

Let $\mathcal{R}_{\mathrm{K,E}}$ be the E-coefficient Robba ring for K. For a continuous character $\chi : \mathrm{K}^\times \to \mathrm{E}^\times$, denote by $\mathcal{R}_{\mathrm{K,E}}(\chi)$ the associated rank one (φ, Γ)-module over $\mathcal{R}_{\mathrm{K,E}}$ (see for example [44, Section 6.2]). Note $\mathcal{R}_{\mathrm{K,E}}(\chi)$ is de Rham if and only if χ is locally algebraic. We write Ext^i (and $\mathrm{Hom} = \mathrm{Ext}^0$) without "$(\varphi, \Gamma)$" in the subscript for the i-th extension group of (φ, Γ)-modules (cf. [47]). For de Rham (φ, Γ)-modules M and N, denote by $\mathrm{Ext}^1_g(\mathrm{M}, \mathrm{N}) \subset \mathrm{Ext}^1(\mathrm{M}, \mathrm{N})$ the subspace of de Rham extensions. For a (φ, Γ)-module M, we identify *elements* in $\mathrm{Ext}^1(\mathrm{M}, \mathrm{M})$ with deformations of M over $\mathcal{R}_{\mathrm{K,E}[\epsilon]/\epsilon^2}$. Indeed, the $\mathrm{E}[\epsilon]/\epsilon^2$-structure on $\widetilde{\mathrm{M}} \in \mathrm{Ext}^1(\mathrm{M}, \mathrm{M})$ is given by letting ϵ act via $\epsilon : \widetilde{\mathrm{M}} \twoheadrightarrow \mathrm{M} \xrightarrow{\mathrm{id}} \mathrm{M} \hookrightarrow \widetilde{\mathrm{M}}$.

We denote by $\mathrm{W}^+_{\mathrm{dR}}(\mathrm{M})$ the (semi-linear) $\mathrm{Gal}_{\mathrm{K}}$-representation over $\mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} \mathrm{E}$ associated to M (cf. [7, Prop. 2.2.6 (2)]). There is a natural decomposition $\mathrm{W}^+_{\mathrm{dR}}(\mathrm{M}) \cong \oplus_{\sigma\in\Sigma_{\mathrm{K}}} \mathrm{W}^+_{\mathrm{dR},\sigma}(\mathrm{M})$ with respect to $\mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} \mathrm{E} \cong \oplus_{\sigma\in\Sigma_{\mathrm{K}}} \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathrm{K},\sigma} \mathrm{E}$. Denote by $\mathrm{D}^+_{\mathrm{dR}}(\mathrm{M}) := \mathrm{W}^+_{\mathrm{dR}}(\mathrm{M})^{\mathrm{Gal}_{\mathrm{K}}} \cong \oplus_{\sigma\in\Sigma_{\mathrm{K}}} \mathrm{W}^+_{\mathrm{dR},\sigma}(\mathrm{M})^{\mathrm{Gal}_{\mathrm{K}}} =: \oplus_{\sigma\in\Sigma_{\mathrm{K}}} \mathrm{D}^+_{\mathrm{dR}}(\mathrm{M})_\sigma$. We will frequently use the following lemma.

Lemma **2.1.** — *Let* M *be a* (φ,Γ)*-module over* $\mathcal{R}_{\mathrm{K,E}}$*,* N *be a* (φ,Γ)*-submodule of* M *such that* $\mathrm{rank}_{\mathcal{R}_{\mathrm{K,E}}}(\mathrm{N})=\mathrm{rank}_{\mathcal{R}_{\mathrm{K,E}}}(\mathrm{M})$*. Then there is a natural isomorphism of* E*-vector spaces:* $\mathrm{H}^0_{(\varphi,\Gamma)}(\mathrm{M}/\mathrm{N})\xrightarrow{\sim}\mathrm{H}^0(\mathrm{Gal}_{\mathrm{K}},\mathrm{W}^+_{\mathrm{dR}}(\mathrm{M})/\mathrm{W}^+_{\mathrm{dR}}(\mathrm{N}))$*. Moreover, when* M *is de Rham, this isomorphism identifies* $\mathrm{H}^0_{(\varphi,\Gamma)}(\mathrm{M}/\mathrm{N})$ *with* $\mathrm{D}^+_{\mathrm{dR}}(\mathrm{M})/\mathrm{D}^+_{\mathrm{dR}}(\mathrm{N})$.

Proof. — The first part follows from a straightforward generalization of the proof of [15, Lem. 5.1] to finite extensions K of $\mathbf{Q}_p$. For the second part, applying $(-)^{\mathrm{Gal}_{\mathrm{K}}}$ to the exact sequence of $\mathrm{B}^+_{\mathrm{dR}}$-representations $0\to\mathrm{W}^+_{\mathrm{dR}}(\mathrm{N})\to\mathrm{W}^+_{\mathrm{dR}}(\mathrm{M})\to\mathrm{W}^+_{\mathrm{dR}}(\mathrm{M})/\mathrm{W}^+_{\mathrm{dR}}(\mathrm{N})\to 0$, it suffices to show the natural map $\mathrm{H}^1(\mathrm{Gal}_{\mathrm{K}},\mathrm{W}^+_{\mathrm{dR}}(\mathrm{N}))\to\mathrm{H}^1(\mathrm{Gal}_{\mathrm{K}},\mathrm{W}^+_{\mathrm{dR}}(\mathrm{M}))$ is injective. But this follows from [50, Lem. 2.6]. □

Let M be a crystabelline (φ,Γ)-module of rank d over $\mathcal{R}_{\mathrm{K,E}}$. We can associate to M a filtered Deligne-Fontaine module $(\mathrm{D}_{\mathrm{pst}}(\mathrm{M}),\mathrm{D}_{\mathrm{dR}}(\mathrm{M}))$ such that

- $\mathrm{D}_{\mathrm{pst}}(\mathrm{M})=(\mathrm{W}_e(\mathrm{M})\otimes_{\mathrm{B}_e}\mathrm{B}_{\mathrm{cris}})^{\mathrm{Gal}_{\mathrm{K}'}}$ which is free of rank d over $\mathrm{K}'_0\otimes_{\mathbf{Q}_p}\mathrm{E}$ equipped with a commuting K'_0-semi-linear action of φ and $\mathrm{Gal}(\mathrm{K}'/\mathrm{K})$, K' is an *abelian* extension of K, and K'_0 is the maximal unramified extension of K' (over $\mathbf{Q}_p$), and where $\mathrm{W}_e(\mathrm{M})$ is the $\mathrm{B}_e=\mathrm{B}_{\mathrm{cris}}^{\varphi=1}$-representation associated to M ([7, Prop. 2.2.6 (1)]),
- $\mathrm{D}_{\mathrm{dR}}(\mathrm{M}):=(\mathrm{W}^+_{\mathrm{dR}}(\mathrm{M})[1/t])^{\mathrm{Gal}_{\mathrm{K}}}\cong(\mathrm{D}_{\mathrm{pst}}(\mathrm{M})\otimes_{\mathrm{K}'_0}\mathrm{K}')^{\mathrm{Gal}(\mathrm{K}'/\mathrm{K})}$ is free of rank d over $\mathrm{K}\otimes_{\mathbf{Q}_p}\mathrm{E}$, equipped with a Hodge filtration $\mathrm{Fil}_{\mathrm{H}}$ of $\mathrm{K}\otimes_{\mathbf{Q}_p}\mathrm{E}$-submodules (not necessarily free).

By [19, Prop. 4.1], to $\mathrm{D}_{\mathrm{pst}}(\mathrm{M})$, one can associate a Weil-Deligne representation $\mathtt{r}(\mathrm{M})$ over E. We call M *generic* if $\mathtt{r}(\mathrm{M})$ generic, which means $\mathtt{r}(\mathrm{M})$ is semi-simple and isomorphic to $\oplus_{i=1}^d\phi_i$ with $\phi_i\phi_j^{-1}\neq 1,|\cdot|_{\mathrm{K}}$ for $i\neq j$. In fact, M being generic crystabelline is equivalent to the existence of smooth characters ϕ_i for $i=1,\dots,d$ such that $\mathrm{M}[1/t]\cong\oplus_{i=1}^d\mathcal{R}_{\mathrm{K,E}}(\phi_i)[1/t]$, and $\phi_i\phi_j^{-1}\neq 1,|\cdot|_{\mathrm{K}}$ for $i\neq j$. An ordering of $(\phi_1,\dots,\phi_d)$ is refereed to as a *refinement* of M. Indeed, an ordering $w(\phi)=(\phi_{w^{-1}(1)},\dots,\phi_{w^{-1}(d)})$ for $w\in\mathrm{S}_d$, corresponds uniquely to a filtration $\mathscr{T}_w=\{\mathscr{T}_w^i\}$, increasing with i, of saturated (φ,Γ)-submodules on M such that $(\mathrm{gr}^i_{\mathscr{T}_w}\mathrm{M})[1/t]\cong\mathcal{R}_{\mathrm{K,E}}(\phi_{w^{-1}(i)})[1/t]$. We frequently view $w(\phi)$ as a (smooth) character of $\mathrm{T}(\mathrm{K})$ (the torus subgroup of $\mathrm{GL}_d(\mathrm{K})$) for any $w\in\mathrm{S}_d$. We also call these characters of $\mathrm{T}(\mathrm{K})$ refinements of M.

Let $\mathbf{h}:=(\mathbf{h}_i)_{i=1,\dots,d}=(\mathbf{h}_\sigma)_{\sigma\in\Sigma_{\mathrm{K}}}=(h_{\sigma,1}\geq\cdots\geq h_{\sigma,d})_{\sigma\in\Sigma_{\mathrm{K}}}$ be the Hodge-Tate-Sen weights of M (normalized such that the weight of the cyclotomic character is 1). Let $w\in\mathrm{S}_d$, we call the refinement $w(\phi)$ (or $\mathscr{T}_w$) *non-critical* if the Hodge-Tate-Sen weights of $\mathrm{gr}^i_{\mathscr{T}_w}\mathrm{M}$ are exactly $\mathbf{h}_i$ (which are hence decreasing with growth of i). We call M non-critical, if all the refinements of M are non-critical. We denote by $\Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$ the set of isomorphism classes of non-critical crystabelline (φ,Γ)-modules of refinement ϕ and of Hodge-Tate-Sen weights $\mathbf{h}$. Finally, we say M has *regular* Hodge-Tate-Sen weights if $\mathbf{h}$ is strictly dominant, i.e. $h_{i,\sigma}>h_{i+1,\sigma}$ for all $\sigma\in\Sigma_{\mathrm{K}}$.

Suppose M is generic crystabelline with refinement ϕ. For a subset $\mathbf{r}=\{r_1,\ldots,r_k\}\subset\{1,\ldots,d\}$, denote by $\mathrm{M}_{\mathbf{r}}$ (resp. $\mathrm{M}^{\mathbf{r}}$) the saturated (φ,Γ)-submodule of M (resp. the quotient of M) which has a refinement given by $(\phi_{r_1},\ldots,\phi_{r_k})$. So $\mathrm{M}^{\mathbf{r}}=\mathrm{M}/\mathrm{M}_{\mathbf{r}^c}$ with $\mathbf{r}^c=\{1,\ldots,d\}\setminus\mathbf{r}$. While $\mathrm{M}_{\mathbf{r}}$ and $\mathrm{M}^{\mathbf{r}}$ depend on the chosen refinement, this will not cause ambiguity: in all instances where they appear, the context will specify the refinement in use. Assuming M is non-critical, $\mathrm{M}_{\mathbf{r}}$ and $\mathrm{M}^{\mathbf{r}}$ are non-critical as well for any $\mathbf{r}$ (noting any triangulation of $\mathrm{M}_{\mathbf{r}}$ or of $\mathrm{M}^{\mathbf{r}}$ extends to a triangulation of M). In this case, the Hodge-Tate-Sen weights of $\mathrm{M}_{\mathbf{r}}$ (resp. $\mathrm{M}^{\mathbf{r}}$) are $(\mathbf{h}_1,\ldots,\mathbf{h}_k)$ (resp. $(\mathbf{h}_{d-k+1},\ldots,\mathbf{h}_d)$).

Throughout the paper, we will use $\bullet\!-\!\!\!-\!\bullet$ to denote an extension of two objects (such as (φ,Γ)-modules, or $\mathrm{GL}_n(\mathrm{K})$-representations etc.), where the left object is the sub and the right the quotient.

2.2. *A reinterpretation of Hodge parameters.* — In this section, we give a reinterpretation of (some) p-adic Hodge parameters of a generic non-critical crystabelline (φ,Γ)-module.

Let $\phi=(\phi_i)_{i=1,\ldots,n}$ be generic, and $\mathbf{h}=(\mathbf{h}_\sigma)_{\sigma\in\Sigma_{\mathrm{K}}}=(\mathbf{h}_i)_{i=1,\ldots,n}=(h_{\sigma,1}>h_{\sigma,2}>\cdots>h_{\sigma,n})$. Let $\mathrm{D}\in\Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$. Let $\mathrm{D}_1:=\mathrm{D}_{\{1,\ldots,n-1\}}$ and $\mathrm{C}_1:=\mathrm{D}^{\{1,\ldots,n-1\}}$, we have two exact sequences:

$$0\longrightarrow \mathrm{D}_1\longrightarrow \mathrm{D}\longrightarrow \mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_n z^{\mathbf{h}_n}\big)\longrightarrow 0, \tag{2.1}$$

$$0\longrightarrow \mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_n z^{\mathbf{h}_1}\big)\longrightarrow \mathrm{D}\longrightarrow \mathrm{C}_1\longrightarrow 0. \tag{2.2}$$

Let ι_{D} be the composition $\mathrm{D}_1\hookrightarrow\mathrm{D}\twoheadrightarrow\mathrm{C}_1$. As $\mathrm{Hom}(\mathrm{D}_1,\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_1}))=0$, ι_{D} is injective.

Proposition **2.2.** — *(1) We have* $\dim_{\mathrm{E}}\mathrm{Hom}(\mathrm{D}_1,\mathrm{C}_1)\leq 2$.

(2) We have $\dim_{\mathrm{E}}\mathrm{Hom}(\mathrm{D}_1,\mathrm{C}_1)=2$ *if and only if* $n\geq 3$, *and for any* $i\in\{1,\ldots,n-1\}$, $\boldsymbol{r}:=\{1,\ldots,n-1\}\setminus\{i\}$, *we have* $(\mathrm{D}_1)^{\boldsymbol{r}}\cong(\mathrm{C}_1)_{\boldsymbol{r}}$ *(for the refinement* $(\phi_1,\ldots,\phi_{n-1})$*). Moreover, if these hold, for* $i\in\{1,\ldots,n-1\}$, *the composition*

$$\alpha_i:\mathrm{D}_1\relbar\!\!\twoheadrightarrow(\mathrm{D}_1)^{\boldsymbol{r}}\cong(\mathrm{C}_1)_{\boldsymbol{r}}\lhook\joinrel\longrightarrow\mathrm{C}_1. \tag{2.3}$$

are pair-wisely linearly independent as elements in $\mathrm{Hom}(\mathrm{D}_1,\mathrm{C}_1)$.

Proof. — (1) If $n=2$, it is clear that $\dim_{\mathrm{E}}\mathrm{Hom}(\mathrm{D}_1,\mathrm{C}_1)=1$. Assume $n\geq 3$, and let $\mathbf{r}=\{1,\ldots,n-3\}$, and consider $(\mathrm{C}_1)_{\mathbf{r}}$, which is the saturated submodule of C_1 of rank $n-3$ over $\mathcal{R}_{\mathrm{K},\mathrm{E}}$ with a refinement $(\phi_1,\ldots,\phi_{n-3})$. As C_1 is non-critical of Hodge-Tate weights $(\mathbf{h}_2,\ldots\mathbf{h}_n)$, $(\mathrm{C}_1)_{\mathbf{r}}$ is non-critical of Hodge-Tate weights $(\mathbf{h}_2,\ldots,\mathbf{h}_{n-2})$. Thus $(\mathrm{C}_1)_{\mathbf{r}}$ is isomorphic to a (non-split) successive extension of $\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_i z^{\mathbf{h}_{i+1}})$ for $i=1,\ldots,n-3$. Consider

$$0\longrightarrow\mathrm{Hom}\big(\mathrm{D}_1,(\mathrm{C}_1)_{\mathbf{r}}\big)\longrightarrow\mathrm{Hom}(\mathrm{D}_1,\mathrm{C}_1)\longrightarrow\mathrm{Hom}\big(\mathrm{D}_1,\mathrm{C}_1/(\mathrm{C}_1)_{\mathbf{r}}\big).$$

Any map in $\mathrm{Hom}(\mathrm{D}_1, (\mathrm{C}_1)_{\mathbf{r}})$ clearly factors through $(\mathrm{D}_1)^{\mathbf{r}}$, the latter being isomorphic to a (non-split) successive extension of $\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_i z^{\mathbf{h}_{i+2}})$ for $i = 1, \dots, n-3$. By an easy dévissage, using $h_{\sigma,i+2} < h_{\sigma,i+1}$ and the fact

$$\mathrm{Hom}\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_1' z^{k_1}\big), \mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_2' z^{k_2}\big)\big) = 0 \text{ if } \phi_1' \neq \phi_2' \text{ or } k_1 < k_2, \tag{2.4}$$

we deduce $\mathrm{Hom}((\mathrm{D}_1)^{\mathbf{r}}, (\mathrm{C}_1)_{\mathbf{r}}) = 0$ hence $\mathrm{Hom}(\mathrm{D}_1, (\mathrm{C}_1)_{\mathbf{r}}) = 0$. Again by an easy dévissage, we have $\dim_{\mathrm{E}} \mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1/(\mathrm{C}_1)_{\mathbf{r}}) = \dim_{\mathrm{E}} \mathrm{Hom}(\mathrm{D}_1, (\mathrm{C}_1)^{\{n-2,n-3\}}) \leq 2$. Hence $\dim_{\mathrm{E}} \mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1) \leq 2$.

(2) We first prove "if". As $(\mathrm{D}_1)^{\mathbf{r}} \cong (\mathrm{C}_1)_{\mathbf{r}}$, it is clear that α_i are well defined (as in (2.3)) and pair-wisely linearly independent. Together with (1), we deduce $\dim_{\mathrm{E}} \mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1) = 2$. Conversely, assume $\mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1) = 2$, and let ι_1, ι_2 be a basis of $\mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1)$. Let i, $\mathbf{r}$ be as in (2). Consider the induced map $f_i : \mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_i z^{\mathbf{h}_1}) \hookrightarrow \mathrm{D}_1 \xrightarrow{\iota_i} \mathrm{C}_1$. As $\dim_{\mathrm{E}} \mathrm{Hom}(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_i z^{\mathbf{h}_1}), \mathrm{C}_1) = 1$, there exists a non-zero linear combination $\iota = a_1\iota_1 + a_2\iota_2$ such that $a_1 f_1 + a_2 f_2 = 0$. So (the non-zero) ι factors through a non-zero map $(\mathrm{D}_1)^{\mathbf{r}} \to \mathrm{C}_1$. As both $(\mathrm{D}_1)^{\mathbf{r}}$ and C_1 are non-critical, by comparing the weights and using (2.4), we deduce the map has to factor through an isomorphism $(\mathrm{D}_1)^{\mathbf{r}} \xrightarrow{\sim} (\mathrm{C}_1)_{\mathbf{r}}$. □

Remark **2.3.** — By Proposition 2.2 (2), $\mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1)$ is always two dimensional when $n = 3$, or $n = 4$ and $\mathrm{K} = \mathbf{Q}_p$. In general, its dimension may be one or two depending on the specific D_1 and C_1.

Consider the cup-product

$$\mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_n z^{\mathbf{h}_n}\big), \mathrm{D}_1\big) \times \mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1) \longrightarrow \mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_n z^{\mathbf{h}_n}\big), \mathrm{C}_1\big). \tag{2.5}$$

Proposition **2.4.** — *Under the cup-product,* $\mathrm{E}[\mathrm{D}] \subset [\iota_{\mathrm{D}}]^{\perp}$*, and we have an equality if* $\mathrm{K} = \mathbf{Q}_p$*. In particular, when* $\mathrm{K} = \mathbf{Q}_p$*,* D *is determined by* D_1*,* C_1*,* ϕ_n *and* ι_{D}*.*

The last statement in Proposition 2.4 can be formulated precisely as follows: for a crystabelline (φ, Γ)-module D' of rank n over $\mathcal{R}_{\mathbf{Q}_p,\mathrm{E}}$, suppose D_1 is a saturated submodule of D', C_1 is a quotient of D' and $(\phi_1, \dots, \phi_n)$ is a refinement of D' (noting $(\phi_1, \dots, \phi_{n-1})$ is already determined by D_1). If the composition $\iota_{\mathrm{D}'} : \mathrm{D}_1 \hookrightarrow \mathrm{D}' \twoheadrightarrow \mathrm{C}_1$ is equal to ι_{D} up to a non-zero scalar, then $\mathrm{D}' \cong \mathrm{D}$.

Proof. — As ι_{D} factors through D, the map induced by the pairing $\langle -, \iota_{\mathrm{D}} \rangle$ (in (2.5)) is equal to the following composition

$$\mathrm{Ext}^1\big(\mathcal{R}\big(\phi_n z^{\mathbf{h}_n}\big), \mathrm{D}_1\big) \longrightarrow \mathrm{Ext}^1\big(\mathcal{R}\big(\phi_n z^{\mathbf{h}_n}\big), \mathrm{D}\big) \longrightarrow \mathrm{Ext}^1\big(\mathcal{R}\big(\phi_n z^{\mathbf{h}_n}\big), \mathrm{C}_1\big). \tag{2.6}$$

The first map sends $[\mathrm{D}]$ to zero, hence $\langle \mathrm{D}, \iota_{\mathrm{D}} \rangle = 0$. In fact, by dévissage, the kernel of the composition is isomorphic to $\mathrm{Hom}(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_n}), \mathrm{C}_1/\mathrm{D}_1)$, which, by Lemma 2.1, is furthermore isomorphic to $\mathrm{D}^+_{\mathrm{dR}}(\mathrm{C}'_1)/\mathrm{D}^+_{\mathrm{dR}}(\mathrm{D}'_1)$, where $\mathrm{C}'_1 = \mathrm{C}_1 \otimes_{\mathcal{R}_{\mathrm{K,E}}} \mathcal{R}_{\mathrm{K,E}}(\phi_n^{-1} z^{-\mathbf{h}_n})$ and $\mathrm{D}'_1 = \mathrm{D}_1 \otimes_{\mathcal{R}_{\mathrm{K,E}}} \mathcal{R}_{\mathrm{K,E}}(\phi_n^{-1} z^{-\mathbf{h}_n})$. As C'_1 (resp. D'_1) has Hodge-Tate-Sen weights $\{\mathbf{h}_i - \mathbf{h}_n\}_{i=2,\dots,n}$ (resp. $\{\mathbf{h}_i - \mathbf{h}_n\}_{i=1,\dots,n-1}$), we have $\dim_{\mathrm{E}} \mathrm{D}^+_{\mathrm{dR}}(\mathrm{C}'_1)/\mathrm{D}^+_{\mathrm{dR}}(\mathrm{D}'_1) = d_{\mathrm{K}}$. In particular, when $\mathrm{K} = \mathbf{Q}_p$, the kernel of (2.6) is exactly generated by [D]. This finishes the proof. □

In the rest of the section, we discuss what information of D can be detected by ι_{D} for general K. The reader who is mainly interested in the $\mathbf{Q}_p$-case can skip to the next section. Fix $\sigma \in \Sigma_{\mathrm{K}}$, and define $\mathfrak{T}_\sigma(\mathbf{h})$ to be the weight such that $\mathfrak{T}_\sigma(\mathbf{h})_{\tau,i} = \begin{cases} h_{\tau,i} & \tau = \sigma \\ h_{\tau,n} & \tau \neq \sigma \end{cases}$ which is in particular constant for $\tau \neq \sigma$. The following proposition is a direct consequence of [6, Thm. A]. We include a proof (of (1)) using similar arguments as in [34, Lem. 2.1].

Proposition **2.5.** — *(1) Let* $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\phi, \boldsymbol{h})$*, and let* $\sigma \in \Sigma_{\mathrm{K}}$*. There exists a unique* (φ, Γ)*-module (up to isomorphism)* D_σ *over* $\mathcal{R}_{\mathrm{K,E}}$ *such that* $\mathrm{D}_\sigma[1/t] \cong \mathrm{D}[1/t]$, $\mathrm{D} \subset \mathrm{D}_\sigma$*, and the Hodge-Tate weights of* D_σ *are* $\mathfrak{T}_\sigma(\boldsymbol{h})$*.*

(2) Let $\mathrm{D}, \mathrm{D}_\sigma$ *be as in (1). The injection* $\mathrm{D} \hookrightarrow \mathrm{D}_\sigma$ *induces a natural isomorphism of Deligne-Fontaine modules* $\mathrm{D}_{\mathrm{pst}}(\mathrm{D}) \xrightarrow{\sim} \mathrm{D}_{\mathrm{pst}}(\mathrm{D}_\sigma)$*, such that the induced map* $\mathrm{D}_{\mathrm{dR}}(\mathrm{D}) \to \mathrm{D}_{\mathrm{dR}}(\mathrm{D}_\sigma)$ *is a morphism of filtered* $\mathrm{K} \otimes_{\mathbf{Q}_p} \mathrm{E}$*-modules, satisfying* $\mathrm{D}_{\mathrm{dR}}(\mathrm{D})_\sigma \xrightarrow{\sim} \mathrm{D}_{\mathrm{dR}}(\mathrm{D}_\sigma)$ *(as filtered* E*-vector space).*

Proof. — Let $(\mathrm{W}_e(\mathrm{D}), \mathrm{W}^+_{\mathrm{dR}}(\mathrm{D}))$ be the B-pair associated to D (cf. [7, Thm. A]). By Fontaine's classification of B_{dR}-representations [40, Thm. 3.19], there is a unique $\mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} \mathrm{E}$-representation $\Lambda \cong \oplus_{\tau \in \Sigma_{\mathrm{K}}} \Lambda_\tau$ such that $\mathrm{W}^+_{\mathrm{dR}}(\mathrm{D}) \subset \Lambda \subset \mathrm{W}^+_{\mathrm{dR}}(\mathrm{D})[\frac{1}{t}]$ and $\Lambda_\tau \cong \begin{cases} \mathrm{W}^+_{\mathrm{dR}}(\mathrm{D})_\tau & \tau = \sigma \\ (t^{h_{\tau,n}} \mathrm{B}^+_{\mathrm{dR}} \otimes_{\mathrm{K},\tau} \mathrm{E})^{\oplus n} & \tau \neq \sigma \end{cases}$. Let D_σ be the (φ, Γ)-module associated to the B-pair $(\mathrm{W}_e(\mathrm{D}), \Lambda)$. This construction satisfies all the claimed properties in (1). (2) follows from (1) and [6, Thm. A]. □

Lemma **2.6.** — *Let* $\mathrm{D}, \mathrm{D}_\sigma$ *be as in Proposition 2.5. For each* $w \in \mathrm{S}_n$*,* $w(\phi) z^{\mathfrak{T}_\sigma(\boldsymbol{h})}$ *is a trianguline parameter of* D_σ*.*

Proof. — Consider the composition $\mathcal{R}_{\mathrm{K,E}}(\phi_{w^{-1}(1)} z^{\mathbf{h}_1}) \hookrightarrow \mathrm{D} \hookrightarrow \mathrm{D}_\sigma$. It is not difficult to see the saturation of the image in D_σ is just $\mathcal{R}_{\mathrm{K,E}}(\phi_{w^{-1}(1)} z^{\mathfrak{T}_\sigma(\mathbf{h})_1})$, and we have $\mathrm{D}/\mathcal{R}_{\mathrm{K,E}}(\phi_{w^{-1}(1)} z^{\mathbf{h}_1}) \hookrightarrow \mathrm{D}_\sigma/\mathcal{R}_{\mathrm{K,E}}(\phi_{w^{-1}(1)} z^{\mathfrak{T}_\sigma(\mathbf{h})_1})$. Continuing with the argument, the lemma follows. □

We have hence a (surjective) map

$$\mathfrak{T}_\sigma : \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h}) \longrightarrow \Phi\Gamma_{\mathrm{nc}}\big(\phi, \mathfrak{T}_\sigma(\mathbf{h})\big), \quad \mathrm{D} \mapsto \mathrm{D}_\sigma. \tag{2.7}$$

Let $\mathrm{D}_{1,\sigma} := (\mathrm{D}_\sigma)_{\{1,\dots,n-1\}}$ and $\mathrm{C}_{1,\sigma} := (\mathrm{D}_\sigma)^{\{1,\dots,n-1\}}$ (for the refinement ϕ). By Lemma 2.6, it is not difficult to see $\mathrm{D}_{1,\sigma}$ (resp. $\mathrm{C}_{1,\sigma}$) has Hodge-Tate-Sen weights $(\mathfrak{T}_\sigma(\mathbf{h})_1, \dots,$

$\mathfrak{T}_\sigma(\mathbf{h})_{n-1}$) (resp. $(\mathfrak{T}_\sigma(\mathbf{h})_2, \dots, \mathfrak{T}_\sigma(\mathbf{h})_n)$). In fact, we have $\mathrm{D}_{1,\sigma} = \mathfrak{T}_\sigma(\mathrm{D}_1)$ and $\mathrm{C}_{1,\sigma} = \mathfrak{T}_\sigma(\mathrm{C}_1)$ (where $\mathfrak{T}_\sigma$ is defined in a similar way as (2.7)). Consider $\mathrm{Hom}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})$. Note it is non-zero as it contains the composition $\iota_{\mathrm{D}_\sigma} : \mathrm{D}_{1,\sigma} \hookrightarrow \mathrm{D}_\sigma \twoheadrightarrow \mathrm{C}_{1,\sigma}$. By similar arguments as in Proposition 2.2, we have:

Proposition **2.7.** — *(1)* $\dim_{\mathrm{E}} \mathrm{Hom}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma}) \leq 2$.

(2) We have $\dim_{\mathrm{E}} \mathrm{Hom}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma}) = 2$ *if and only if* $n \geq 3$, *and for any* $i \in \{1, \dots, n-1\}$, $\boldsymbol{r} := \{1, \dots, n-1\} \setminus \{i\}$, *we have* $(\mathrm{D}_{1,\sigma})^{\boldsymbol{r}} \cong (\mathrm{C}_{1,\sigma})_{\boldsymbol{r}}$ *(for the refinement* $(\phi_1, \dots, \phi_{n-1})$*). Moreover, if these hold, for* $i \in \{1, \dots, n-1\}$, *the composition*

$$\alpha_{i,\sigma} : \mathrm{D}_{1,\sigma} \twoheadrightarrow (\mathrm{D}_{1,\sigma})^{\boldsymbol{r}} \cong (\mathrm{C}_{1,\sigma})_{\boldsymbol{r}} \hookrightarrow \mathrm{C}_{1,\sigma}. \tag{2.8}$$

are pair-wisely linearly independent as elements in $\mathrm{Hom}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})$.

Proposition **2.8.** — *For the cup-product*

$$\mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K,E}}(\phi_n z^{\boldsymbol{h}_n}), \mathrm{D}_{1,\sigma}\big) \times \mathrm{Hom}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma}) \to \mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K,E}}(\phi_n z^{\boldsymbol{h}_n}), \mathrm{C}_{1,\sigma}\big),$$

we have $[\iota_{\mathrm{D}_\sigma}]^\perp = \mathrm{E}[\mathrm{D}_\sigma]$. *In particular,* D_σ *is determined by* $\mathrm{D}_{1,\sigma}$, $\mathrm{C}_{1,\sigma}$, ϕ_n *and* $\iota_{\mathrm{D}_\sigma}$ *in a similar sense to that discussed following Proposition 2.4.*

Proof. — Taking the cup-product with $\iota_{\mathrm{D}_\sigma}$ is equal to the following composition

$$\mathrm{Ext}^1\big(\mathcal{R}(\phi_n z^{\mathbf{h}_n}), \mathrm{D}_{1,\sigma}\big) \to \mathrm{Ext}^1\big(\mathcal{R}(\phi_n z^{\mathbf{h}_n}), \mathrm{D}_\sigma\big) \to \mathrm{Ext}^1\big(\mathcal{R}(\phi_n z^{\mathbf{h}_n}), \mathrm{C}_{1,\sigma}\big), \tag{2.9}$$

which is the push-forward map via $\iota_{\mathrm{D}_\sigma}$. We see $\langle \mathrm{D}_\sigma, \iota_{\mathrm{D}_\sigma}\rangle = 0$. On the other hand, by dévissage and Lemma 2.1, Ker(2.9) is isomorphic to

$$\mathrm{D}_{\mathrm{dR}}^+\big(\mathrm{C}_{1,\sigma} \otimes_{\mathcal{R}_{\mathrm{K,E}}} \mathcal{R}_{\mathrm{K,E}}(\phi_n^{-1} z^{-\mathbf{h}_n})\big)/\mathrm{D}_{\mathrm{dR}}^+\big(\mathrm{D}_{1,\sigma} \otimes_{\mathcal{R}_{\mathrm{K,E}}} \mathcal{R}_{\mathrm{K,E}}(\phi_n^{-1} z^{-\mathbf{h}_n})\big). \tag{2.10}$$

By comparing the Hodge-Tate-Sen weights (and noting the weights of $\mathrm{D}_{1,\sigma}$ and $\mathrm{C}_{1,\sigma}$ for embeddings different from σ are the same), we easily see that (2.10) is one dimensional. Hence Ker(2.9) is generated by $[\mathrm{D}_\sigma]$. □

Example **2.9.** — We give an example to illustrate how $\iota_{\mathrm{D}_\sigma}$ determines D_σ (or equivalently the Hodge σ-filtration of D). Suppose $n = 3$, K unramified and D is crystalline (generic non-critical) of regular Hodge-Tate-Sen weights $\mathbf{h}$. In this case we have $\mathrm{D}_{\mathrm{cris}}(\mathrm{D}) \cong \mathrm{D}_{\mathrm{dR}}(\mathrm{D}) \cong \oplus_{\tau \in \Sigma_{\mathrm{K}}} \mathrm{D}_{\mathrm{cris}}(\mathrm{D})_\tau$, where each $\mathrm{D}_{\mathrm{cris}}(\mathrm{D})_\tau$ is a filtered $\varphi^{d_{\mathrm{K}}}$-module. Fix $\sigma \in \Sigma_{\mathrm{K}}$. Note that we have an isomorphism of filtered $\varphi^{d_{\mathrm{K}}}$-module $\mathrm{D}_{\mathrm{cris}}(\mathrm{D}_\sigma)_\sigma \cong \mathrm{D}_{\mathrm{cris}}(\mathrm{D})_\sigma$.

Let $\alpha_1, \alpha_2, \alpha_3$ be the three distinct eigenvalues of $\varphi^{d_{\mathrm{K}}}$ on $\mathrm{D}_{\mathrm{cris}}(\mathrm{D}_\sigma)_\tau$ (for any τ). Let $e_{i,\sigma}$ be an α_i-eigenvector in $\mathrm{D}_{\mathrm{cris}}(\mathrm{D}_\sigma)_\sigma$, hence $\mathrm{D}_{\mathrm{cris}}(\mathrm{D}_\sigma)_\sigma \cong \mathrm{E}e_{1,\sigma} \oplus \mathrm{E}e_{2,\sigma} \oplus \mathrm{E}e_{3,\sigma}$. For $j = 0, \dots, d_{\mathrm{K}} - 1$, we have $\mathrm{D}_{\mathrm{cris}}(\mathrm{D}_\sigma)_{\sigma \circ \mathrm{Frob}^{-j}} \cong \mathrm{E}\varphi^j(e_{1,\sigma}) \oplus \mathrm{E}\varphi^j(e_{2,\sigma}) \oplus \mathrm{E}\varphi^j(e_{3,\sigma})$ (where Frob denotes the absolute Frobenius), and $\mathrm{D}_{\mathrm{cris}}(\mathrm{D}_{1,\sigma})_{\sigma \circ \mathrm{Frob}^{-j}} \cong \mathrm{E}\varphi^j(e_{1,\sigma}) \oplus \mathrm{E}\varphi^j(e_{2,\sigma})$ for $j =$

$0, \dots, d_K - 1$, which is equipped with the induced Hodge filtration. As $D_{1,\sigma}$ is non-critical, multiplying $e_{1,\sigma}$, $e_{2,\sigma}$ by non-zero scalars, we can and do assume $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D_{1,\sigma})_\sigma = \mathrm{Fil}^j D_{\mathrm{cris}}(D_{1,\sigma})_\sigma$, $-h_{1,\sigma} < j \leq -h_{2,\sigma}$, is generated by $e_{1,\sigma} + e_{2,\sigma}$. As D_σ is non-critical for all the refinements, multiplying $e_{3,\sigma}$ by a non-zero scalar, we can and do assume $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D_\sigma)_\sigma = \mathrm{Fil}^j D_{\mathrm{cris}}(D_\sigma)_\sigma$, $-h_{2,\sigma} < j \leq -h_{3,\sigma}$, is generated by $e_1 + a_{D_\sigma} e_2 + e_3$. The filtered φ^{d_K}-module $D_{\mathrm{cris}}(D_\sigma)_\sigma$ is in fact parametrized (and determined) by $a_{D_\sigma} \in E \setminus \{0, 1\}$: we have

$$\mathrm{Fil}^j D_{\mathrm{cris}}(D_\sigma)_\sigma = \begin{cases} D_{\mathrm{cris}}(D_\sigma)_\sigma & j \leq -h_{1,\sigma} \\ E(e_{1,\sigma} + e_{2,\sigma}) & \\ \quad \oplus E(e_{1,\sigma} + a_{D_\sigma} e_{2,\sigma} + e_{3,\sigma}) & -h_{1,\sigma} < j \leq -h_{2,\sigma} \\ E(e_{1,\sigma} + a_{D_\sigma} e_{2,\sigma} + e_{3,\sigma}) & -h_{2,\sigma} < j \leq -h_{3,\sigma} \\ 0 & j > -h_{3,\sigma} \end{cases}$$

For $\tau \neq \sigma$, we have $\mathrm{Fil}^j D_{\mathrm{cris}}(D_\sigma)_\tau = \begin{cases} D_{\mathrm{cris}}(D_\sigma)_\tau & j \leq -h_{n,\tau} \\ 0 & j > -h_{n,\tau} \end{cases}$. So D_σ is indeed determined by the single parameter a_{D_σ} (in contrast, D itself has many more parameters, when $K \neq \mathbf{Q}_p$). Note that for $-h_{2,\sigma} < j \leq -h_{3,\sigma}$, $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(C_{1,\sigma})_\sigma = \mathrm{Fil}^j D_{\mathrm{cris}}(C_{1,\sigma})_\sigma$ is generated by $e_{1,\sigma} + a_{D_\sigma} e_{2,\sigma}$ (as it is equipped with the quotient filtration). The map ι_{D_σ} uniquely corresponds to the morphism of filtered φ^{d_K}-modules $\iota_{D_\sigma} : D_{\mathrm{cris}}(D_{1,\sigma})_\sigma \to D_{\mathrm{cris}}(C_{1,\sigma})_\sigma$ sending $e_{i,\sigma}$ to $e_{i,\sigma}$ for $i = 1, 2$. We see a_{D_σ} can be read out from the relative position of the two lines $\mathrm{Fil}^{\max} D_{\mathrm{cris}}(C_{1,\sigma})_\sigma$ and $\iota_{D_\sigma}(\mathrm{Fil}^{\max} D_{\mathrm{cris}}(D_{1,\sigma})_\sigma)$ in $D_{\mathrm{cris}}(C_{1,\sigma})_\sigma$. Thus a_{D_σ} (hence D_σ) is determined by ι_{D_σ}.

2.3. *Deformations of crystabelline* (φ, Γ)*-modules.* — Let $D \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$. In this section, we collect some facts on certain deformations of D.

2.3.1. *Trianguline and paraboline deformations, I.* — We first consider trianguline deformations. For a character $\chi : K^\times \to E^\times$, recall we have natural isomorphisms

$$(\mathbf{2.11}) \qquad \mathrm{Hom}\big(K^\times, E\big) \xrightarrow{\sim} \mathrm{Ext}^1_{K^\times}(\chi, \chi) \xrightarrow{\sim} \mathrm{Ext}^1\big(\mathcal{R}_{K,E}(\chi), \mathcal{R}_{K,E}(\chi)\big),$$

sending ψ to $\chi(1 + \psi\epsilon)$ then to $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\chi(1 + \psi\epsilon))$.

For $w \in S_n$, denote by $\mathrm{Ext}^1_w(D, D) \subset \mathrm{Ext}^1(D, D)$ the subspace of trianguline deformations with respect to the refinement $w(\phi)$. More precisely, for $\widetilde{D} \in \mathrm{Ext}^1(D, D)$ (viewed as a (φ, Γ)-module over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$)), $\widetilde{D} \in \mathrm{Ext}^1_w(D, D)$ if and only if $\widetilde{D}$ is isomorphic to a successive extension of $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}(\phi_{w^{-1}(i)} z^{\mathbf{h}_i}(1 + \psi_i \epsilon))$ for $\psi_i \in \mathrm{Hom}(K^\times, E)$. In this case, we call the character $w(\phi) z^{\mathbf{h}}(1 + \psi\epsilon)$ (with $\psi := (\psi_1, \dots, \psi_n)$) of $T(K)$ over $E[\epsilon]/\epsilon^2$ the *trianguline parameter* of $\widetilde{D}$ with respect to $w(\phi)$. Let κ_w be the following composition:

$$(\mathbf{2.12}) \qquad \kappa_w : \mathrm{Ext}^1_w(D, D) \longrightarrow \mathrm{Ext}^1_{T(K)}\big(w(\phi) z^{\mathbf{h}}, w(\phi) z^{\mathbf{h}}\big) \xrightarrow{\sim} \mathrm{Hom}\big(T(K), E\big),$$

where the first map sends $\widetilde{D}$ to its trianguline parameter with respect to $w(\phi)$, and the second map is induced by (2.11). We also denote $\mathrm{Ext}^1_w(D,D)$ by $\mathrm{Ext}^1_{w(\phi)}(D,D)$ or $\mathrm{Ext}^1_{\mathscr{T}_w}(D,D)$ where $\mathscr{T}_w$ is the filtration on D associated to $w(\phi)$ whenever it is convenient for the context. The following proposition is well-known (cf. [3, Section 2] [51, Section 2]).

Proposition **2.10.** — *(1)* $\dim_E \mathrm{Ext}^1(D,D) = 1 + n^2 d_K$, $\dim_E \mathrm{Ext}^1_g(D,D) = 1 + \frac{n(n-1)}{2} d_K$ *and* $\dim_E \mathrm{Ext}^1_w(D,D) = 1 + \frac{n(n+1)}{2} d_K$ *for all* $w \in S_n$.

(2) For $w \in S_n$, κ_w *is surjective.*

(3) For $w \in S_n$, $\mathrm{Ext}^1_g(D,D) \subset \mathrm{Ext}^1_w(D,D)$ *and is equal to the preimage of the subspace* $\mathrm{Hom}_{\mathrm{sm}}(T(K),E)$ *via* κ_w.

Proof. — The $K = \mathbf{Q}_p$-case is given in [3, Prop. 2.3.10, Thm. 2.5.10]. We sketch a proof for general K. As D is non-critical, $\mathrm{Hom}(D,D) = E$. We also have $\mathrm{Ext}^2(D,D) = 0$ since D is generic. By [47, Thm. 1.2(1)], $\dim_E \mathrm{Ext}^1(D,D) = 1 + n^2 d_K$. By [51, Cor. 2.53] (noting any de Rham deformation of D is automatically potentially crystalline), $\dim_E \mathrm{Ext}^1_g(D,D) = 1 + \frac{n(n-1)}{2} d_K$. By [51, Prop. 2.41] and the proof, $\dim_E \mathrm{Ext}^1_w(D,D) = 1 + \frac{n(n+1)}{2} d_K$ and κ_w is surjective for all $w \in S_n$. Hence $\dim_E \mathrm{Ker}\,\kappa_w = \frac{n(n-1)}{2} d_K + 1 - n$. By [51, Lem. 2.56], $\mathrm{Ext}^1_g(D,D) \subset \mathrm{Ext}^1_w(D,D)$ for all w. It is also clear $\kappa_w(\mathrm{Ext}^1_g(D,D)) \subset \mathrm{Hom}_{\mathrm{sm}}(T(K),E)$. By comparing dimensions: $\dim_E \mathrm{Ext}^1_g(D,D) = \dim_E \mathrm{Hom}_{\mathrm{sm}}(T(K),E) + \dim_E \mathrm{Ker}\,\kappa_w$, (3) follows. □

Recall there is a right action of S_n on $T(K)$: $w(a_1, \dots, a_n) = (a_{w(1)}, \dots, a_{w(n)})$ for $w \in S_n$. It induces a left action of S_n on $\mathrm{Hom}(T(K),E)$: $(w\psi)(a_1,\dots,a_n) = \psi(a_{w(1)},\dots,a_{w(n)})$. It is clear that $\mathrm{Hom}_{\mathrm{sm}}(T(K),E)$ is stabilized by the action.

Lemma **2.11.** — *Let* $w_1, w_2 \in S_n$, *the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Ext}^1_g(D,D) & \xrightarrow{\kappa_{w_1}} & \mathrm{Hom}_{\mathrm{sm}}(T(K),E) \\ \| & & \downarrow{\scriptstyle w_2 w_1^{-1}} \wr \\ \mathrm{Ext}^1_g(D,D) & \xrightarrow{\kappa_{w_2}} & \mathrm{Hom}_{\mathrm{sm}}(T(K),E). \end{array} \tag{2.13}$$

Proof. — The lemma is well-known, but we include a proof for the convenience of the reader. It suffices to prove the statement for the case where $w_2 w_1^{-1}$ is a simple reflection, say, s_k. Let $\widetilde{D} \in \mathrm{Ext}^1_g(D,D)$ and suppose $\kappa_{w_i}(\widetilde{D}) = (\psi_{i,1}, \dots, \psi_{i,n})$. By definition, $\widetilde{D}$ admits triangulations:

$$\mathcal{R}_{K,E[\epsilon]/\epsilon^2}\big(\phi_{w_i^{-1}(1)} z^{\mathbf{h}_1}(1+\psi_{i,1}\epsilon)\big) - \cdots - \mathcal{R}_{K,E[\epsilon]/\epsilon^2}\big(\phi_{w_i^{-1}(n)} z^{\mathbf{h}_1}(1+\psi_{i,n}\epsilon)\big).$$

Note by assumption $w_1^{-1}(j) = w_2^{-1}(j)$ for $j \neq k, k+1$. Consequently, for $j < k$ or $j > k+1$, we have $\mathrm{Fil}^j_{\mathscr{T}_{w_1}} \widetilde{D} \cong \mathrm{Fil}^j_{\mathscr{T}_{w_2}} \widetilde{D}$, since $\mathrm{Hom}(\mathrm{Fil}^j_{\mathscr{T}_{w_1}} \widetilde{D}, \widetilde{D}/\mathrm{Fil}^j_{\mathscr{T}_{w_2}} \widetilde{D}) = 0$.

As $\mathrm{Hom}(\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(\phi_{w_1^{-1}(1)}z^{\mathbf{h}_1}(1+\psi_{1,1}\epsilon)),\widetilde{\mathrm{D}})\cong \mathrm{E}[\epsilon]/\epsilon^2$, using dévissage for $\mathscr{T}_{w_2}$, we easily deduce that (using $\mathcal{R}$ for $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$ for short) if $k>1$,

$$\mathrm{Hom}\big(\mathcal{R}\big(\phi_{w_1^{-1}(1)}z^{\mathbf{h}_1}(1+\psi_{1,1}\epsilon)\big),\mathcal{R}\big(\phi_{w_2^{-1}(1)}z^{\mathbf{h}_1}(1+\psi_{2,1}\epsilon)\big)\big)\cong \mathrm{E}[\epsilon]/\epsilon^2,$$

hence $\mathrm{H}^0_{(\varphi,\Gamma)}(\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(1+(\psi_{1,1}-\psi_{2,1})\epsilon))\cong \mathrm{E}[\epsilon]/\epsilon^2$ (noting $w_1^{-1}(1)=w_2^{-1}(1)$). So $\psi_{1,1}=\psi_{2,1}$. We can then consider the $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$-module $\widetilde{\mathrm{D}}/\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(\phi_{w_1^{-1}(1)}z^{\mathbf{h}_1}(1+\psi_{1,1}\epsilon))$ equipped with the filtrations induced by $\mathscr{T}_{w_1}$ and $\mathscr{T}_{w_2}$. Continuing with the above argument, we have $\psi_{1,j}=\psi_{2,j}$ for $j<k$.

For $j=k$, we have (noting $\mathrm{Fil}^{k-1}_{\mathscr{T}_{w_1}}\widetilde{\mathrm{D}}=\mathrm{Fil}^{k-1}_{\mathscr{T}_{w_2}}\widetilde{\mathrm{D}}$)

$$\mathrm{Hom}\big(\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}\big(\phi_{w_1^{-1}(k)}z^{\mathbf{h}_k}(1+\psi_{1,k\epsilon})\big),\widetilde{\mathrm{D}}/\mathrm{Fil}^{k-1}_{\mathscr{T}_{w_2}}\widetilde{\mathrm{D}}\big)\cong \mathrm{E}[\epsilon]/\epsilon^2.$$

Using dévissage for $\mathscr{T}_{w_2}$ (and the fact $w_2w_1^{-1}=s_k$), we get

$$\begin{aligned}&\mathrm{Hom}\big(\mathcal{R}\big(\phi_{w_1^{-1}(k)}z^{\mathbf{h}_k}(1+\psi_{1,k}\epsilon)\big),\mathcal{R}\big(\phi_{w_2^{-1}(k+1)}z^{\mathbf{h}_{k+1}}(1+\psi_{2,k+1}\epsilon)\big)\big)\\&\quad\cong \mathrm{E}[\epsilon]/\epsilon^2,\end{aligned}$$

hence $\psi_{1,k}=\psi_{2,k+1}$. Exchanging $\mathscr{T}_{w_1}$ and $\mathscr{T}_{w_2}$, we get $\psi_{2,k}=\psi_{1,k+1}$.

For $j>k+1$, using the same argument as in the case of $j<k$ with $\widetilde{\mathrm{D}}$ replaced by $\widetilde{\mathrm{D}}/\mathrm{Fil}^{k+1}_{\mathscr{T}_{w_1}}\widetilde{\mathrm{D}}$, we see $\psi_{1,j}=\psi_{2,j}$. This concludes the proof. □

Let $\mathrm{Ext}^1_0(\mathrm{D},\mathrm{D}):=\mathrm{Ker}\,\kappa_w$ (for some $w\in S_n$ *a priori*). By Proposition 2.10 (3), $\mathrm{Ext}^1_0(\mathrm{D},\mathrm{D})\subset \mathrm{Ext}^1_g(\mathrm{D},\mathrm{D})$. Using Lemma 2.11, we see $\mathrm{Ext}^1_0(\mathrm{D},\mathrm{D})=\mathrm{Ker}\,\kappa_w$ for all $w\in S_n$. Moreover, by Proposition 2.10 (1) (2), we have

$$\dim_{\mathrm{E}}\mathrm{Ext}^1_0(\mathrm{D},\mathrm{D})=\frac{n(n-1)}{2}d_{\mathrm{K}}+1-n. \tag{2.14}$$

For $\mathrm{Ext}^1_*(\mathrm{D},\mathrm{D})\subset \mathrm{Ext}^1(\mathrm{D},\mathrm{D})$ (with $*=g,w,\dots$), if $\mathrm{Ext}^1_*(\mathrm{D},\mathrm{D})\supset \mathrm{Ext}^1_0(\mathrm{D},\mathrm{D})$, we set

$$\overline{\mathrm{Ext}}^1_*(\mathrm{D},\mathrm{D}):=\mathrm{Ext}^1_*(\mathrm{D},\mathrm{D})/\mathrm{Ext}^1_0(\mathrm{D},\mathrm{D}).$$

We have hence isomorphisms

$$\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})\xrightarrow[\sim]{\kappa_w}\mathrm{Hom}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big),\quad \overline{\mathrm{Ext}}^1_g(\mathrm{D},\mathrm{D})\xrightarrow[\sim]{\kappa_w}\mathrm{Hom}_{\mathrm{sm}}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big). \tag{2.15}$$

Note also

$$\dim_{\mathrm{E}}\overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})=\frac{n(n+1)}{2}d_{\mathrm{K}}+n. \tag{2.16}$$

Let $\mathrm{Ext}^1_{g'}(\mathrm{D},\mathrm{D}) \subset \mathrm{Ext}^1(\mathrm{D},\mathrm{D})$ be the subspace of de Rham deformations up to twist by characters of $\mathrm{K}^\times$ over $(\mathrm{E}[\epsilon]/\epsilon^2)^\times$. Similarly, set

$$\mathrm{Hom}_{g'}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big) := \Big\{\psi \in \mathrm{Hom}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big) \mid \exists \psi_0 : \mathrm{K}^\times \to \mathrm{E} \text{ such that } \psi - \psi_0 \circ \det \in \mathrm{Hom}_{\mathrm{sm}}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big)\Big\}. \tag{2.17}$$

One easily deduces from Proposition 2.10 (3) that for all $w \in \mathrm{S}_n$, $\mathrm{Ext}^1_{g'}(\mathrm{D},\mathrm{D}) \subset \mathrm{Ext}^1_w(\mathrm{D},\mathrm{D})$ and is equal to the preimage of $\mathrm{Hom}_{g'}(\mathrm{T}(\mathrm{K}),\mathrm{E})$ under κ_w. Thus $\dim_{\mathrm{E}} \mathrm{Ext}^1_{g'}(\mathrm{D},\mathrm{D}) = 1 + (\frac{n(n-1)}{2}+1)d_{\mathrm{K}}$. Moreover, (2.13) holds with "g" and "sm" replaced by "g'". Using the fact that D is non-critical, by [26, Thm. 3.19] (for $\mathrm{K} = \mathbf{Q}_p$) and [51, Thm. 2.62] (for general K) (see also [46] for the $(n=2)$-case, noting the proposition also follows from Corollary 2.33 below and an easy induction argument), we have

Proposition **2.12.** — *The natural map* $\oplus_{w\in \mathrm{S}_n} \mathrm{Ext}^1_w(\mathrm{D},\mathrm{D}) \to \mathrm{Ext}^1(\mathrm{D},\mathrm{D})$ *is surjective and induces a surjective map* $\oplus_{w\in \mathrm{S}_n} \overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D}) \twoheadrightarrow \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})$.

Now we consider general paraboline deformations of D. Let B the Borel subgroup of GL_n of upper triangular matrices, $\mathrm{P} \supset \mathrm{B}$ be a standard parabolic subgroup of GL_n with the standard Levi subgroup $\mathrm{L}_{\mathrm{P}} \supset \mathrm{T}$ equal to $\mathrm{diag}(\mathrm{GL}_{n_1}, \dots, \mathrm{GL}_{n_r})$. A filtration

$$\mathscr{F}_{\mathrm{P}} : 0 = \mathrm{Fil}^0_{\mathscr{F}_{\mathrm{P}}} \mathrm{D} \subsetneq \mathrm{Fil}^1_{\mathscr{F}_{\mathrm{P}}} \mathrm{D} \subsetneq \cdots \subsetneq \mathrm{Fil}^r_{\mathscr{F}_{\mathrm{P}}} \mathrm{D} = \mathrm{D}$$

of saturated (φ,Γ)-submodules of D is called a P-*filtration* if $\mathrm{M}_i := \mathrm{rank}\, \mathrm{gr}^i_{\mathscr{F}_{\mathrm{P}}} \mathrm{D} = n_i$. A deformation $\widetilde{\mathrm{D}}$ of D over $\mathrm{E}[\epsilon/\epsilon^2]$ is called an $\mathscr{F}_{\mathrm{P}}$-*deformation*, if $\widetilde{\mathrm{D}}$ admits a filtration $\mathrm{Fil}^i_{\mathscr{F}_{\mathrm{P}}} \widetilde{\mathrm{D}}$ of saturated (φ,Γ)-submodules of D over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$ (which means $\mathrm{Fil}^i_{\mathscr{F}_{\mathrm{P}}} \widetilde{\mathrm{D}}$ is free over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$) such that $\mathrm{gr}^i_{\mathscr{F}_{\mathrm{P}}} \widetilde{\mathrm{D}}$ is a deformation of M_i over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$. Denote by $\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}) \subset \mathrm{Ext}^1(\mathrm{D},\mathrm{D})$ the subspace of $\mathscr{F}_{\mathrm{P}}$-deformations. By [26, Prop. 3.6, Prop. 3.7] (which is for $\mathrm{K} = \mathbf{Q}_p$, but all the arguments generalize directly to general K, see also the proof of Proposition 2.17 below), we have

Proposition **2.13.** — $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}) = 1 + d_{\mathrm{K}} \dim \mathrm{P} = 1 + d_{\mathrm{K}} \sum_{1\leq i\leq j\leq r} n_i n_j$. *The natural map*

$$\kappa_{\mathscr{F}_{\mathrm{P}}} : \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}) \longrightarrow \prod_{i=1}^r \mathrm{Ext}^1(\mathrm{M}_i,\mathrm{M}_i), \tag{2.18}$$

sending $\widetilde{\mathrm{D}}$ *to* $(\mathrm{gr}^i_{\mathscr{F}_{\mathrm{P}}} \widetilde{\mathrm{D}})_{i=1,\dots,r}$, *is surjective.*

For $w \in \mathrm{S}_n$, we call the B-filtration $\mathscr{T}_w$ (associated to $w(\phi)$) compatible with $\mathscr{F}_{\mathrm{P}}$, if $\mathscr{T}_w$ induces a complete flag on $\mathrm{Fil}^i_{\mathscr{F}_{\mathrm{P}}} \mathrm{D}$ for all i. In this case, we have $\mathrm{Ext}^1_{\mathscr{T}_w}(\mathrm{D},\mathrm{D}) \subset \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D})$. For $i = 1, \dots, r$, we let $\mathscr{T}_{w,i}$ be the induced filtration on M_i $(= \mathrm{gr}^i_{\mathscr{F}_{\mathrm{P}}} \mathrm{D})$.

Corollary **2.14.** — *Keep the above situation.*

(1) $\mathrm{Ext}^1_w(\mathrm{D},\mathrm{D})$ *is the preimage of* $\prod_{i=1}^r \mathrm{Ext}^1_{\mathscr{T}_{w,i}}(\mathrm{M}_i,\mathrm{M}_i)$ *via* $\kappa_{\mathscr{F}_\mathrm{P}}$. *In particular,* $\kappa_{\mathscr{F}_\mathrm{P}}$ *induces a surjective map* $\kappa_{\mathscr{F}_\mathrm{P}}:\mathrm{Ext}^1_w(\mathrm{D},\mathrm{D})\twoheadrightarrow\prod_{i=1}^r \mathrm{Ext}^1_{\mathscr{T}_{w,i}}(\mathrm{M}_i,\mathrm{M}_i)$.

(2) The map $\kappa_{\mathscr{F}_\mathrm{P}}$ *sends* $\mathrm{Ext}^1_0(\mathrm{D},\mathrm{D})$ *to* $\prod_{i=1}^r \mathrm{Ext}^1_0(\mathrm{M}_i,\mathrm{M}_i)$ *and induces isomorphisms*

$$\kappa_{\mathscr{F}_\mathrm{P}}:\overline{\mathrm{Ext}}^1_{\mathscr{F}_\mathrm{P}}(\mathrm{D},\mathrm{D})\xrightarrow{\sim}\prod_{i=1}^r\overline{\mathrm{Ext}}^1(\mathrm{M}_i,\mathrm{M}_i), \tag{2.19}$$

and $\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})\xrightarrow{\sim}\prod_{i=1}^r\overline{\mathrm{Ext}}^1_{\mathscr{T}_{w,i}}(\mathrm{M}_i,\mathrm{M}_i)$.

Proof. — The first part of (1) is by definition, and the second part follows from Proposition 2.13. It is clear that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Ext}^1_w(\mathrm{D},\mathrm{D}) & \twoheadrightarrow & \prod_{i=1}^r \mathrm{Ext}^1_{\mathscr{T}_{w,i}}(\mathrm{M}_i,\mathrm{M}_i) \\ \downarrow\!\!\downarrow \kappa_w & & \downarrow\!\!\downarrow (\kappa_{\mathscr{T}_{w,i}}) \\ \mathrm{Hom}(\mathrm{T}(\mathrm{K}),\mathrm{E}) & \xrightarrow{\sim} & \prod_{i=1}^r \mathrm{Hom}(\mathrm{T}_i(\mathrm{K}),\mathrm{E}) \end{array} \tag{2.20}$$

where T_i is the torus subgroup of GL_{n_i}. The first part of (2) follows. By (1) and Proposition 2.12, (2.19) is surjective. However, by Proposition 2.13, (2.14) and (2.16) (applied to the M_i's), we have $\dim_\mathrm{E}\overline{\mathrm{Ext}}^1_{\mathscr{F}_\mathrm{P}}(\mathrm{D},\mathrm{D})=d_\mathrm{K}\dim(\mathrm{B}\cap\mathrm{L}_\mathrm{P})-n=\sum_{i=1}^r\dim_\mathrm{E}\overline{\mathrm{Ext}}^1(\mathrm{M}_i,\mathrm{M}_i)$. Hence (2.19) is bijective. The final isomorphism follows by similar arguments. □

Let $\mathrm{Ext}^1_{\mathscr{F}_\mathrm{P},g'}(\mathrm{D},\mathrm{D})$ be the preimage of $\prod_{i=1}^r\mathrm{Ext}^1_{g'}(\mathrm{M}_i,\mathrm{M}_i)$ via (2.18). Set

$$\mathrm{Hom}_{\mathrm{P},g'}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big):=\big\{\psi\in\mathrm{Hom}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big)\mid\exists\psi_\mathrm{P}:\mathrm{Z}_{\mathrm{L}_\mathrm{P}}(\mathrm{K})\to\mathrm{E} \\ \text{such that } \psi-\psi_\mathrm{P}\circ\det_{\mathrm{L}_\mathrm{P}}\in\mathrm{Hom}_{\mathrm{sm}}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big)\big\}. \tag{2.21}$$

It is straightforward to see $\dim_\mathrm{E}\mathrm{Hom}_{\mathrm{P},g'}(\mathrm{T}(\mathrm{K}),\mathrm{E})=n+rd_\mathrm{K}$. The following corollary generalizes (2.13).

Corollary **2.15.** — *(1) Let* $w\in\mathrm{S}_n$ *such that* $\mathscr{T}_w$ *is compatible with* $\mathscr{F}_\mathrm{P}$, *then* $\mathrm{Ext}^1_{\mathscr{F}_\mathrm{P},g'}(\mathrm{D},\mathrm{D})\subset\mathrm{Ext}^1_w(\mathrm{D},\mathrm{D})$.

(2) Let $w_1,w_2\in\mathrm{S}_n$ *such that* $\mathscr{T}_{w_1}$, $\mathscr{T}_{w_2}$ *are compatible with* $\mathscr{F}_\mathrm{P}$ *(so* $w_2w_1^{-1}$ *lies in the Weyl group* $\mathscr{W}_\mathrm{P}$ *of* L_P*), we have a commutative diagram*

$$\begin{array}{ccc} \overline{\mathrm{Ext}}^1_{\mathscr{F}_\mathrm{P},g'}(\mathrm{D},\mathrm{D}) & \xrightarrow[\sim]{\kappa_{w_1}} & \mathrm{Hom}_{\mathrm{P},g'}(\mathrm{T}(\mathrm{K}),\mathrm{E}) \\ \| & & w_2w_1^{-1}\downarrow\sim \\ \overline{\mathrm{Ext}}^1_{\mathscr{F}_\mathrm{P},g'}(\mathrm{D},\mathrm{D}) & \xrightarrow[\sim]{\kappa_{w_2}} & \mathrm{Hom}_{\mathrm{P},g'}(\mathrm{T}(\mathrm{K}),\mathrm{E}). \end{array}$$

Proof. — (1) follows from the fact $\mathrm{Ext}^1_{g'}(\mathrm{M}_i, \mathrm{M}_i) \subset \mathrm{Ext}^1_{\mathscr{T}_{w,i}}(\mathrm{M}_i, \mathrm{M}_i)$ and Corollary 2.14 (1). By Corollary 2.14 (2), we have $\overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}},g'}(\mathrm{D}, \mathrm{D}) \xrightarrow{\sim} \prod_{i=1}^r \overline{\mathrm{Ext}}^1_{g'}(\mathrm{M}_i, \mathrm{M}_i)$. (2) then follows from the commutative diagram (2.20) and Lemma 2.11 (applied to each M_i, and with "g", "sm" replaced by "g'"). □

2.3.2. *Trianguline and paraboline deformations, II.* — Let $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$. We consider some partially de Rham deformations of D. The reader who is mainly interested in the $\mathbf{Q}_p$-case can skip this section. Recall for $\mathrm{J} \subset \Sigma_{\mathrm{K}}$, and a (φ, Γ)-module M over $\mathcal{R}_{\mathrm{K,E}}$, M is called J-*de Rham*, if $\dim_{\mathrm{E}} \mathrm{D}_{\mathrm{dR}}(\mathrm{M})_\tau = \mathrm{rank}_{\mathcal{R}_{\mathrm{K,E}}} \mathrm{M}$ for all $\tau \in \mathrm{J}$, where $\mathrm{D}_{\mathrm{dR}}(\mathrm{M})_\tau = \mathrm{H}^0(\mathrm{Gal}_{\mathrm{K}}, \mathrm{W}^+_{\mathrm{dR}}(\mathrm{M})_\tau[1/t])$. Note the property is clearly inherited by taking subquotients. For a (φ, Γ)-module M over $\mathcal{R}_{\mathrm{K,E}}$, denote by $\mathrm{W}(\mathrm{M}) = (\mathrm{W}_e(\mathrm{M}), \mathrm{W}^+_{\mathrm{dR}}(\mathrm{M}))$ its associated B-pair ([7]). By [51, Thm. 5.11], there are natural isomorphisms for $i = 0, 1, 2$,

$$\mathrm{H}^i_{(\varphi,\Gamma)}(\mathrm{M}) \xrightarrow{\sim} \mathrm{H}^i\big(\mathrm{Gal}_{\mathrm{K}}, \mathrm{W}(\mathrm{M})\big) \tag{2.22}$$

where $\mathrm{H}^i(\mathrm{Gal}_{\mathrm{K}}, \mathrm{W}(\mathrm{M}))$ denotes the i-th Galois cohomology of the B-pair M, see [50, Section 2.1].

Throughout the section, we fix $\sigma \in \Sigma_{\mathrm{K}}$. For an extension group $\mathrm{Ext}^1_?(\mathrm{D}, \mathrm{D})$, we denote by $\mathrm{Ext}^1_{\sigma,?}(\mathrm{D}, \mathrm{D}) \subset \mathrm{Ext}^1_?(\mathrm{D}, \mathrm{D})$ the subspace consisting of $\widetilde{\mathrm{D}}$ that are $\Sigma_{\mathrm{K}} \setminus \{\sigma\}$-de Rham. If $\mathrm{Ext}^1_?(\mathrm{D}, \mathrm{D}) \supset \mathrm{Ext}^1_0(\mathrm{D}, \mathrm{D})$, then it is clear that $\mathrm{Ext}^1_{\sigma,?}(\mathrm{D}, \mathrm{D}) \supset \mathrm{Ext}^1_0(\mathrm{D}, \mathrm{D})$ and we set

$$\overline{\mathrm{Ext}}^1_{\sigma,?}(\mathrm{D}, \mathrm{D}) := \mathrm{Ext}^1_{\sigma,?}(\mathrm{D}, \mathrm{D}) / \mathrm{Ext}^1_0(\mathrm{D}, \mathrm{D}) \subset \overline{\mathrm{Ext}}^1_?(\mathrm{D}, \mathrm{D}).$$

Lemma **2.16.** — *We have* $\dim_{\mathrm{E}} \mathrm{Ext}^1_\sigma(\mathrm{D}, \mathrm{D}) = 1 + \frac{n(n-1)}{2}(d_{\mathrm{K}} - 1) + n^2$.

Proof. — Using the notation of [31, Section A], the isomorphism (2.22) (for $i = 1$, $\mathrm{M} = \mathrm{D} \otimes_{\mathcal{R}_{\mathrm{K,E}}} \mathrm{D}^\vee$) induces an isomorphism $\mathrm{Ext}^1_\sigma(\mathrm{D}, \mathrm{D}) \cong \mathrm{H}^1_{g,\Sigma_{\mathrm{K}} \setminus \{\sigma\}}(\mathrm{Gal}_{\mathrm{K}}, \mathrm{W}(\mathrm{D} \otimes_{\mathcal{R}_{\mathrm{K,E}}} \mathrm{D}^\vee))$ where $\mathrm{D}^\vee := \mathrm{Hom}_{\mathcal{R}_{\mathrm{K,E}}}(\mathrm{D}, \mathcal{R}_{\mathrm{K,E}})$. The lemma follows then from [31, Cor. A.4] (noting $\mathrm{D} \otimes_{\mathcal{R}_{\mathrm{K,E}}} \mathrm{D}^\vee$ has Hodge-Tate-Sen weights $\{h_{\tau,i} - h_{\tau,j}\}_{\substack{\tau \in \Sigma_{\mathrm{K}} \\ i,j=1,\dots,r}}$). The required assumption holds because D is generic. □

Let P be a standard parabolic subgroup, and $\mathscr{F}_{\mathrm{P}}$ be a P-filtration on D with $\mathrm{gr}^i_{\mathscr{F}_{\mathrm{P}}} \mathrm{D} =: \mathrm{M}_i$. The surjection $\kappa_{\mathscr{F}_{\mathrm{P}}}$ (2.18) induces a map

$$\kappa_{\mathscr{F}_{\mathrm{P}}} : \mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D}, \mathrm{D}) \longrightarrow \prod_{i=1}^r \mathrm{Ext}^1_\sigma(\mathrm{M}_i, \mathrm{M}_i). \tag{2.23}$$

Proposition **2.17.** — *(1) We have* $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D}, \mathrm{D}) = 1 + (d_{\mathrm{K}} - 1)\frac{n(n-1)}{2} + \dim \mathrm{P}$.

(2) The map (2.23) is surjective and induces an isomorphism

$$\overline{\mathrm{Ext}}_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}) \xrightarrow{\sim} \prod_{i=1}^{r} \overline{\mathrm{Ext}}^1_{\sigma}(\mathrm{M}_i,\mathrm{M}_i). \tag{2.24}$$

Proof. — Let $\mathrm{Hom}_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D})$ be the (φ,Γ)-submodule of $\mathrm{Hom}_{\mathcal{R}_{\mathrm{K,E}}}(\mathrm{D},\mathrm{D}) \cong \mathrm{D}\otimes_{\mathcal{R}_{\mathrm{K,E}}}\mathrm{D}^\vee$ consisting of the maps f such that $f(\mathrm{Fil}^i_{\mathscr{F}_{\mathrm{P}}}) \subset \mathrm{Fil}^i_{\mathscr{F}_{\mathrm{P}}}$ for $i=1,\dots,r$. Similarly as in [26, Prop. 3.6 (ii)] and using the notation of [31, Section A], we have $\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}) \cong \mathrm{H}^1_{g,\Sigma_{\mathrm{K}}\setminus\{\sigma\}}(\mathrm{Gal}_{\mathrm{K}}, \mathrm{W}(\mathrm{Hom}_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D})))$. Since D is non-critical, it is straightforward to see $\mathrm{Hom}_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D})$ has Hodge-Tate-Sen weights $\{h_{\tau,i}-h_{\tau,j}\}_{\tau\in\Sigma_{\mathrm{K}}}$ where the indices (i,j) correspond to entries of the matrix $\mathfrak{gl}_n$ lying in $\mathfrak{p}$, the Lie algebra of P. By [31, Cor. A.4] (noting that the (φ,Γ)-module $\mathrm{Hom}_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D})$ satisfies the assumptions in *loc. cit.* as D is generic), we calculate $\dim_{\mathrm{E}} \mathrm{H}^1_{g,\Sigma_{\mathrm{K}}\setminus\{\sigma\}}(\mathrm{Gal}_{\mathrm{K}}, \mathrm{W}^+_{\mathrm{dR}}(\mathrm{Hom}_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}))) = 1 + d_{\mathrm{K}}\dim \mathrm{P} - \sum_{\tau\neq\sigma}\dim(\mathrm{B}\cap\mathrm{L}_{\mathrm{P}}) = 1 + (d_{\mathrm{K}}-1)\frac{n(n-1)}{2} + \dim \mathrm{P}$. (1) follows. For any $\widetilde{\mathrm{D}} \in \mathrm{Ker}(2.18)$, using Corollary 2.14 (1), (2.20) and Proposition 2.10 (3), we see $\widetilde{\mathrm{D}}$ is de Rham. Hence $\mathrm{Ker}(2.18) \subset \mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D})$. Let N be the unipotent radical of B. As $\dim_{\mathrm{E}}\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}) - \dim_{\mathrm{E}}\mathrm{Ker}(2.18) = r + (d_{\mathrm{K}}-1)\dim(\mathrm{N}\cap\mathrm{L}_{\mathrm{P}}) + \dim \mathrm{L}_{\mathrm{P}} = \sum_{i=1}^r \dim_{\mathrm{E}}\mathrm{Ext}^1_{\sigma}(\mathrm{M}_i,\mathrm{M}_i)$, (2.23) hence (2.24) are surjective. Finally we have equalities $\dim_{\mathrm{E}}\overline{\mathrm{Ext}}_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}) = n + \dim(\mathrm{B}\cap\mathrm{L}_{\mathrm{P}}) = \sum_{i=1}^r \dim \overline{\mathrm{Ext}}^1_{\sigma}(\mathrm{M}_i,\mathrm{M}_i)$ which complete the proof of (2). □

Combining Proposition 2.17 (2) with Corollary 2.14 (2), we get:

Corollary **2.18.** — *Let $\mathscr{T}_w$ be a B-filtration compatible with $\mathscr{F}_{\mathrm{P}}$ (see Corollary 2.14). The map $\kappa_{\mathscr{F}_{\mathrm{P}}}$ induces a bijection $\overline{\mathrm{Ext}}^1_{\sigma,w}(\mathrm{D},\mathrm{D}) \xrightarrow{\sim} \prod_{i=1}^r \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{T}_{w,i}}(\mathrm{M}_i,\mathrm{M}_i)$.*

For a rank one de Rham (φ,Γ)-module $\mathcal{R}_{\mathrm{K,E}}(\chi)$ (implying χ is locally algebraic), by [31, Lem. 1.15], (2.11) induces by restriction an isomorphism

$$\mathrm{Ext}^1_{\sigma}\big(\mathcal{R}_{\mathrm{K,E}}(\chi), \mathcal{R}_{\mathrm{K,E}}(\chi)\big) \cong \mathrm{Hom}_{\sigma}\big(\mathrm{K}^\times, \mathrm{E}\big). \tag{2.25}$$

By Proposition 2.17 (2) applied to $\mathrm{P}=\mathrm{B}$, we obtain:

Corollary **2.19.** — *For $w\in \mathrm{S}_n$, κ_w (2.12) induces an isomorphism $\overline{\mathrm{Ext}}_{\sigma,w}(\mathrm{D},\mathrm{D}) \xrightarrow{\sim} \mathrm{Hom}_{\sigma}(\mathrm{T}(\mathrm{K}),\mathrm{E})$.*

We will show later (in Corollary 2.40 below) the induced map

$$\oplus_{w\in \mathrm{S}_n}\mathrm{Ext}^1_{\sigma,w}(\mathrm{D},\mathrm{D}) \longrightarrow \mathrm{Ext}^1_{\sigma}(\mathrm{D},\mathrm{D}) \tag{2.26}$$

is surjective (and the same holds with Ext^1 replaced by $\overline{\mathrm{Ext}}^1$). Consider now certain extension groups of $\mathrm{D}_\sigma := \mathfrak{T}_\sigma(\mathrm{D})$ (cf. (2.7)).

Proposition **2.20.** — *(1) We have* $\dim_{\mathrm{E}} \mathrm{Ext}^1(\mathrm{D}_\sigma, \mathrm{D}_\sigma) = 1 + n^2 d_{\mathrm{K}}$.

(2) We have $\dim_{\mathrm{E}} \mathrm{Ext}^1_g(\mathrm{D}_\sigma, \mathrm{D}_\sigma) = 1 + \frac{n(n-1)}{2}$.

(3) Let P *be a standard parabolic subgroup of* GL_n*, and* $\mathscr{F}_{\mathrm{P}}$ *be a* P*-filtration of* D_σ *with* $\mathrm{gr}_i \mathscr{F}_{\mathrm{P}} \cong \mathrm{M}_{i,\sigma}$*. We have* $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma) = 1 + d_{\mathrm{K}} \dim \mathrm{P}$ *and* $\mathrm{Ext}^1_g(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \subset \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$*. Moreover, the following natural map (defined similarly as in (2.18)) is surjective*

$$\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \twoheadrightarrow \prod_{i=1}^r \mathrm{Ext}^1(\mathrm{M}_{i,\sigma}, \mathrm{M}_{i,\sigma}). \tag{2.27}$$

Proof. — (1) follows from [47, Thm. 1.2 (1)] as $\mathrm{Hom}(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \cong \mathrm{E}$, $\mathrm{Ext}^2(\mathrm{D}_\sigma, \mathrm{D}_\sigma) = 0$. (2) follows from (1), [31, Cor. A.4] and $\dim \mathrm{H}^0(\mathrm{Gal}_{\mathrm{K}}, \mathrm{W}^+_{\mathrm{dR}}(\mathrm{D}_\sigma \otimes_{\mathcal{R}_{\mathrm{K,E}}} \mathrm{D}_\sigma^\vee)_\tau) = \begin{cases} n^2 & \tau \neq \sigma \\ \frac{n(n+1)}{2} & \tau = \sigma. \end{cases}$ The statements in (3) except $\mathrm{Ext}^1_g(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \subset \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$ follow by the same argument as in the proof of [26, Prop. 3.6, Prop. 3.7]. By [31, Cor. A.4], $\mathrm{Ext}^1_g(\mathrm{Fil}^i_{\mathscr{F}_{\mathrm{P}}} \mathrm{D}_\sigma, \mathrm{D}_\sigma / \mathrm{Fil}^i_{\mathscr{F}_{\mathrm{P}}} \mathrm{D}_\sigma) = 0$ for $i = 1, \dots, r-1$. Hence if $\widetilde{\mathrm{D}}_\sigma \in \mathrm{Ext}^1_g(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$, it must map to zero under the natural map

$$\mathrm{Ext}^1(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \longrightarrow \mathrm{Ext}^1\big(\mathrm{Fil}^1_{\mathscr{F}_{\mathrm{P}}} \mathrm{D}_\sigma, \mathrm{D}_\sigma / \mathrm{Fil}^1_{\mathscr{F}_{\mathrm{P}}} \mathrm{D}_\sigma\big).$$

Thus $\widetilde{\mathrm{D}}_\sigma$ has the form $[\widetilde{\mathrm{M}}_1 — \widetilde{\mathrm{M}}_2]$ where $\widetilde{\mathrm{M}}_1$ (resp. $\widetilde{\mathrm{M}}_2$) is a deformation of $\mathrm{Fil}^1_{\mathscr{F}_{\mathrm{P}}} \mathrm{D}_\sigma$ (resp. of $\mathrm{D}_\sigma / \mathrm{Fil}^1_{\mathscr{F}_{\mathrm{P}}} \mathrm{D}_\sigma$). Iterating the argument for $\widetilde{\mathrm{M}}_2$, we inductively deduce $\widetilde{\mathrm{D}}_\sigma \in \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$. □

Remark **2.21.** — Recall for each $w \in \mathrm{S}_n$, $w(\phi)$ is also a refinement of D_σ and we still use $\mathscr{T}_w$ to denote the associated B-filtration on D_σ. Applying Proposition 2.20 (3) for $\mathrm{P} = \mathrm{B}$ and $\mathscr{F}_{\mathrm{P}} = \mathscr{T}_w$, we have $\dim_{\mathrm{E}} \mathrm{Ext}^1_w(\mathrm{D}_\sigma, \mathrm{D}_\sigma) = 1 + d_{\mathrm{K}} \frac{n(n+1)}{2}$ and a natural surjection

$$\kappa_w : \mathrm{Ext}^1_w(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \twoheadrightarrow \mathrm{Hom}\big(\mathrm{T}(\mathrm{K}), \mathrm{E}\big). \tag{2.28}$$

The preimage of $\mathrm{Hom}_{\mathrm{sm}}(\mathrm{T}(\mathrm{K}), \mathrm{E})$ hence has dimension equal to $(1 + d_{\mathrm{K}} \frac{n(n+1)}{2}) - n d_{\mathrm{K}} = 1 + d_{\mathrm{K}} \frac{n(n-1)}{2}$. Together with Proposition 2.20 (2), we see when $\mathrm{K} \neq \mathbf{Q}_p$, $\mathrm{Ext}^1_g(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$ is properly contained in the preimage of $\mathrm{Hom}_{\mathrm{sm}}(\mathrm{T}(\mathrm{K}), \mathrm{E})$.

For $\Sigma_{\mathrm{K}} \setminus \{\sigma\}$-de Rham deformations of D_σ, we have:

Proposition **2.22.** — *(1) We have* $\dim_{\mathrm{E}} \mathrm{Ext}^1_\sigma(\mathrm{D}_\sigma, \mathrm{D}_\sigma) = 1 + n^2$.

(2) Let P *be a standard parabolic subgroup of* GL_n*, and* $\mathscr{F}_{\mathrm{P}}$ *be a* P*-filtration of* D_σ *with* $\mathrm{gr}^i_{\mathscr{F}_{\mathrm{P}}} \mathrm{D}_\sigma \cong \mathrm{M}_{i,\sigma}$*. Then* $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\sigma, \mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma) = 1 + \dim \mathrm{P}$.

Proof. — By [31, Cor. A.4], (1) (resp. (2)) follows from Proposition 2.20 (1) (resp. (3)) and the fact that for $\tau \neq \sigma$, $\dim_{\mathrm{E}} \mathrm{H}^0(\mathrm{Gal}_{\mathrm{K}}, \mathrm{W}^+_{\mathrm{dR}}(\mathrm{Hom}_{\mathcal{R}_{\mathrm{K,E}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma))_\tau) = n^2$ (resp.

$\dim_{\mathrm{E}} \mathrm{H}^0(\mathrm{Gal}_{\mathrm{K}}, \mathrm{W}_{\mathrm{dR}}^+(\mathrm{Hom}_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma))_\tau) = \dim \mathrm{P})$. Here $\mathrm{Hom}_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$ is defined in a similar way as in the proof of Proposition 2.17. □

Now we consider the relation between deformations of D and those of D_σ. The following proposition follows from the same argument as in the proof of Proposition 2.5, accounting for the $\mathrm{E}[\epsilon]/\epsilon^2$-structure. We leave the details to the reader.

Proposition **2.23.** — *For any (φ, Γ)-module $\widetilde{\mathrm{D}} \in \mathrm{Ext}^1(\mathrm{D}, \mathrm{D})$ over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$, there is a unique (φ, Γ)-module $\widetilde{\mathrm{D}}_\sigma \in \mathrm{Ext}^1(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$ over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$ satisfying that $\widetilde{\mathrm{D}} \subset \widetilde{\mathrm{D}}_\sigma$, $\widetilde{\mathrm{D}}[1/t] \cong \widetilde{\mathrm{D}}_\sigma[1/t]$, and the Sen σ-weights of $\widetilde{\mathrm{D}}_\sigma$ are equal to those of $\widetilde{\mathrm{D}}$, and the Sen τ-weights (over E) of $\widetilde{\mathrm{D}}_\sigma$ are constantly $h_{\tau,n}$ for $\tau \neq \sigma$.*

We obtain hence a natural map

$$\mathfrak{T}_\sigma : \mathrm{Ext}^1(\mathrm{D}, \mathrm{D}) \longrightarrow \mathrm{Ext}^1(\mathrm{D}_\sigma, \mathrm{D}_\sigma), \quad \widetilde{\mathrm{D}} \mapsto \widetilde{\mathrm{D}}_\sigma. \tag{2.29}$$

It is clear that this operation preserves (partial) de Rhamness and filtrations of saturated submodules. In particular, $\mathfrak{T}_\sigma$ restricts to a map $\mathrm{Ext}^1_\sigma(\mathrm{D}, \mathrm{D}) \to \mathrm{Ext}^1_\sigma(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$, and to a map $\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}, \mathrm{D}) \to \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$, where $\mathscr{F}_{\mathrm{P}}$ on D_σ is defined by $\mathrm{Fil}^i_{\mathscr{F}_{\mathrm{P}}} \mathrm{D}_\sigma = \mathfrak{T}_\sigma(\mathrm{Fil}^i_{\mathscr{F}_{\mathrm{P}}} \mathrm{D})$.

Proposition **2.24.** — *(1) For $* \in \{g, \sigma, \{\sigma, \mathscr{F}_{\mathrm{P}}\}\}$, the induced map $\mathfrak{T}_\sigma : \mathrm{Ext}^1_*(\mathrm{D}, \mathrm{D}) \to \mathrm{Ext}^1_*(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$ is surjective, and has the same kernel as (2.29).*

(2) The following diagram commutes

$$\begin{array}{ccc} \mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D}, \mathrm{D}) & \xrightarrow{(2.24)} & \prod_{i=1}^r \mathrm{Ext}^1_\sigma(\mathrm{M}_i, \mathrm{M}_i) \\ \downarrow \mathfrak{T}_\sigma & & \downarrow \mathfrak{T}_\sigma \\ \mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma) & \xrightarrow{(2.27)} & \prod_{i=1}^r \mathrm{Ext}^1_\sigma(\mathrm{M}_{i,\sigma}, \mathrm{M}_{i,\sigma}). \end{array} \tag{2.30}$$

Moreover, the map $\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \to \prod_{i=1}^r \mathrm{Ext}^1_\sigma(\mathrm{M}_{i,\sigma}, \mathrm{M}_{i,\sigma})$ is surjective.

Proof. — First, any $\widetilde{\mathrm{D}} \in \mathrm{Ker}(2.29)$ is de Rham, as it is contained in the de Rham (φ, Γ)-module $\mathrm{D}_\sigma \oplus \mathrm{D}_\sigma$. Hence Ker(2.29) coincides with the kernel of any maps in (1) (see also Proposition 2.20 (3)). Consider the composition (where the second map is the natural pull-back)

$$\mathrm{Ext}^1(\mathrm{D}, \mathrm{D}) \xrightarrow{\mathfrak{T}_\sigma} \mathrm{Ext}^1(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \longrightarrow \mathrm{Ext}^1(\mathrm{D}, \mathrm{D}_\sigma). \tag{2.31}$$

As $\mathrm{Hom}(\mathrm{D}, \mathrm{D}) \cong \mathrm{Hom}(\mathrm{D}, \mathrm{D}_\sigma) \cong \mathrm{Hom}(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \cong \mathrm{E}$, the kernel of the second map in (2.31) is isomorphic to $\mathrm{H}^0_{(\varphi,\Gamma)}((\mathrm{D}_\sigma \otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}}} \mathrm{D}^\vee)/(\mathrm{D}_\sigma \otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}}} \mathrm{D}_\sigma^\vee)) \xrightarrow[\sim]{\text{Lem. 2.1}} \mathrm{D}^+_{\mathrm{dR}}(\mathrm{D}_\sigma \otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}}} \mathrm{D}^\vee)/\mathrm{D}^+_{\mathrm{dR}}(\mathrm{D}_\sigma \otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}}} \mathrm{D}_\sigma^\vee) = 0$ where the vanishing follows easily by comparing the weights. Thus Ker(2.29) $\xrightarrow{\sim}$ Ker(2.31). The composition (2.31) coincides with the natural push-forward map via $D \hookrightarrow D_\sigma$. We deduce by dévissage that Ker(2.31) is isomorphic to $H^0_{(\varphi,\Gamma)}((D_\sigma \otimes_{\mathcal{R}_{K,E}} D^\vee)/(D \otimes_{\mathcal{R}_{K,E}} D^\vee))$. Using Lemma 2.1 and the easy fact $\dim_E D^+_{dR}(D_\sigma \otimes_{\mathcal{R}_{K,E}} D^\vee) = \frac{n(n+1)}{2} + (d_K - 1)n^2$ and $\dim_E D^+_{dR}(D \otimes_{\mathcal{R}_{K,E}} D^\vee) = \frac{n(n+1)}{2} d_K$, we deduce $\dim_E \mathrm{Ker}(2.29) = \dim_E \mathrm{Ker}(2.31) = (d_K - 1)\frac{n(n-1)}{2}$. By the dimension results in Proposition 2.10 (1) (resp. Proposition 2.13, resp. Proposition 2.17 (1)) and Proposition 2.20 (2) (resp. Proposition 2.22 (1), resp. Proposition 2.22 (2)), the difference in dimensions between the source and target spaces in (1) is exactly $(d_K - 1)\frac{n(n-1)}{2}$ for $* = g$ (resp. $* = \sigma$, resp. $* = \{\sigma, \mathscr{F}_P\}$). This proves (1). The commutativity of (2.30) follows directly from the definition of $\mathfrak{T}_\sigma$. The second part of (2) is then a consequence of (1) applied to each M_i (with $* = \sigma$) and of the surjectivity of (2.23) (see the first part of Proposition 2.17 (2)). □

Corollary **2.25.** — *Let* $w_1, w_2 \in S_n$, *the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Ext}^1_g(D_\sigma, D_\sigma) & \xrightarrow{\kappa_{w_1}} & \mathrm{Hom}_{sm}(T(K), E) \\ \| & & \downarrow{\scriptstyle w_2 w_1^{-1}} \\ \mathrm{Ext}^1_g(D_\sigma, D_\sigma) & \xrightarrow{\kappa_{w_2}} & \mathrm{Hom}_{sm}(T(K), E), \end{array}$$

and the horizontal maps are surjective.

Proof. — The commutativity follows by the same argument as in Lemma 2.11. For $w \in S_n$, we have a commutative diagram (where the right square corresponds to (2.30) for $P = B$ and $\mathscr{F}_P = \mathscr{T}_w$)

$$\begin{array}{ccccc} \mathrm{Ext}^1_g(D, D) & \hookrightarrow & \mathrm{Ext}^1_{\sigma,w}(D, D) & \twoheadrightarrow & \mathrm{Hom}_\sigma\big(T(K), E\big) \\ \downarrow{\scriptstyle \mathfrak{T}_\sigma} & & \downarrow{\scriptstyle \mathfrak{T}_\sigma} & & \| \\ \mathrm{Ext}^1_g(D_\sigma, D_\sigma) & \hookrightarrow & \mathrm{Ext}^1_{\sigma,w}(D_\sigma, D_\sigma) & \longrightarrow & \mathrm{Hom}_\sigma\big(T(K), E\big). \end{array} \qquad (\mathbf{2.32})$$

The surjectivity of κ_{w_i} in the corollary follows from Proposition 2.10 (3). □

Let $\mathrm{Ext}^1_0(D_\sigma, D_\sigma) \subset \mathrm{Ext}^1_g(D_\sigma, D_\sigma)$ be the kernel of $\kappa_w : \mathrm{Ext}^1_g(D_\sigma, D_\sigma) \to \mathrm{Hom}_{sm}(T(K), E)$ (for one or equivalently any $w \in S_n$, by Corollary 2.25). Note that unlike the case for D, this subspace is strictly contained in the kernel of (2.28) when $K \neq \mathbf{Q}_p$ (see the last sentence in Remark 2.21).

Corollary **2.26.** — *We have* $\mathrm{Ext}^1_0(D, D) = \mathfrak{T}_\sigma^{-1}(\mathrm{Ext}^1_0(D_\sigma, D_\sigma))$, *and* $\mathfrak{T}_\sigma$ *restricts to a surjection* $\mathrm{Ext}^1_0(D, D) \twoheadrightarrow \mathrm{Ext}^1_0(D_\sigma, D_\sigma)$.

Proof. — By Proposition 2.24 (1) (and the proof), $\mathrm{Ext}^1_g(\mathrm{D},\mathrm{D}) = \mathfrak{T}_\sigma^{-1}(\mathrm{Ext}^1_g(\mathrm{D}_\sigma, \mathrm{D}_\sigma))$. The corollary then follows from the definition of Ext^1_0's and (2.32). ☐

For $\mathrm{Ext}^1_?(\mathrm{D}_\sigma,\mathrm{D}_\sigma) \supset \mathrm{Ext}^1_0(\mathrm{D}_\sigma,\mathrm{D}_\sigma)$, set $\overline{\mathrm{Ext}}^1_?(\mathrm{D}_\sigma,\mathrm{D}_\sigma) := \frac{\mathrm{Ext}^1_?(\mathrm{D}_\sigma,\mathrm{D}_\sigma)}{\mathrm{Ext}^1_0(\mathrm{D}_\sigma,\mathrm{D}_\sigma)}$. By (the first statement of) Corollary 2.26 and Proposition 2.24, we easily deduce:

Corollary **2.27.** — *For* $* \in \{\sigma, g, \{\mathscr{F}_{\mathrm{P}},\sigma\}\}$, *the (surjective) map* $\mathfrak{T}_\sigma : \mathrm{Ext}^1_*(\mathrm{D},\mathrm{D}) \to \mathrm{Ext}^1_*(\mathrm{D}_\sigma,\mathrm{D}_\sigma)$ *induces an isomorphism* $\mathfrak{T}_\sigma : \overline{\mathrm{Ext}}^1_*(\mathrm{D},\mathrm{D}) \xrightarrow{\sim} \overline{\mathrm{Ext}}^1_*(\mathrm{D}_\sigma,\mathrm{D}_\sigma)$. *Moreover, there is a natural commutative diagram*

$$\begin{array}{ccc}
\overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}) & \xrightarrow[\sim]{(2.24)} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(\mathrm{M}_i,\mathrm{M}_i) \\
{\scriptstyle \mathfrak{T}_\sigma}\downarrow{\scriptstyle\sim} & & {\scriptstyle \mathfrak{T}_\sigma}\downarrow{\scriptstyle\sim} \\
\overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma,\mathrm{D}_\sigma) & \xrightarrow{\sim} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(\mathrm{M}_{i,\sigma},\mathrm{M}_{i,\sigma}).
\end{array}$$

2.4. *Hodge filtration and higher intertwining.* — Let $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$. The existence of S_n-distinct trianguline filtrations of D corresponds to an intertwining phenomenon on the automorphic side. We adapt the term "intertwining" to describe the non-uniqueness of saturated (φ,Γ)-submodules in such modules. Analogously, *higher intertwining* in this section refers to the non-uniqueness of filtrations of saturated (φ,Γ)-submodules over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$ for a (φ,Γ)-module over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$. By Corollary 2.15 (2), higher intertwining relations exist for $\widetilde{\mathrm{D}} \in \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}},g'}(\mathrm{D},\mathrm{D})$. In this section, we show a special class of paraboline deformations of D admits higher intertwining (cf. Theorem 2.32 below). Moreover, the Hodge parameter, reinterpreted as in Section 2.2, can be revealed in such higher intertwining relations.

Let D_1, C_1 be as in Section 2.2. Let $\mathscr{F}$ be the filtration $\mathrm{D}_1 \subset \mathrm{D}$, and $\mathscr{G}$ be the filtration $\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_1}) \subset \mathrm{D}$, which correspond to the exact sequences (2.1) (2.2) respectively. By Proposition 2.13, we have $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D}) = \dim_{\mathrm{E}} \mathrm{Ext}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D}) = 1+(n^2-n+1)d_{\mathrm{K}}$. And there are natural surjections (identifying $\mathrm{Ext}^1_{\mathrm{K}^\times}(\delta,\delta)$ with $\mathrm{Hom}(\mathrm{K}^\times,\mathrm{E})$):

$$\begin{aligned}
\kappa_{\mathscr{F}} &= (\kappa_{\mathscr{F},1},\kappa_{\mathscr{F},2}) : \mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D}) \twoheadrightarrow \mathrm{Ext}^1(\mathrm{D}_1,\mathrm{D}_1) \times \mathrm{Hom}\big(\mathrm{K}^\times,\mathrm{E}\big), \\
\kappa_{\mathscr{G}} &= (\kappa_{\mathscr{G},1},\kappa_{\mathscr{G},2}) : \mathrm{Ext}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D}) \twoheadrightarrow \mathrm{Ext}^1(\mathrm{C}_1,\mathrm{C}_1) \times \mathrm{Hom}\big(\mathrm{K}^\times,\mathrm{E}\big).
\end{aligned} \tag{2.33}$$

We introduce certain subspaces of $\mathrm{Ext}^1(\mathrm{D}_1,\mathrm{D}_1)$ and $\mathrm{Ext}^1(\mathrm{C}_1,\mathrm{C}_1)$. For $\iota \in \mathrm{Hom}(\mathrm{D}_1,\mathrm{C}_1)$. Consider the pull-back and push-forward maps:

$$\iota^- : \mathrm{Ext}^1(\mathrm{C}_1,\mathrm{D}_1) \longrightarrow \mathrm{Ext}^1(\mathrm{D}_1,\mathrm{D}_1),\ \ \iota^+ : \mathrm{Ext}^1(\mathrm{C}_1,\mathrm{D}_1) \longrightarrow \mathrm{Ext}^1(\mathrm{C}_1,\mathrm{C}_1). \tag{2.34}$$

Set $\mathrm{Ext}^1_\iota(\mathrm{D}_1,\mathrm{D}_1) := \iota^-(\mathrm{Ext}^1(\mathrm{C}_1,\mathrm{D}_1))$, $\mathrm{Ext}^1_\iota(\mathrm{C}_1,\mathrm{C}_1) := \iota^+(\mathrm{Ext}^1(\mathrm{C}_1,\mathrm{D}_1))$.

Lemma **2.28.** — *Suppose* $\dim_{\mathrm{E}} \mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1) = 2$, *and for* $i \in \{1, \dots, n-1\}$, *let* α_i *be as in (2.3). We have* $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\alpha_i}(\mathrm{D}_1, \mathrm{D}_1) = (n-1)(n-2)d_{\mathrm{K}}$. *Moreover for* $j \in \{1, \dots, n-1\}$, $j \neq i$,

$$\dim_{\mathrm{E}}\big(\mathrm{Ext}^1_{\alpha_i}(\mathrm{D}_1, \mathrm{D}_1) \cap \mathrm{Ext}^1_{\alpha_j}(\mathrm{D}_1, \mathrm{D}_1)\big) = (n-1)(n-3)d_{\mathrm{K}} + d_{\mathrm{K}} - 1.$$

Consequently, $\dim_{\mathrm{E}}(\mathrm{Ext}^1_{\alpha_i}(\mathrm{D}_1, \mathrm{D}_1) + \mathrm{Ext}^1_{\alpha_j}(\mathrm{D}_1, \mathrm{D}_1)) = 1 + n(n-2)d_{\mathrm{K}}$. *Finally, the same statement holds with* D_1 *replaced by* C_1.

Proof. — We only prove it for D_1, C_1 being similar. Fix the refinement $(\phi_1, \dots, \phi_{n-1})$ of D_1 and C_1. Let $\mathbf{r} := \{1, \dots, n-1\} \setminus \{i\}$. The map α_i^- factors through $\mathrm{Ext}^1(\mathrm{C}_1, \mathrm{D}_1) \twoheadrightarrow \mathrm{Ext}^1((\mathrm{D}_1)^{\mathbf{r}}, \mathrm{D}_1) \hookrightarrow \mathrm{Ext}^1(\mathrm{D}_1, \mathrm{D}_1)$ where the corresponding surjectivity and injectivity follow easily by dévissage. So $\mathrm{Ext}^1_{\alpha_i}(\mathrm{D}_1, \mathrm{D}_1)$ is just the image of $\mathrm{Ext}^1((\mathrm{D}_1)^{\mathbf{r}}, \mathrm{D}_1)$ in $\mathrm{Ext}^1(\mathrm{D}_1, \mathrm{D}_1)$, and is the kernel of the natural pull-back map $\kappa_i : \mathrm{Ext}^1(\mathrm{D}_1, \mathrm{D}_1) \to \mathrm{Ext}^1(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_i z^{\mathbf{h}_1}), \mathrm{D}_1)$. We directly calculate $\dim_{\mathrm{E}} \mathrm{Ext}^1((\mathrm{D}_1)^{\mathbf{r}}, \mathrm{D}_1) = (n-1)(n-2)d_{\mathrm{K}}$, and the first part follows. For $i \neq j$, consider the following composition (of natural pull-back maps)

$$\begin{aligned}(\mathbf{2.35}) \qquad \mathrm{Ext}^1(\mathrm{D}_1, \mathrm{D}_1) &\xrightarrow{\kappa_{i,j}} \mathrm{Ext}^1\big((\mathrm{D}_1)_{\{i,j\}}, \mathrm{D}_1\big) \\ &\xrightarrow{f_{i,j}} \mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_i z^{\mathbf{h}_1}\big) \oplus \mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_j, z^{\mathbf{h}_1}\big), \mathrm{D}_1\big) \\ &\cong \mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_i z^{\mathbf{h}_1}\big), \mathrm{D}_1\big) \oplus \mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}\big(\phi_j z^{\mathbf{h}_1}\big), \mathrm{D}_1\big),\end{aligned}$$

whose kernel is clearly $\mathrm{Ext}^1_{\alpha_i}(\mathrm{D}_1, \mathrm{D}_1) \cap \mathrm{Ext}^1_{\alpha_j}(\mathrm{D}_1, \mathrm{D}_1)$. By dévissage, $\kappa_{i,j}$ is surjective and $\mathrm{Ker}(\kappa_{i,j}) \cong \mathrm{Ext}^1((\mathrm{D}_1)^{i,j}, \mathrm{D}_1)$, hence has dimension equal to $(n-3)(n-1)d_{\mathrm{K}}$. Let $\mathrm{M}_1 := \mathrm{D}_1 \otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}}} (\mathrm{D}_1)^\vee_{\{i,j\}}$ and $\mathrm{M}_2 := \mathrm{D}_1 \otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}}} (\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_i^{-1} z^{-\mathbf{h}_1}) \oplus \mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_j^{-1} z^{-\mathbf{h}_1}))$. By dévissage, we have

$$\begin{aligned}0 &\to \mathrm{H}^0_{(\varphi,\Gamma)}(\mathrm{M}_1) \to \mathrm{H}^0_{(\varphi,\Gamma)}(\mathrm{M}_2) \to \mathrm{H}^0_{(\varphi,\Gamma)}(\mathrm{M}_2/\mathrm{M}_1) \\ &\to \mathrm{H}^1_{(\varphi,\Gamma)}(\mathrm{M}_1) \to \mathrm{H}^1_{(\varphi,\Gamma)}(\mathrm{M}_2)\end{aligned}$$

where the last map coincides with $f_{i,j}$ in (2.35). We have $\dim_{\mathrm{E}} \mathrm{H}^0_{(\varphi,\Gamma)}(\mathrm{M}_1) = 1$, $\dim_{\mathrm{E}} \mathrm{H}^0_{(\varphi,\Gamma)}(\mathrm{M}_2) = 2$, and by Lemma 2.1, $\dim_{\mathrm{E}} \mathrm{H}^0_{(\varphi,\Gamma)}(\mathrm{M}_2/\mathrm{M}_1) = \dim_{\mathrm{E}} \mathrm{D}^+_{\mathrm{dR}}(\mathrm{M}_2) - \dim_{\mathrm{E}} \mathrm{D}^+_{\mathrm{dR}}(\mathrm{M}_1) = 2(n-1)d_{\mathrm{K}} - (n-1+n-2)d_{\mathrm{K}} = d_{\mathrm{K}}$. So $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\alpha_i}(\mathrm{D}_1, \mathrm{D}_1) \cap \mathrm{Ext}^1_{\alpha_j}(\mathrm{D}_1, \mathrm{D}_1) = \dim_{\mathrm{E}} \mathrm{Ker}(\kappa_{i,j}) + \dim_{\mathrm{E}} \mathrm{Ker}(f_{i,j}) = (n-1)(n-3)d_{\mathrm{K}} + d_{\mathrm{K}} - 1$. This proves the second part of the lemma. □

Proposition **2.29.** — *Let* $\iota \in \mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1)$ *be an injection.*

(1) $\dim_{\mathrm{E}} \mathrm{Ext}^1_\iota(\mathrm{D}_1, \mathrm{D}_1) = \dim_{\mathrm{E}} \mathrm{Ext}^1_\iota(\mathrm{C}_1, \mathrm{C}_1) = 1 + (n-1)(n-2)d_{\mathrm{K}}$.

(2) $\mathrm{Ext}^1_g(\mathrm{D}_1, \mathrm{D}_1) \subset \mathrm{Ext}^1_\iota(\mathrm{D}_1, \mathrm{D}_1)$ *and* $\mathrm{Ext}^1_g(\mathrm{C}_1, \mathrm{C}_1) \subset \mathrm{Ext}^1_\iota(\mathrm{C}_1, \mathrm{C}_1)$.

(3) For $\iota' \in \mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1)$, $\mathrm{Ext}^1_{\iota'}(\mathrm{D}_1, \mathrm{D}_1) = \mathrm{Ext}^1_\iota(\mathrm{D}_1, \mathrm{D}_1)$ *if and only if* $\mathrm{Ext}^1_{\iota'}(\mathrm{C}_1, \mathrm{C}_1) = \mathrm{Ext}^1_\iota(\mathrm{C}_1, \mathrm{C}_1)$ *if and only if* $\iota' = a\iota$ *for some* $a \in \mathrm{E}^\times$.

Proof. — We only prove it for D_1 with C_1 being similar.

(1) By dévissage, we have

$$0 \longrightarrow H^0_{(\varphi,\Gamma)}(D_1 \otimes_{\mathcal{R}_{K,E}} D_1^\vee) \longrightarrow H^0_{(\varphi,\Gamma)}\big((D_1 \otimes_{\mathcal{R}_{K,E}} D_1^\vee)/(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee)\big) \longrightarrow H^1_{(\varphi,\Gamma)}(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee) \longrightarrow H^1_{(\varphi,\Gamma)}(D_1 \otimes_{\mathcal{R}_{K,E}} D_1^\vee),$$

where the last map can be identified with ι^-. By Lemma 2.1, we have

$$\begin{aligned}\dim_E H^0_{(\varphi,\Gamma)}\big((D_1 \otimes_{\mathcal{R}_{K,E}} D_1^\vee)/(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee)\big) &= \dim_E D^+_{\mathrm{dR}}(D_1 \otimes_{\mathcal{R}_{K,E}} D_1^\vee) - \dim_E D^+_{\mathrm{dR}}(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee)\\ &= \frac{n(n-1)}{2} d_K - \frac{(n-1)(n-2)}{2} d_K = (n-1)d_K.\end{aligned}$$

Hence $\dim_E \operatorname{Im} \iota^- = (n-1)^2 d_K - (n-1)d_K + 1 = 1 + (n-1)(n-2)d_K$.

(2) The map ι^- clearly induces $\iota_g^- : \mathrm{Ext}^1_g(C_1, D_1) \to \mathrm{Ext}^1_g(D_1, D_1)$. For any $M \in \mathrm{Ker}(\iota^-)$, $D_1 \oplus D_1 \subset M$ implies M is de Rham. So $\mathrm{Ker}\, \iota^- \subset \mathrm{Ext}^1_g(C_1, D_1)$ and is equal to $\mathrm{Ker}\, \iota_g^-$. By [31, Cor. A.4] applied to the B-pair associated to $D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee$ (which satisfies the assumptions of *loc. cit.* by the generic assumption on D), we have $\dim_E H^1_g(D_1 \otimes_{\mathcal{R}_{K,E}} C_1^\vee) = (n-1)^2 d_K - \frac{(n-1)(n-2)}{2} d_K = \frac{n(n-1)}{2} d_K$. Together with $\dim_E \mathrm{Ext}^1_g(D_1, D_1) = 1 + \frac{(n-1)(n-2)}{2} d_K$ (cf. Proposition 2.10 (1)) and (1), we see ι_g^- is surjective.

(3) The case where $\dim_E \mathrm{Hom}(D_1, C_1) = 1$ is trivial. Assume henceforth $\dim_E \mathrm{Hom}(D_1, C_1) = 2$ (which implies $n \geq 3$ and Lemma 2.28 can apply). Suppose $\iota' \notin E[\iota]$, then ι' and ι form a basis of $\mathrm{Hom}(D_1, C_1)$. If $\mathrm{Ext}^1_{\iota'}(D_1, D_1) = \mathrm{Ext}^1_\iota(D_1, D_1)$, we then easily deduce $\mathrm{Ext}^1_{\alpha_i}(D_1, D_1) \subset \mathrm{Ext}^1_\iota(D_1, D_1)$ for all $i = \{1, \dots, n-1\}$. However, for $i \neq j$, by Lemma 2.28, $\dim_E(\mathrm{Ext}^1_{\alpha_i}(D_1, D_1) + \mathrm{Ext}^1_{\alpha_j}(D_1, D_1)) = 1 + n(n-2)d_K > \dim_E \mathrm{Ext}^1_\iota(D_1, D_1)$, a contradiction. □

Let T_1 be the torus subgroup of GL_{n-1}, and $\phi^1 := \phi_1 \boxtimes \cdots \boxtimes \phi_{n-1}$. Let $\mathbf{h}^1 := (\mathbf{h}_1, \dots, \mathbf{h}_{n-1})$, and $\mathbf{h}^2 := (\mathbf{h}_2, \dots, \mathbf{h}_n)$. For the refinement ϕ^1 (of D_1 and C_1), we have maps

$$\mathrm{Ext}^1_g(D_1, D_1) \xrightarrow{\kappa_{\phi^1}} \mathrm{Hom}_{\mathrm{sm}}(T_1(K), E),$$

$$\mathrm{Ext}^1_g(C_1, C_1) \xrightarrow{\kappa_{\phi^1}} \mathrm{Hom}_{\mathrm{sm}}(T_1(K), E).$$

Lemma **2.30.** — *For* $M \in \mathrm{Ext}^1_g(C_1, D_1)$, $\kappa_{\phi^1} \circ \iota_g^-(M) = \kappa_{\phi^1} \circ \iota_g^+(M)$, *where* $\iota_g^\pm$ *is the restriction of* $\iota^\pm$ *to* $\mathrm{Ext}^1_g(C_1, D_1)$ *(see the proof of Proposition 2.29 (2)).*

Proof. — By definition, there is a natural injection $\tilde{\iota}: \iota_g^-(\mathrm{M}) \hookrightarrow \iota_g^+(\mathrm{M})$ which sits in the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{D}_1 & \longrightarrow & \iota_g^-(\mathrm{M}) & \longrightarrow & \mathrm{D}_1 & \longrightarrow & 0 \\ & & \downarrow \iota & & \downarrow \tilde{\iota} & & \downarrow \iota & & \\ 0 & \longrightarrow & \mathrm{C}_1 & \longrightarrow & \iota_g^+(\mathrm{M}) & \longrightarrow & \mathrm{C}_1 & \longrightarrow & 0. \end{array}$$

Moreover, $\tilde{\iota}$ is $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$-linear if $\iota_g^-(\mathrm{M})$ and $\iota_g^+(\mathrm{M})$ are equipped with the natural $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$-action. Suppose

$$\kappa_{\phi^1} \circ \iota_g^-(\mathrm{M}) = (\psi_1, \dots, \psi_{n-1}), \quad \kappa_{\phi^1} \circ \iota_g^+(\mathrm{M}) = \big(\psi_1', \dots, \psi_{n-1}'\big).$$

Then $\iota_g^-(\mathrm{M})$ (resp. $\iota_g^+(\mathrm{M})$) is isomorphic, as (φ,Γ)-module over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$, to a successive extension of $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(\phi_i z^{\mathbf{h}_i}(1+\psi_i\epsilon))$ (resp. $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(\phi_i z^{\mathbf{h}_{i+1}}(1+\psi_i'\epsilon))$) for $i = 1, \dots, n-1$. One sees inductively that $\tilde{\iota}$ induces injections $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(\phi_i z^{\mathbf{h}_i}(1+\psi_i\epsilon)) \hookrightarrow \mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(\phi_i z^{\mathbf{h}_{i+1}}(1+\psi_i'\epsilon))$ of (φ,Γ)-modules over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$. Hence $\psi_i = \psi_i'$ for all i. □

We fix crystabelline (φ,Γ)-modules D_1 and C_1, where D_1 has Hodge-Tate-Sen weights $\mathbf{h}^1$ and C_1 has weights $\mathbf{h}^2$, and both have a generic refinement ϕ^1. Denote by $\Phi\Gamma_{\mathrm{nc}}(\mathrm{D}_1, \mathrm{C}_1, \phi_n) \subset \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ the subset of isomorphism classes of (φ,Γ)-modules D such that $\mathrm{Hom}(\mathrm{D}_1, \mathrm{D}) = \mathrm{Hom}(\mathrm{D}, \mathrm{C}_1) \cong \mathrm{E}$. Assume $\Phi\Gamma_{\mathrm{nc}}(\mathrm{D}_1, \mathrm{C}_1, \phi_n)$ is non-empty. For an injection $\iota \in \mathrm{Hom}(\mathrm{D}_1, \mathrm{C}_1)$, we set $\mathscr{I}_\iota$ to be the following set

$$\begin{aligned}(\mathbf{2.36}) \qquad & \big\{(\widetilde{\mathrm{D}}_1, \widetilde{\mathrm{C}}_1) \in \mathrm{Ext}^1_\iota(\mathrm{D}_1, \mathrm{D}_1) \times \mathrm{Ext}^1_\iota(\mathrm{C}_1, \mathrm{C}_1) \mid \\ & \quad \exists \mathrm{M} \in \mathrm{Ext}^1(\mathrm{C}_1, \mathrm{D}_1) \text{ s.t. } \iota^-(\mathrm{M}) = \widetilde{\mathrm{D}}_1, -\iota^+(\mathrm{M}) = \widetilde{\mathrm{C}}_1\big\}.\end{aligned}$$

If $\iota = \iota_{\mathrm{D}}$ for some $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\mathrm{D}_1, \mathrm{C}_1, \phi_n)$, we write $\mathscr{I}_{\mathrm{D}} := \mathscr{I}_{\iota_{\mathrm{D}}}$. The following corollary is a direct consequence of Proposition 2.29 (3) and Proposition 2.4.

Corollary **2.31.** — *We have $\mathscr{I}_\iota = \mathscr{I}_{\iota'}$ if and only if $\iota' = a\iota$ for some $a \in \mathrm{E}^\times$. In particular, for $\mathrm{D}, \mathrm{D}' \in \Phi\Gamma_{\mathrm{nc}}(\mathrm{D}_1, \mathrm{C}_1, \phi_n)$ we have $\mathscr{I}_{\mathrm{D}} = \mathscr{I}_{\mathrm{D}'}$ if and only if $\iota_{\mathrm{D}} = a\iota_{\mathrm{D}'}$ for $a \in \mathrm{E}^\times$. When $\mathrm{K} = \mathbf{Q}_p$, this is equivalent to $\mathrm{D} \cong \mathrm{D}'$.*

Theorem **2.32** *(Higher intertwining). — Let $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\mathrm{D}_1, \mathrm{C}_1, \phi_n)$ and $\widetilde{\mathrm{D}} \in \mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D}, \mathrm{D})$ with $\kappa_{\mathscr{F}}(\widetilde{\mathrm{D}}) = (\widetilde{\mathrm{D}}_1, \psi)$ (cf. (2.33)). The followings are equivalent:*

(1) $\widetilde{\mathrm{D}} \in \mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D}, \mathrm{D}) \cap \mathrm{Ext}^1_{\mathscr{G}}(\mathrm{D}, \mathrm{D})$.
(2) $\widetilde{\mathrm{D}}_1 \otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}} \mathcal{R}_{\mathrm{E}[\epsilon]/\epsilon^2}(1 - \psi\epsilon) \in \mathrm{Ext}^1_{\iota_{\mathrm{D}}}(\mathrm{D}_1, \mathrm{D}_1)$.

Moreover, if the equivalent conditions hold, then $\kappa_{\mathscr{G},2}(\widetilde{\mathrm{D}}) = \psi$ and there exists $\mathrm{M} \in \mathrm{Ext}^1(\mathrm{C}_1, \mathrm{D}_1)$ such that $\widetilde{\mathrm{D}}_1 = \iota_{\mathrm{D}}^-(\mathrm{M}) \otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}} \mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(1+\psi\epsilon)$ and $\kappa_{\mathscr{G},1}(\widetilde{\mathrm{D}}) = \iota_{\mathrm{D}}^+(\mathrm{M}) \otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}} \mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(1+\psi\epsilon)$.

Proof. — Twisting $\widetilde{\mathrm{D}}$ by $1-\psi\epsilon$, we can and do assume $\kappa_{\mathscr{F},2}(\widetilde{\mathrm{D}})=0$. By definition, $\widetilde{\mathrm{D}}\in \mathrm{Ext}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D})$ if and only if it lies in the kernel of the composition

$$\mathrm{Ext}^1(\mathrm{D},\mathrm{D})\longrightarrow \mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_1}),\mathrm{D}\big)\longrightarrow \mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_1}),\mathrm{C}_1\big). \tag{2.37}$$

Similarly, $\mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})$ is equal to the kernel of the composition

$$\mathrm{Ext}^1(\mathrm{D},\mathrm{D})\longrightarrow \mathrm{Ext}^1\big(\mathrm{D},\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_n})\big)\longrightarrow \mathrm{Ext}^1\big(\mathrm{D}_1,\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_n})\big).$$

By dévissage, one can deduce an exact sequence ($\mathcal{R}$ standing for $\mathcal{R}_{\mathrm{K},\mathrm{E}}$)

$$0\to \mathrm{Ext}^1(\mathrm{D},\mathrm{D}_1)\to \mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})\to \mathrm{Ext}^1\big(\mathcal{R}(\phi_n z^{\mathbf{h}_n}),\mathcal{R}(\phi_n z^{\mathbf{h}_n})\big)\to 0.$$

As $\kappa_{\mathscr{F},2}(\widetilde{\mathrm{D}})=0$, $\widetilde{\mathrm{D}}$ lies in the image of $\mathrm{Ext}^1(\mathrm{D},\mathrm{D}_1)\to \mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})$. Let $\mathrm{M}_1\in \mathrm{Ext}^1(\mathrm{D},\mathrm{D}_1)$ be the preimage of $\widetilde{\mathrm{D}}$. Consider the composition

$$\begin{aligned}\mathrm{Ext}^1(\mathrm{D},\mathrm{D}_1)\hookrightarrow \mathrm{Ext}^1(\mathrm{D},\mathrm{D})\to \mathrm{Ext}^1\big(\mathcal{R}(\phi_n z^{\mathbf{h}_1}),\mathrm{D}\big)\\ \to \mathrm{Ext}^1\big(\mathcal{R}(\phi_n z^{\mathbf{h}_1}),\mathrm{C}_1\big).\end{aligned}$$

It is straightforward to see it is equal to the composition

$$\mathrm{Ext}^1(\mathrm{D},\mathrm{D}_1)\longrightarrow \mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_1}),\mathrm{D}_1\big)\xrightarrow{\iota_{\mathrm{D}}} \mathrm{Ext}^1\big(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_1}),\mathrm{C}_1\big). \tag{2.38}$$

So $\widetilde{\mathrm{D}}$ lies in the kernel of (2.37) if and only if M_1 is sent to zero via (2.38). However, using dévissage, we see the kernel of ι_{D} in (2.38) is isomorphic to $\mathrm{H}^0_{(\varphi,\Gamma)}(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n^{-1}z^{-\mathbf{h}_1})\otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}}}(\mathrm{C}_1/\mathrm{D}_1))$, which, by Lemma 2.1, has dimension $\dim_{\mathrm{E}}\mathrm{D}^+_{\mathrm{dR}}(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n^{-1}z^{-\mathbf{h}_1})\otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}}}\mathrm{C}_1)-\dim_{\mathrm{E}}\mathrm{D}^+_{\mathrm{dR}}(\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n^{-1}z^{-\mathbf{h}_1})\otimes_{\mathcal{R}_{\mathrm{K},\mathrm{E}}}\mathrm{D}_1)=0$. So ι_{D} in (2.38) is injective. We see (under the assumption $\psi=0$) that (1) is equivalent to that M_1 lies in the kernel of the first map of (2.38), which is equal to $\mathrm{Ext}^1(\mathrm{C}_1,\mathrm{D}_1)$ by dévissage. This is furthermore equivalent to that $\widetilde{\mathrm{D}}_1$ lies in the image of the composition $\mathrm{Ext}^1(\mathrm{C}_1,\mathrm{D}_1)\hookrightarrow \mathrm{Ext}^1(\mathrm{D},\mathrm{D}_1)\to \mathrm{Ext}^1(\mathrm{D}_1,\mathrm{D}_1)$, which is no other than ι^-_{D}. The other parts are straightforward. □

Corollary **2.33.** — *We have* $\dim_{\mathrm{E}}(\mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})\cap \mathrm{Ext}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D}))=1+(n^2-2n+2)d_{\mathrm{K}}$. *Consequently, the following natural map is surjective:*

$$\mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})\oplus \mathrm{Ext}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D})\twoheadrightarrow \mathrm{Ext}^1(\mathrm{D},\mathrm{D}). \tag{2.39}$$

Proof. — By Theorem 2.32, $\dim_{\mathrm{E}}(\mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})\cap \mathrm{Ext}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D}))=\dim_{\mathrm{E}}\mathrm{Hom}(\mathrm{K}^\times,\mathrm{E})+\dim_{\mathrm{E}}\mathrm{Ext}^1_{\iota_{\mathrm{D}}}(\mathrm{D}_1,\mathrm{D}_1)+\dim_{\mathrm{E}}\mathrm{Ker}(\kappa_{\mathscr{F}})$ (cf. (2.33)). By Proposition 2.29 (1) and Proposition 2.13, it is equal to $(1+d_{\mathrm{K}})+(1+(n-1)(n-2)d_{\mathrm{K}})+(-1+(n-1)d_{\mathrm{K}})=1+(n^2-2n+2)d_{\mathrm{K}}$. Together with Proposition 2.13, we see $\dim_{\mathrm{E}}(\mathrm{Ext}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})+\mathrm{Ext}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D}))=2(1+(n(n-1)+1)d_{\mathrm{K}}-1+(n^2-2n+2)d_{\mathrm{K}}=1+n^2d_{\mathrm{K}}\overset{\text{Prop. 2.10(1)}}{=}\dim_{\mathrm{E}}\mathrm{Ext}^1(\mathrm{D},\mathrm{D})$. The second part follows. □

Let $V(D_1, C_1) := \big(\overline{\mathrm{Ext}}^1(D_1, D_1) \times \mathrm{Hom}(K^\times, E)\big) \oplus \big(\overline{\mathrm{Ext}}^1(C_1, C_1) \times \mathrm{Hom}(K^\times, E)\big)$ $\big(\xleftarrow[\sim]{(\kappa_{\mathscr{F}}, \kappa_{\mathscr{G}})} \overline{\mathrm{Ext}}^1_{\mathscr{F}}(D, D) \oplus \overline{\mathrm{Ext}}^1_{\mathscr{G}}(D, D)\big)$, and $\mathcal{L}(D, D_1, C_1)$ be the subspace consisting of those $\big((\overline{\widetilde{D}_1}, \psi), (\overline{\widetilde{C}_1}, -\psi)\big) \in V(D_1, C_1)$ such that $\big(\widetilde{D}_1 \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1-\psi\epsilon), \widetilde{C}_1 \otimes_{\mathcal{R}_{K,E[\epsilon]/\epsilon^2}} \mathcal{R}_{K,E[\epsilon]/\epsilon^2}(1+\psi\epsilon)\big) \in \mathscr{I}_D$ (cf. (2.36)).

Corollary **2.34.** — *(1) Let* $D, D' \in \Phi\Gamma_{nc}(D_1, C_1, \phi_n)$, $\mathcal{L}(D', D_1, C_1) = \mathcal{L}(D, D_1, C_1)$ *if and only if* $\iota_{D'} = a\iota_D$ *for some* $a \in E^\times$. *When* $K = \mathbf{Q}_p$, *this is equivalent to* $D \cong D'$.

(2) For $D \in \Phi\Gamma_{nc}(D_1, C_1, \phi_n)$, *there is a natural exact sequence*

$$0 \longrightarrow \mathcal{L}(D, D_1, C_1) \longrightarrow V(D_1, C_1) \longrightarrow \overline{\mathrm{Ext}}^1(D, D) \longrightarrow 0. \tag{2.40}$$

Proof. — (1): The "if" part is trivial. Suppose $\mathcal{L}(D', D_1, C_1) = \mathcal{L}(D, D_1, C_1)$. Let $\widetilde{D}_1 \in \mathrm{Ext}^1_{\iota_D}(D_1, D_1)$, $M \in \mathrm{Ext}^1(C_1, D_1)$ be a preimage of $\widetilde{D}_1$ (via ι_D^-) and $\widetilde{C}_1 := -\iota_D^+(M) \in \mathrm{Ext}^1_{\iota_D}(C_1, C_1)$. We have by definition and assumption

$$\big((\overline{\widetilde{D}_1}, 0), (\overline{\widetilde{C}_1}, 0)\big) \in \mathcal{L}(D, D_1, C_1) = \mathcal{L}\big(D', D_1, C_1\big).$$

There exists hence $\widetilde{D}_1' \in \mathrm{Ext}^1_{\iota_{D'}}(D_1, D_1)$ such that $[\widetilde{D}_1'] - [\widetilde{D}_1] \in \mathrm{Ext}^1_0(D_1, D_1)$. As $\mathrm{Ext}^1_0(D_1, D_1) \subset \mathrm{Ext}^1_{\iota_{D'}}(D_1, D_1)$ (by Proposition 2.29 (2)), this implies $\widetilde{D}_1 \in \mathrm{Ext}^1_{\iota'_D}(D_1, D_1)$. So $\mathrm{Ext}^1_{\iota_D}(D_1, D_1) \subset \mathrm{Ext}^1_{\iota_{D'}}(D_1, D_1)$ hence $\mathrm{Ext}^1_{\iota_{D'}}(D_1, D_1) = \mathrm{Ext}^1_{\iota_D}(D_1, D_1)$ by symmetry and $\iota_{D'} \in E^\times \iota_D$ by Proposition 2.29 (3).

(2) Quotienting (2.39) by $\mathrm{Ext}^1_0(D, D)$ yields a surjection $V(D_1, C_1) \twoheadrightarrow \overline{\mathrm{Ext}}^1(D, D)$. By Theorem 2.32, the kernel is exactly $\mathcal{L}(D, D_1, C_1)$. □

Now we consider $\Sigma_K \setminus \{\sigma\}$-de Rham deformations for general K. Let $D_{1,\sigma} = \mathfrak{T}_\sigma(D_1)$ and $C_{1,\sigma} = \mathfrak{T}_\sigma(C_1)$ (cf. (2.7)). Let $\iota_\sigma \in \mathrm{Hom}(D_{1,\sigma}, C_{1,\sigma})$. We have similar maps as in (2.34), which induce, by restricting to $\Sigma_K \setminus \{\sigma\}$-de Rham extension groups,

$$\iota_\sigma^- : \mathrm{Ext}^1_\sigma(C_{1,\sigma}, D_{1,\sigma}) \to \mathrm{Ext}^1_\sigma(D_{1,\sigma}, D_{1,\sigma}),$$
$$\iota_\sigma^+ : \mathrm{Ext}^1_\sigma(C_{1,\sigma}, D_{1,\sigma}) \to \mathrm{Ext}^1_\sigma(C_{1,\sigma}, C_{1,\sigma}).$$

Let $\mathrm{Ext}^1_{\iota_\sigma}(D_{1,\sigma}, D_{1,\sigma}) := \mathrm{Im}(\iota_\sigma^-)$, $\mathrm{Ext}^1_{\iota_\sigma}(C_{1,\sigma}, C_{1,\sigma}) := \mathrm{Im}(\iota_\sigma^+)$. Denote by

$$\mathscr{I}_{\iota_\sigma} := \big\{(\widetilde{D}_{1,\sigma}, \widetilde{C}_{1,\sigma}) \in \mathrm{Ext}^1_{\iota_\sigma}(D_{1,\sigma}, D_{1,\sigma}) \times \mathrm{Ext}^1_{\iota_\sigma}(C_{1,\sigma}, C_{1,\sigma}) \mid \exists M \in \mathrm{Ext}^1_\sigma(C_{1,\sigma}, D_{1,\sigma}) \text{ with } \iota_\sigma^-(M) = \widetilde{D}_{1,\sigma}, -\iota_\sigma^+(M) = \widetilde{C}_{1,\sigma}\big\}. \tag{2.41}$$

Similarly as in Proposition 2.29, we have:

Proposition **2.35.** — *Let* $\iota_\sigma \in \mathrm{Hom}(D_{1,\sigma}, C_{1,\sigma})$ *be an injection.*

(1) $\dim_E \mathrm{Ext}^1_{\iota_\sigma}(D_{1,\sigma}, D_{1,\sigma}) = \dim_E \mathrm{Ext}^1_{\iota_\sigma}(C_{1,\sigma}, C_{1,\sigma}) = 1 + (n-1)(n-2)$.

(2) $\mathrm{Ext}^1_g(\mathrm{D}_{1,\sigma},\mathrm{D}_{1,\sigma})\subset \mathrm{Ext}^1_{\iota_\sigma}(\mathrm{D}_{1,\sigma},\mathrm{D}_{1,\sigma})$ *and* $\mathrm{Ext}^1_g(\mathrm{C}_{1,\sigma},\mathrm{C}_{1,\sigma})\subset \mathrm{Ext}^1_{\iota_\sigma}(\mathrm{C}_{1,\sigma},\mathrm{C}_{1,\sigma})$.

(3) For $\iota'_\sigma\in \mathrm{Hom}(\mathrm{D}_{1,\sigma},\mathrm{C}_{1,\sigma})$, $\mathrm{Ext}^1_{\iota'_\sigma}(\mathrm{D}_{1,\sigma},\mathrm{D}_{1,\sigma})=\mathrm{Ext}^1_{\iota_\sigma}(\mathrm{D}_{1,\sigma},\mathrm{D}_{1,\sigma})$ *if and only if* $\mathrm{Ext}^1_{\iota'_\sigma}(\mathrm{C}_{1,\sigma},\mathrm{C}_{1,\sigma})=\mathrm{Ext}^1_{\iota_\sigma}(\mathrm{C}_{1,\sigma},\mathrm{C}_{1,\sigma})$ *if and only if* $\iota'_\sigma=a\iota_\sigma$ *for some* $a\in \mathrm{E}^\times$.

Proof. — We still only prove the statements for $\mathrm{D}_{1,\sigma}$. By [31, Cor. A.4], $\dim_{\mathrm{E}}\mathrm{Ext}^1_\sigma(\mathrm{C}_{1,\sigma},\mathrm{D}_{1,\sigma})=(n-1)^2 d_{\mathrm{K}}-\sum_{\tau\in\Sigma_{\mathrm{K}}\setminus\{\sigma\}}\dim_{\mathrm{E}}\mathrm{D}^+_{\mathrm{dR}}(\mathrm{D}_{1,\sigma}\otimes_{\mathcal{R}_{\mathrm{K,E}}}\mathrm{C}^\vee_{1,\sigma})_\tau=(n-1)^2$. By similar arguments as in the proof of Proposition 2.29 (1), the kernel of $\mathrm{Ext}^1(\mathrm{C}_{1,\sigma},\mathrm{D}_{1,\sigma})\to\mathrm{Ext}^1(\mathrm{D}_{1,\sigma},\mathrm{D}_{1,\sigma})$ has dimension $(n-1)-1$. But any element in this kernel contains $\mathrm{D}_{1,\sigma}\oplus\mathrm{D}_{1,\sigma}$ hence is de Rham. We see it is the same as $\mathrm{Ker}\,\iota^-_\sigma$ and $\mathrm{Ker}(\iota^-_\sigma|_{\mathrm{Ext}^1_g(\mathrm{C}_{1,\sigma},\mathrm{D}_{1,\sigma})})$. (1) follows. Using [31, Cor. A.4], $\dim_{\mathrm{E}}\mathrm{Ext}^1_g(\mathrm{C}_{1,\sigma},\mathrm{D}_{1,\sigma})=\frac{n(n-1)}{2}$. Together with Proposition 2.20 (2) and comparing dimensions, the induced map $\mathrm{Ext}^1_g(\mathrm{C}_{1,\sigma},\mathrm{D}_{1,\sigma})\to\mathrm{Ext}^1_g(\mathrm{D}_{1,\sigma},\mathrm{D}_{1,\sigma})$ is surjective. (2) follows. (3) follows from similar arguments as in the proof of Proposition 2.29 (3) using an analogue of Lemma 2.28 for $\mathrm{Ext}^1_{\alpha_{i,\sigma}}$ with $\alpha_{i,\sigma}$ given as in (2.8) (when $\dim_{\mathrm{E}}\mathrm{Hom}(\mathrm{D}_{1,\sigma},\mathrm{C}_{1,\sigma})=2$). Note the dévissage arguments in the proof of Lemma 2.28 work when Ext^1's are all replaced by Ext^1_σ's, by [31, Prop. A.5]. We leave the details to the reader. □

For $\mathrm{D}_\sigma\in\Phi\Gamma_{\mathrm{nc}}(\mathrm{D}_{1,\sigma},\mathrm{C}_{1,\sigma},\phi_n)$ (which is the subset of $\Phi\Gamma_{\mathrm{nc}}(\phi,\mathscr{T}_\sigma(\mathbf{h}))$ defined similarly as $\Phi\Gamma_{\mathrm{nc}}(\mathrm{D}_1,\mathrm{C}_1,\phi_n)$), set $\mathscr{I}_{\mathrm{D}_\sigma}:=\mathscr{I}_{\iota_{\mathrm{D}_\sigma}}$ (cf. (2.41)) where $\iota_{\mathrm{D}_\sigma}$ is the composition $\mathrm{D}_{1,\sigma}\hookrightarrow\mathrm{D}_\sigma\twoheadrightarrow\mathrm{C}_{1,\sigma}$. We have by Proposition 2.35 (3) and Proposition 2.8:

Corollary **2.36.** — *For* $\mathrm{D}_\sigma,\mathrm{D}'_\sigma\in\Phi\Gamma_{\mathrm{nc}}(\mathrm{D}_{1,\sigma},\mathrm{C}_{1,\sigma},\phi_n)$, *we have* $\mathscr{I}_{\mathrm{D}_\sigma}=\mathscr{I}_{\mathrm{D}'_\sigma}$ *if and only if* $\mathrm{D}_\sigma\cong\mathrm{D}'_\sigma$.

Consider $\kappa_{\mathscr{F}}:\mathrm{Ext}^1_{\sigma,\mathscr{F}}(\mathrm{D}_\sigma,\mathrm{D}_\sigma)\to\mathrm{Ext}^1_\sigma(\mathrm{D}_{1,\sigma},\mathrm{D}_{1,\sigma})\times\mathrm{Hom}_\sigma(\mathrm{K}^\times,\mathrm{E})$, and $\kappa_{\mathscr{G}}:\mathrm{Ext}^1_{\sigma,\mathscr{G}}(\mathrm{D}_\sigma,\mathrm{D}_\sigma)\to\mathrm{Ext}^1_\sigma(\mathrm{C}_{1,\sigma},\mathrm{C}_{1,\sigma})\times\mathrm{Hom}_\sigma(\mathrm{K}^\times,\mathrm{E})$ (cf. (2.27) and (2.25)). The following theorem follows by the same argument as in the proof of Theorem 2.32 (note that all the dévissage arguments used in *loc. cit.* work if Ext^1's are all replaced by Ext^1_σ's by [31, Prop. A.5]).

Theorem **2.37.** — *Let* $\widetilde{\mathrm{D}}_\sigma\in\mathrm{Ext}^1_{\sigma,\mathscr{F}}(\mathrm{D}_\sigma,\mathrm{D}_\sigma)$ *with* $\kappa_{\mathscr{F}}(\widetilde{\mathrm{D}}_\sigma)=(\widetilde{\mathrm{D}}_{\sigma,1},\psi)$. *The followings are equivalent:*

(1) $\widetilde{\mathrm{D}}_\sigma\in\mathrm{Ext}^1_{\sigma,\mathscr{F}}(\mathrm{D}_\sigma,\mathrm{D}_\sigma)\cap\mathrm{Ext}^1_{\sigma,\mathscr{G}}(\mathrm{D}_\sigma,\mathrm{D}_\sigma)$,
(2) $\widetilde{\mathrm{D}}_{1,\sigma}\otimes_{\mathcal{R}_{\mathrm{K,E}[\epsilon]/\epsilon^2}}\mathcal{R}_{\mathrm{E}[\epsilon]/\epsilon^2}(1-\psi\epsilon)\in\mathrm{Ext}^1_{\iota_{\mathrm{D}_\sigma}}(\mathrm{D}_{1,\sigma},\mathrm{D}_{1,\sigma})$.

Moreover, if the equivalent conditions hold, then $\kappa_{\mathscr{G},2}(\widetilde{\mathrm{D}}_\sigma)=\psi$ *and there exists* $\mathrm{M}\in\mathrm{Ext}^1_{\mathfrak{g}}(\mathrm{C}_{1,\sigma},\mathrm{D}_{1,\sigma})$ *such that* $\widetilde{\mathrm{D}}_{1,\sigma}\cong\iota^-_{\mathrm{D}_\sigma}(\mathrm{M})\otimes_{\mathcal{R}_{\mathrm{K,E}[\epsilon]/\epsilon^2}}\mathcal{R}_{\mathrm{K,E}[\epsilon]/\epsilon^2}(1+\psi\epsilon)$ *and* $\kappa_{\mathscr{G},1}(\widetilde{\mathrm{D}}_\sigma)=\iota^+_{\mathrm{D}_\sigma}(\mathrm{M})\otimes_{\mathcal{R}_{\mathrm{K,E}[\epsilon]/\epsilon^2}}\mathcal{R}_{\mathrm{K,E}[\epsilon]/\epsilon^2}(1+\psi\epsilon)$.

Set $\mathrm{V}(\mathrm{D}_{1,\sigma},\mathrm{C}_{1,\sigma})_\sigma:=\big(\overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}_{1,\sigma},\mathrm{D}_{1,\sigma})\times\mathrm{Hom}_\sigma(\mathrm{K}^\times,\mathrm{E})\big)\oplus\big(\overline{\mathrm{Ext}}^1_\sigma(\mathrm{C}_{1,\sigma},\mathrm{C}_{1,\sigma})\times\mathrm{Hom}_\sigma(\mathrm{K}^\times,\mathrm{E})\big)$ and $\mathcal{L}(\mathrm{D}_\sigma,\mathrm{D}_{1,\sigma},\mathrm{D}_{2,\sigma})_\sigma$ to be the subspace consisting of those $\big((\overline{\widetilde{\mathrm{D}}_{1,\sigma}},\psi),$

$(\widetilde{\mathrm{C}}_{1,\sigma}, -\psi)\big) \in \mathrm{V}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})_\sigma$ such that (cf. (2.41))

$$\big(\widetilde{\mathrm{D}}_{1,\sigma} \otimes \mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(1-\psi\epsilon), \widetilde{\mathrm{C}}_{1,\sigma} \otimes \mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}(1+\psi\epsilon)\big) \in \mathscr{I}_{\mathrm{D}_\sigma}.$$

By Proposition 2.8 and the same arguments as in Corollary 2.34, we have:

Corollary **2.38.** — *(1) Let* $\mathrm{D}_\sigma, \mathrm{D}'_\sigma \in \Phi\Gamma_{\mathrm{nc}}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma}, \phi_n)$*, then* $\mathcal{L}(\mathrm{D}'_\sigma, \mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma}) = \mathcal{L}(\mathrm{D}_\sigma, \mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})$ *if and only if* $\mathrm{D}_\sigma \cong \mathrm{D}'_\sigma$.

(2) There is a natural exact sequence

$$0 \longrightarrow \mathcal{L}(\mathrm{D}_\sigma, \mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma}) \longrightarrow \mathrm{V}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})_\sigma \longrightarrow \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \longrightarrow 0. \tag{2.42}$$

Set $\mathrm{V}(\mathrm{D}_1, \mathrm{C}_1)_\sigma \subset \mathrm{V}(\mathrm{D}_1, \mathrm{C}_1)$ to be

$$\big(\overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}_1, \mathrm{D}_1) \times \mathrm{Hom}_\sigma(\mathrm{K}^\times, \mathrm{E})\big) \oplus \big(\overline{\mathrm{Ext}}^1_\sigma(\mathrm{C}_1, \mathrm{C}_1) \times \mathrm{Hom}_\sigma(\mathrm{K}^\times, \mathrm{E})\big),$$

and $\mathcal{L}(\mathrm{D}, \mathrm{D}_1, \mathrm{C}_1)_\sigma := \mathcal{L}(\mathrm{D}, \mathrm{D}_1, \mathrm{C}_1) \cap \mathrm{V}(\mathrm{D}_1, \mathrm{C}_1)_\sigma \subset \mathrm{V}(\mathrm{D}_1, \mathrm{C}_1)$. Note $\mathrm{V}(\mathrm{D}_1, \mathrm{C}_1)_\sigma \cong \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}}(\mathrm{D}, \mathrm{D}) \oplus \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{G}}(\mathrm{D}, \mathrm{D})$ by Proposition 2.17 (2).

Proposition **2.39.** — *The functor* $\mathfrak{T}_\sigma$ *induces a commutative diagram of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}(\mathrm{D}, \mathrm{D}_1, \mathrm{C}_1)_\sigma & \longrightarrow & \mathrm{V}(\mathrm{D}_1, \mathrm{C}_1)_\sigma & \longrightarrow & \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}, \mathrm{D}) & \longrightarrow & 0 \\ & & \mathfrak{T}_\sigma \big\downarrow \sim & & \mathfrak{T}_\sigma \big\downarrow \sim & & \mathfrak{T}_\sigma \big\downarrow \sim & & \\ 0 & \longrightarrow & \mathcal{L}(\mathrm{D}_\sigma, \mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})_\sigma & \longrightarrow & \mathrm{V}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})_\sigma & \longrightarrow & \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}_\sigma, \mathrm{D}_\sigma) & \longrightarrow & 0 \end{array}$$

where the top sequence is induced by (2.40).

Proof. — All the maps are clear, and we have seen in the above corollary that the bottom sequence is exact. The left exactness of the top sequence is clear. It is also exact in the middle because of the definition of $\mathcal{L}(\mathrm{D}, \mathrm{D}_1, \mathrm{C}_1)_\sigma$. By Corollary 2.27, the two right vertical maps are both isomorphisms. The proposition follows. □

Corollary **2.40.** — *The map (2.26) is surjective. And the same holds with* D *replaced by* D_σ.

Proof. — By the above proposition, $\overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}}(\mathrm{D}, \mathrm{D}) \oplus \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{G}}(\mathrm{D}, \mathrm{D}) \cong \mathrm{V}(\mathrm{D}_1, \mathrm{C}_1)_\sigma \to \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}, \mathrm{D})$ is surjective. Using Proposition 2.17 (2), Corollary 2.18 and induction on the rank n, one deduces $\oplus_{w\in S_n}\overline{\mathrm{Ext}}^1_{\sigma,w}(\mathrm{D}, \mathrm{D}) \to \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}, \mathrm{D})$ is surjective. As $\mathrm{Ext}^1_0(\mathrm{D}, \mathrm{D}) \subset \mathrm{Ext}^1_{\sigma,w}(\mathrm{D}, \mathrm{D})$ for any $w \in S_n$, we see (2.26) is also surjective. The statement for D_σ follows by similar arguments or using Corollary 2.27. □

3. Locally analytic crystabelline representations of $\mathrm{GL}_n(\mathrm{K})$

3.1. *Locally analytic representations of* $\mathrm{GL}_n(\mathrm{K})$ *and extensions.*

3.1.1. *Notation and preliminaries.* — We introduce some (more) notation on the GL_n-side. Recall T is the torus subgroup of GL_n, and $\mathrm{B} \supset \mathrm{T}$ is the Borel subgroup of upper triangular matrices. For a standard parabolic subgroup $\mathrm{P} \supset \mathrm{B}$ of GL_n, let $\mathrm{L_P} \supset \mathrm{T}$ be its standard Levi subgroup and P^- its opposite parabolic subgroup. Denote by $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{gl}_n$ the corresponding Lie algebras over K. Let $\theta := (0, \dots, 1-i, \dots, 1-n)$. For a parabolic subgroup P, let $n_i \in \mathbf{Z}_{\geq 1}$ such that the simple roots of $\mathrm{L_P}$ are given by $\{1, \dots, n-1\} \setminus \{n_1, n_1+n_2, \dots, n_1+\cdots+n_{r-1}\}$ (so $\mathrm{L_P} \cong \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \mathrm{GL}_{n_r}$). Let $\theta^{\mathrm{P}} := (\underbrace{0, \dots, 0}_{n_1}, \underbrace{-n_1, \dots, -n_1}_{n_2}, \dots, \underbrace{-(n_1+\cdots+n_{r-1}), \dots, -(n_1+\cdots+n_{r-1})}_{n_r})$ (so $\theta = \theta^{\mathrm{B}}$), that we view as an algebraic character of $\mathrm{L_P}$. For simplicity, for $i \in \{1, \dots, n-1\}$, we denote by P_i the associated maximal parabolic subgroup such that its standard Levi subgroup $\mathrm{L}_i \supset \mathrm{T}$ has simple roots $\{1, \dots, n-1\} \setminus \{i\}$.

For a Lie algebra $\mathfrak{g}$ over K, denote by $\mathfrak{g}_{\Sigma_\mathrm{K}} := \mathfrak{g} \otimes_{\mathbf{Q}_p} \mathrm{E} \cong \prod_{\sigma \in \Sigma_\mathrm{K}} \mathfrak{g} \otimes_{\mathrm{K},\sigma} \mathrm{E} =: \prod_{\sigma \in \Sigma_\mathrm{K}} \mathfrak{g}_\sigma$. For a weight μ of $\mathfrak{t}_{\Sigma_\mathrm{K}}$, denote by $\mathrm{M}^-(\mu) := \mathrm{U}(\mathfrak{gl}_{n,\Sigma_\mathrm{K}}) \otimes_{\mathrm{U}(\mathfrak{b}^-_{\Sigma_\mathrm{K}})} \mu$, and let $\mathrm{L}^-(\mu)$ be its unique simple quotient. If μ is anti-dominant (i.e. $\mu_{\sigma,1} < \mu_{\sigma,2} < \cdots < \mu_{\sigma,n}$ for all $\sigma \in \Sigma_\mathrm{K}$, where $\mu = (\mu_{\sigma,i})_{\substack{\sigma \in \Sigma_\mathrm{K} \\ i=1,\dots,n}}$), then $\mathrm{L}^-(\mu)$ is finite dimensional and isomorphic to the dual $\mathrm{L}(-\mu)^\vee$, where $\mathrm{L}(-\mu)$ is the algebraic representation of $\mathrm{Res}^{\mathrm{K}}_{\mathbf{Q}_p} \mathrm{GL}_n$ of highest weight $-\mu$ with respect to $\mathrm{Res}^{\mathrm{K}}_{\mathbf{Q}_p} \mathrm{B}$.

For an admissible locally $\mathbf{Q}_p$-analytic representation V of $\mathrm{GL}_n(\mathrm{K})$, by [56], its continuous dual $\mathrm{V}^\vee$ is naturally a module over the ($\mathbf{Q}_p$-analytic) distribution algebra $\mathcal{D}(\mathrm{GL}_n(\mathrm{K}), \mathrm{E})$, which, equipped with the strong topology, is a coadmissible module over $\mathcal{D}(\mathrm{H}, \mathrm{E})$ for a(ny) compact open subgroup H of $\mathrm{GL}_n(\mathrm{K})$. For admissible locally $\mathbf{Q}_p$-analytic representations V_1, V_2 of $\mathrm{GL}_n(\mathrm{K})$, set $\mathrm{Ext}^i_{\mathrm{GL}_n(\mathrm{K})}(\mathrm{V}_1, \mathrm{V}_2) := \mathrm{Ext}^i_{\mathcal{D}(\mathrm{GL}_n(\mathrm{K}),\mathrm{E})}(\mathrm{V}_2^\vee, \mathrm{V}_1^\vee)$, where the latter is defined in the abelian category of abstract $\mathcal{D}(\mathrm{GL}_n(\mathrm{K}), \mathrm{E})$-modules. By [13, Lem. 2.1.1], $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\mathrm{V}_1, \mathrm{V}_2)$ is equal to the extension group of admissible locally $\mathbf{Q}_p$-analytic representations of V_1 by V_2. If V_1, V_2 are locally algebraic, set $\mathrm{Ext}^1_{\mathrm{lalg}}(\mathrm{V}_1, \mathrm{V}_2)$ to be the subgroup of locally algebraic extensions. Any representation $\widetilde{\mathrm{V}}$ in $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\mathrm{V}, \mathrm{V})$ is equipped with a natural $\mathrm{E}[\epsilon]/\epsilon^2$ structure where ϵ acts via $\widetilde{\mathrm{V}} \twoheadrightarrow \mathrm{V} \xrightarrow{\mathrm{id}} \mathrm{V} \hookrightarrow \widetilde{\mathrm{V}}$.

Suppose $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\mathrm{V}_1, \mathrm{V}_2)$ is finite dimensional over E. For a subspace $\mathrm{U} \subset \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\mathrm{V}_1, \mathrm{V}_2)$, we can associate a tautological extension of $\mathrm{V}_1 \otimes_\mathrm{E} \mathrm{U}$ by V_2 (for example see the discussion below Theorem 1.3). When $\mathrm{U} = \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\mathrm{V}_1, \mathrm{V}_2)$, we call the corresponding extension the *universal* extension of V_1 (or $\mathrm{V}_1 \otimes_\mathrm{E} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\mathrm{V}_1, \mathrm{V}_2)$) by V_2.

Let $\phi = \phi_1 \boxtimes \cdots \boxtimes \phi_n : \mathrm{T}(\mathrm{K}) \to \mathrm{E}^\times$ be a smooth character. We call ϕ generic if $\phi_i \phi_j^{-1} \neq 1, |\cdot|_\mathrm{K}$ for $i \neq j$. For $w \in \mathrm{S}_n$, let $w(\phi) := \phi_{w^{-1}(1)} \boxtimes \cdots \boxtimes \phi_{w^{-1}(n)}$. Let $\delta_\mathrm{B} = |\cdot|_\mathrm{K}^{n-1} \boxtimes \cdots \boxtimes |\cdot|_\mathrm{K}^{n+1-2i} \boxtimes \cdots \boxtimes |\cdot|_\mathrm{K}^{1-n}$ be the modulus character of $\mathrm{B}(\mathrm{K})$ and $\eta := 1 \boxtimes |\cdot|_\mathrm{K} \boxtimes \cdots \boxtimes |\cdot|_\mathrm{K}^{n-1} = |\cdot|_\mathrm{K}^{-1} \circ \theta$. Let $\mathrm{I_{sm}}(\phi) := (\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} \phi\eta)^\infty$, which is an absolutely irreducible smooth admissible representation of $\mathrm{GL}_n(\mathrm{K})$ when ϕ is generic. Moreover, when ϕ is generic, $\mathrm{I_{sm}}(\phi) \cong \mathrm{I_{sm}}(w(\phi)) =: \pi_{\mathrm{sm}}(\phi)$ for all $w \in \mathrm{S}_n$, which is in fact the smooth representation of $\mathrm{GL}_n(\mathrm{K})$ corresponding to the Weil-Deligne representation $\oplus_{i=1}^n \phi_i$ in the classical local Langlands correspondence.

3.1.2. *Principal series.* — We collect some facts on the locally $\mathbf{Q}_p$-analytic principal series of $\mathrm{GL}_n(\mathrm{K})$.

Let $\mathbf{h}$ be a strictly dominant weight of $\mathfrak{t}_{\Sigma_{\mathrm{K}}}$, put $\lambda := \mathbf{h} - \theta^{[\mathrm{K}:\mathbf{Q}_p]} = (\lambda_{i,\sigma} = h_{i,\sigma} + i - 1)_{\substack{\sigma\in\Sigma_{\mathrm{K}}\\ i=1,\dots,n}}$, which is a dominant weight of $\mathfrak{t}$. Let ϕ be a generic smooth character of $\mathrm{T}(\mathrm{K})$. Put $\pi_{\mathrm{alg}}(\phi, \mathbf{h}) := \pi_{\mathrm{sm}}(\phi) \otimes_{\mathrm{E}} \mathrm{L}(\lambda)$ ($\cong \mathrm{I}_{\mathrm{sm}}(w(\phi)) \otimes_{\mathrm{E}} \mathrm{L}(\lambda)$ for all $w \in \mathrm{S}_n$), which is an irreducible locally algebraic representation of $\mathrm{GL}_n(\mathrm{K})$. For $w \in \mathrm{S}_n$, put $\mathrm{PS}(w(\phi), \mathbf{h}) := (\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} w(\phi)\eta z^{\lambda})^{\mathbf{Q}_p-\mathrm{an}} = (\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} w(\phi) z^{\mathbf{h}}(\varepsilon^{-1}\circ\theta))^{\mathbf{Q}_p-\mathrm{an}}$. We have (where $\mathcal{F}_{\mathrm{B}^-}^{\mathrm{GL}_n}(-,-)$ denotes Orlik-Strauch functor [55]):

Proposition **3.1.** — *Let* $w \in \mathrm{S}_n$.

(1) The irreducible constituents of $\mathrm{PS}(w(\phi), \boldsymbol{h})$ *are given by* $\{\mathscr{C}(w,u) := \mathcal{F}_{\mathrm{B}^-}^{\mathrm{GL}_n}(\mathrm{L}^-(-u\cdot\lambda), w(\phi)\eta)\}_{u=(u_\sigma)\in \mathrm{S}_n^{|\Sigma_{\mathrm{K}}|}}$, *which are pairwisely distinct. Moreover, if* $\lg(u) = 1$, *then* $\mathscr{C}(w,u)$ *has multiplicity one.*

(2) $\mathrm{soc}_{\mathrm{GL}_n(\mathrm{K})} \mathrm{PS}(w(\phi), \boldsymbol{h}) \cong \mathrm{I}_{\mathrm{sm}}(w(\phi)) \otimes_{\mathrm{E}} \mathrm{L}(\lambda) \cong \pi_{\mathrm{alg}}(\phi, \boldsymbol{h})$.

(3) $\mathrm{soc}_{\mathrm{GL}_n(\mathrm{K})}(\mathrm{PS}(w(\phi), \boldsymbol{h})/\pi_{\mathrm{alg}}(\phi, \boldsymbol{h})) \cong \oplus_{\substack{u\in \mathrm{S}_n^{|\Sigma_{\mathrm{K}}|},\\ \lg(u)=1}} \mathscr{C}(w,u)$.

(4) For $w' \in \mathrm{S}_n$, *and* $u, u' \in \mathrm{S}_n^{|\Sigma_{\mathrm{K}}|}$ *with* $\lg(u) = \lg(u') = 1$, $\mathscr{C}(w,u) \cong \mathscr{C}(w',u')$ *if and only if* $u = u' = s_{i,\sigma}$ *for some* $i \in \{1,\dots,n-1\}$ *and* $\sigma \in \Sigma_{\mathrm{K}}$, *and* $w(w')^{-1}$ *lies in the Weyl group of* $\mathrm{L}_{\mathrm{P}_i}$.

Proof. — (1) and (4) follow from [55, Thm.] (together with some standard facts on the constituents of the Verma module, see for example [43, Chap. 6]). (2) (3) follow from [54, Thm. 1]. □

For $i \in \{1,\dots,n-1\}$, let $\mathrm{I} \subset \{1,\dots,n\}$ be a subset of cardinality i. By Proposition 3.1 (4), all the representations $\mathscr{C}(w, s_{i,\sigma})$ with $w(\{1,\dots,i\}) = \mathrm{I}$ are isomorphic, which we denote by $\mathscr{C}(\mathrm{I}, s_{i,\sigma})$. Moreover, $\mathscr{C}(\mathrm{I}, s_{i,\sigma})$ are pairwisely distinct for different $s_{i,\sigma}$ or I. For $w \in \mathrm{S}_n$ with $w(\{1,\dots,i\}) = \mathrm{I}$, we have (by [54, Thm. 1])

$$\mathscr{C}(\mathrm{I}, s_{i,\sigma}) \cong \mathrm{soc}_{\mathrm{GL}_n(\mathrm{K})}\big(\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta\big)^{\mathbf{Q}_p-\mathrm{an}}. \tag{3.1}$$

Lemma **3.2.** — *Let* $w \in \mathrm{S}_n$ *such that* $w(\{1,\dots,i\}) = \mathrm{I}$.

(1) We have $\mathrm{Hom}_{\mathrm{T}(\mathbf{Q}_p)}(z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta\delta_{\mathrm{B}}, \mathrm{J}_{\mathrm{B}}(\mathscr{C}(\mathrm{I}, s_{i,\sigma}))) \cong \mathrm{E}$, *where* $\mathrm{J}_{\mathrm{B}}(-)$ *denotes the Jacquet-Emerton functor for* B *(cf. [36]).*

(2) We have $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta) \cong \mathscr{C}(\mathrm{I}, s_{i,\sigma})$, *where* $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(-)$ *is Emerton's induction functor [37, Section (2.8)].*

Proof. — By [55, Thm.], it is easy to see any irreducible constituent of $\mathrm{PS}(w(\phi), \mathbf{h})$ is a subrepresentation of a certain locally $\mathbf{Q}_p$-analytic principal series, hence is very strongly admissible by [37, Prop. 2.1.2]. (1) then follows by [12, Thm. 4.3, Rem. 4.4 (i)]. By *loc. cit.* and [54, Thm. 1], $\mathrm{Hom}_{\mathrm{T}(\mathbf{Q}_p)}(z^{-s_{i,\sigma}\cdot\lambda}w(\phi)\eta\delta_{\mathrm{B}}, \mathrm{J}_{\mathrm{B}}(\mathscr{C}))=0$ for any irreducible constituent $\mathscr{C}$ of $(\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} z^{-s_{i,\sigma}\cdot\lambda}w(\phi)\eta)^{\mathbf{Q}_p-\mathrm{an}}$ with $\mathscr{C}\neq\mathscr{C}(\mathrm{I}, s_{i,\sigma})$. The natural map $z^{-s_{i,\sigma}\cdot\lambda}w(\phi)\eta\delta_{\mathrm{B}}\hookrightarrow \mathrm{J}_{\mathrm{B}}((\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} z^{-s_{i,\sigma}\cdot\lambda}w(\phi)\eta)^{\mathbf{Q}_p-\mathrm{an}})$ hence has image contained in $\mathrm{J}_{\mathrm{B}}(\mathscr{C}(\mathrm{I}, s_{i,\sigma}))$. By definition of $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(-)$ (cf. [36, Section (2.8)]), (2) follows. □

Let $\mathrm{PS}_1(w(\phi),\mathbf{h})$ be the unique subrepresentation of $\mathrm{PS}(w(\phi),\mathbf{h})$ of socle $\mathrm{I}_{\mathrm{sm}}(w(\phi))\otimes_{\mathrm{E}}\mathrm{L}(\lambda)$ and cosocle $\oplus_{\substack{i=1,\dots,n-1\\ \sigma\in\Sigma_{\mathrm{K}}}}\mathscr{C}(w, s_{i,\sigma})$ (with the tautological injection $\mathrm{PS}_1(w(\phi),\mathbf{h})\hookrightarrow \mathrm{PS}(w(\phi),\mathbf{h})$). Throughout the section, we fix isomorphisms

$$\pi_{\mathrm{alg}}(\phi,\mathbf{h})\cong \mathrm{L}(\lambda)\otimes_{\mathrm{E}}\mathrm{I}_{\mathrm{sm}}\big(w(\phi)\big)\big(\hookrightarrow \mathrm{PS}_1\big(w(\phi),\mathbf{h}\big)\big). \tag{3.2}$$

for all $w\in \mathrm{S}_n$. The amalgamated sum $\oplus^{w\in\mathrm{S}_n}_{\pi_{\mathrm{alg}}(\phi,\lambda)}\mathrm{PS}_1(w(\phi),\mathbf{h})$ admits a unique quotient, denoted by $\pi_1(\phi,\mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$. By Lemma 3.1 (3) (4), $\pi_1(\phi,\mathbf{h})$ is given by an extension of $\oplus_{\substack{i=1,\dots,n-1,\sigma\in\Sigma_{\mathrm{K}}\\ \mathrm{I}\subset\{1,\dots,n\},\#\mathrm{I}=i}}\mathscr{C}(\mathrm{I}, s_{i,\sigma})$ ($(2^n-2)d_{\mathrm{K}}$ constituents in total) by $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$. Note we have a tautological injection

$$\pi_{\mathrm{alg}}(\phi,\mathbf{h})\hookrightarrow \pi_1(\phi,\mathbf{h}). \tag{3.3}$$

We study the extension group of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ by $\pi_1(\phi,\mathbf{h})$.

Proposition **3.3.** — *(1) For $w\in\mathrm{S}_n$ and $\psi\in\mathrm{Hom}_{g'}(\mathrm{T}(\mathrm{K}),\mathrm{E})$ (cf. (2.17)), we have $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(w(\phi)\eta z^{\lambda}(1+\psi\epsilon))\in\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))$ (using (3.2)). Moreover, the following map is a bijection:*

$$\begin{aligned}\zeta_w:\mathrm{Hom}_{g'}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big)&\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big)\\ \psi&\mapsto \mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\big(w(\phi)\eta z^{\lambda}(1+\psi\epsilon)\big),\end{aligned} \tag{3.4}$$

and induces $\mathrm{Hom}_{\mathrm{sm}}(\mathrm{T}(\mathrm{K}),\mathrm{E})\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{lalg}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))$. In particular, $\dim_{\mathrm{E}}\mathrm{Ext}^1_{\mathrm{lalg}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))=n$, $\dim_{\mathrm{E}}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))=n+d_{\mathrm{K}}$.

(2) For $w_1, w_2\in\mathrm{S}_n$, the following diagram commutes:

$$\begin{array}{ccc}\mathrm{Hom}_{g'}(\mathrm{T}(\mathrm{K}),\mathrm{E}) & \xrightarrow[\sim]{\zeta_{w_1}} & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))\\ {\scriptstyle w_2w_1^{-1}}\downarrow\wr & & \Vert\\ \mathrm{Hom}_{g'}(\mathrm{T}(\mathrm{K}),\mathrm{E}) & \xrightarrow[\sim]{\zeta_{w_2}} & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})).\end{array} \tag{3.5}$$

Proof. — For $\psi=\psi_1+\psi_0\circ\det$ with $\psi_1\in\mathrm{Hom}_{\mathrm{sm}}(\mathrm{T}(\mathrm{K}),\mathrm{E})$ and $\psi_0\in\mathrm{Hom}(\mathrm{K}^\times,\mathrm{E})$, it is easy to see the natural map

$$w(\phi)\eta z^{\lambda}(1+\psi\epsilon)\delta_{\mathrm{B}}\hookrightarrow \mathrm{J}_{\mathrm{B}}\Big(\big(\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}w(\phi)\eta z^{\lambda}(1+\psi\epsilon)\big)^{\mathbf{Q}_p-\mathrm{an}}\Big)$$

$$\hookrightarrow \big(\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} w(\phi)z^\lambda\eta(1+\psi\epsilon)\big)^{\mathbf{Q}_p-\mathrm{an}}$$

factors through the subrepresentation $(\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} w(\phi)\eta(1+\psi_1\epsilon))^{\mathrm{sm}}\otimes_{\mathrm{E}}\mathrm{L}(\lambda)\otimes_{\mathrm{E}[\epsilon]/\epsilon^2}(1+\epsilon\psi_0\circ\det)$. By definition ([37, Section (2.8)]), we see

$$\begin{aligned}&\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\big(w(\phi)\eta z^\lambda(1+\psi\epsilon)\big)\\ &\quad\cong \big(\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} w(\phi)\eta(1+\psi_1\epsilon)\big)^{\mathrm{sm}}\otimes_{\mathrm{E}}\mathrm{L}(\lambda)\otimes_{\mathrm{E}[\epsilon]/\epsilon^2}(1+\epsilon\psi_0\circ\det).\end{aligned}$$

Together with the isomorphism $\mathrm{I}_{\mathrm{sm}}(w(\phi))\otimes_{\mathrm{E}}\mathrm{L}(\lambda)\cong\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ (3.2), it gives a well-defined element in $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$. By [58, Prop. 4.7], we have (where $\mathrm{Z}\subset\mathrm{GL}_n$ denotes the centre, and the subscript "Z" stands for fixing central character)

$$(\mathbf{3.6})\qquad \mathrm{Ext}^1_{\mathrm{lalg},\mathrm{Z}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h})\big)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K}),\mathrm{Z}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h})\big).$$

By classical smooth representation theory, the restriction of ζ_w induces an isomorphism $\mathrm{Hom}_{\mathrm{sm}}(\mathrm{T}(\mathrm{K})/\mathrm{Z}(\mathrm{K}),\mathrm{E})\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{lalg},\mathrm{Z}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$ (so the latter has dimension $n-1$). Using similar arguments as in [14, Lem. 3.16] (and the aforementioned discussion), we obtain a commutative diagram of short exact sequences (we omit $\mathrm{GL}_n(\mathrm{K})$, $(\phi,\mathbf{h})$)

$$\begin{array}{ccccc}\mathrm{Hom}_{\mathrm{sm}}(\mathrm{T}(\mathrm{K})/\mathrm{Z}(\mathrm{K}),\mathrm{E}) & \hookrightarrow & \mathrm{Hom}_{g'}(\mathrm{T}(\mathrm{K}),\mathrm{E}) & \twoheadrightarrow & \mathrm{Hom}(\mathrm{Z}(\mathrm{K}),\mathrm{E})\\ \downarrow\sim & & \downarrow\zeta_w & & \|\\ \mathrm{Ext}^1_{\mathrm{Z}}(\pi_{\mathrm{alg}},\pi_{\mathrm{alg}}) & \hookrightarrow & \mathrm{Ext}^1(\pi_{\mathrm{alg}},\pi_{\mathrm{alg}}) & \twoheadrightarrow & \mathrm{Hom}(\mathrm{Z}(\mathrm{K}),\mathrm{E})\end{array}$$

So ζ_w is a bijection and $\dim_{\mathrm{E}}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))=n+d_{\mathrm{K}}$. For (2), it suffices to prove the statement for g' replaced by "sm". This is a classical fact. Indeed, let $\mathcal{H}$ (resp. $\mathcal{H}_i\cong\mathbf{G}_m$) be the Bernstein centre over E associated to the smooth representation $\pi_{\mathrm{sm}}(\phi)$ of $\mathrm{GL}_n(\mathrm{K})$ (resp. ϕ_i of $\mathrm{K}^\times$) (cf. [23, Section 3.13]). By [23, Lem. 3.22], for each $w\in\mathrm{S}_n$, there is a natural map $\mathcal{J}_w:\prod_{i=1}^n\mathrm{Spec}\,\mathcal{H}_{w^{-1}(i)}\to\mathrm{Spec}\,\mathcal{H}$ sending a point $(\phi_i')\in\prod_{i=1}^n\mathrm{Spec}\,\mathcal{H}_{w^{-1}(i)}$ to the point (associated to) $(\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(\boxtimes_{i=1}^r\phi_i')\eta)^{\mathrm{sm}}$ of $\mathcal{H}$. Moreover, the tangent map of $\mathcal{J}_w$ at $(\phi_{w^{-1}(i)})$ coincides with ζ_w. The intertwining property implies that for $w_1,w_2\in\mathrm{S}_n$, $\mathcal{J}_{w_2}=(w_2w_1^{-1})\circ\mathcal{J}_{w_1}$ where $w_2w_1^{-1}$ here denotes the morphism $\prod_{i=1}^n\mathrm{Spec}\,\mathcal{H}_{w_1^{-1}(i)}\to\prod_{i=1}^n\mathrm{Spec}\,\mathcal{H}_{w_2^{-1}(i)}$, $(\phi_i')\mapsto(\phi'_{(w_2w_1^{-1})^{-1}(i)})$. By considering the corresponding tangent maps of $\mathcal{J}_{w_1}$, $\mathcal{J}_{w_2}$, we deduce the commutativity of (3.5) (with g' replaced by "sm"). This concludes the proof. □

Remark **3.4.** — Note that ζ_w is in fact independent of the choice of (3.2).

Lemma **3.5.** — *For any* $\mathscr{C}(\mathrm{I},s_{i,\sigma})$, *we have:*
(1) $\dim_{\mathrm{E}}\mathrm{Ext}^1_{\mathrm{GL}_n}(\mathscr{C}(\mathrm{I},s_{i,\sigma}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}))=\dim_{\mathrm{E}}\mathrm{Ext}^1_{\mathrm{GL}_n}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathscr{C}(\mathrm{I},s_{i,\sigma}))=1.$

(2) Let $\tilde{\pi}_{\mathrm{alg}}(\phi, \boldsymbol{h}) \in \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\mathrm{alg}}(\phi, \boldsymbol{h}))$ *be non-split, then the following pull-back map (via* $\tilde{\pi}_{\mathrm{alg}}(\phi, \boldsymbol{h}) \twoheadrightarrow \pi_{\mathrm{alg}}(\phi, \boldsymbol{h})$*) is a bijection:*

$$\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \mathscr{C}(\mathrm{I}, s_{i,\sigma})\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\tilde{\pi}_{\mathrm{alg}}(\phi, \boldsymbol{h}), \mathscr{C}(\mathrm{I}, s_{i,\sigma})\big). \tag{3.7}$$

Proof. — (1) follows from [18, Prop. 5.1.14] together with [18, Lem. 3.2.4 (ii)] (when $\mathrm{K} = \mathbf{Q}_p$, the part on $\mathrm{Ext}^1_{\mathrm{GL}_n}(\mathscr{C}(\mathrm{I}, s_{i,\sigma}), \pi_{\mathrm{alg}}(\phi, \mathbf{h}))$ was proved in [15, Cor. 5.9]). We give a proof of (2) and an alternative proof of the second equality in (1) using Schraen's spectral sequence [58, Cor. 4.9] (for $\mathrm{G} = \mathrm{Res}^{\mathrm{K}}_{\mathbf{Q}_p} \mathrm{GL}_n$). First, note by the same argument below [58, Cor. 4.9], the separatedness assumption in [58, Cor. 4.9] is satisfied for either $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ or any $\widetilde{\pi}_{\mathrm{alg}}(\phi, \mathbf{h})$ in (2) (noting by Proposition 3.3 (1) and the proof, $\widetilde{\pi}_{\mathrm{alg}}(\phi, \mathbf{h})|_{\mathrm{SL}_n(\mathrm{K})}$ is locally algebraic). Let $\delta := z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta$. By [58, Cor. 4.9], we have a spectral sequence

$$\mathrm{Ext}^p_{\mathrm{T(K)}}\big(\mathrm{H}_q\big(\mathrm{N}^-(\mathrm{K}), \pi_{\mathrm{alg}}(\phi, \mathbf{h})\big), \delta\big) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \big(\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{B}^-(\mathrm{K})} \delta\big)^{\mathbf{Q}_p-\mathrm{an}}\big) \tag{3.8}$$

where N^- is the unipotent radical of B^-. Recall for characters χ, χ' of T(K) over E, we have $\mathrm{Ext}^i_{\mathrm{T(K)}}(\chi, \chi') = 0$ for all i if $\chi \neq \chi'$. This, together with [58, (4.40), (4.41), (4.42)] and the classical fact $\mathrm{J}_{\mathrm{N}^-}(\mathrm{I}_{\mathrm{sm}}(\phi)) \cong \oplus_{w' \in \mathrm{S}_n} w'(\phi)\eta$ (where $\mathrm{J}_{\mathrm{N}^-}(-)$ denotes the classical Jacquet module for N^-), imply that for $p + q = 1$, the only non-zero term on the left hand side of (3.8) is $\mathrm{Hom}_{\mathrm{T(K)}}(\mathrm{H}_1(\mathrm{N}^-(\mathrm{K}), \pi_{\mathrm{alg}}(\phi, \mathbf{h})), \delta) \cong \mathrm{Hom}_{\mathrm{T(K)}}(\delta, \delta) \cong$ E. So $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), (\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{B}^-(\mathrm{K})} \delta)^{\mathbf{Q}_p-\mathrm{an}}) \cong$ E. By (3.1) and [32, Lem. 2.26], the natural push-forward map is an isomorphism: $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathscr{C}(\mathrm{I}, s_{i,\sigma})) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), (\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{B}^-(\mathrm{K})} \delta)^{\mathbf{Q}_p-\mathrm{an}})$. This proves the second equality in (1). Similarly with $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ replaced by $\tilde{\pi}_{\mathrm{alg}}(\phi, \mathbf{h})$, we get

$$\begin{aligned} &\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\tilde{\pi}_{\mathrm{alg}}(\phi, \mathbf{h}), \mathscr{C}(\mathrm{I}, s_{i,\sigma})\big) \\ &\cong \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\tilde{\pi}_{\mathrm{alg}}(\phi, \mathbf{h}), \big(\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{B}^-(\mathrm{K})} z^{-s_{i,\sigma}\cdot\lambda} w(\phi)\eta\big)^{\mathbf{Q}_p-\mathrm{an}}\big) \\ &\cong \mathrm{Hom}_{\mathrm{T(K)}}\big(w(\phi)\eta z^{-s_{i,\sigma}\cdot\lambda}(1 + \psi\epsilon), w(\phi)\eta z^{-s_{i,\sigma}\cdot\lambda}\big) \end{aligned} \tag{3.9}$$

where $\psi = \zeta_w^{-1}(\tilde{\pi}_{\mathrm{alg}}(\phi, \mathbf{h})) \in \mathrm{Hom}_{g'}(\mathrm{T(K)}, \mathrm{E})$. As $\widetilde{\pi}_{\mathrm{alg}}(\phi, \mathbf{h})$ is non-split, $\psi \neq 0$ hence the right hand side of (3.9) is one dimensional over E. However, by an easy dévissage, (3.7) is injective hence has to be bijective. □

For $w \in \mathrm{S}_n$, consider the natural map

$$\mathrm{Hom}\big(\mathrm{T(K)}, \mathrm{E}\big) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\mathrm{PS}\big(w(\phi), \mathbf{h}\big), \mathrm{PS}\big(w(\phi), \mathbf{h}\big)\big)$$

sending ψ to $(\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{B}^-(\mathrm{K})} w(\phi)\eta z^{\lambda}(1 + \psi\epsilon))^{\mathbf{Q}_p-\mathrm{an}}$. Composed with the pull-back map for (3.2) and using [32, Lem. 2.26], it induces:

$$\mathrm{Hom}\big(\mathrm{T(K)}, \mathrm{E}\big) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathrm{PS}_1\big(w(\phi), \mathbf{h}\big)\big). \tag{3.10}$$

Composed furthermore with the push-forward map via the injection $\mathrm{PS}_1(w(\phi), \mathbf{h}) \hookrightarrow \pi_1(\phi, \mathbf{h})$ (associated to (3.2), see also (3.3)), we finally obtain a map

$$\zeta_w : \mathrm{Hom}\big(\mathrm{T}(\mathrm{K}), \mathrm{E}\big) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big). \tag{3.11}$$

Note that the map ζ_w does not depend on the choice of (3.2).

Proposition **3.6.** — *(1) For $w \in \mathrm{S}_n$, the map (3.10) is bijective. In particular, we have* $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \mathrm{PS}_1(w(\phi), \boldsymbol{h})) = n + nd_{\mathrm{K}}$.

(2) For $w \in \mathrm{S}_n$, $\zeta_w|_{\mathrm{Hom}_{g'}(\mathrm{T}(\mathrm{K}),\mathrm{E})}$ is equal to the composition of (3.4) with the push-forward map $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\mathrm{alg}}(\phi, \boldsymbol{h})) \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h}))$.

Proof. — (1) follows from similar arguments as in the proof of Lemma 3.5, using Schraen's spectral sequence [58, Cor. 4.9] and [31, Lem. 2.26]. We leave the details to the reader. (2) is clear (see also Remark 3.7 below). □

Remark **3.7.** — The map ζ_w can also be obtained by using Emerton's functor $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(-)$. In fact, by definition (cf. [37, Section (2.8)]) and using [32, Lem. 2.26], it is straightforward to see for $\psi \in \mathrm{Hom}(\mathrm{T}(\mathrm{K}), \mathrm{E})$, $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(w(\phi)\eta z^{\lambda}(1+\psi\epsilon)) \subset (\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} w(\phi)\eta z^{\lambda}(1+\psi\epsilon))^{\mathbf{Q}_p-\mathrm{an}}$ is an extension of $\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \cong \mathrm{I}_{\mathrm{sm}}(w(\phi)) \otimes_{\mathrm{E}} \mathrm{L}(\lambda)$ by a certain subrepresentation V of $\mathrm{PS}_1(w(\phi), \mathbf{h})$. Then $\zeta_w(\psi)$ is just its image of the push-forward map via $\mathrm{V} \hookrightarrow \mathrm{PS}_1(w(\phi), \mathbf{h}) \hookrightarrow \pi_1(\phi, \mathbf{h})$.

Proposition **3.8.** — *(1) We have an exact sequence*

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\mathrm{alg}}(\phi, \boldsymbol{h})\big) &\to \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})\big) \\ &\longrightarrow \oplus_{\substack{i=1,\dots,n-1, \sigma\in\Sigma_{\mathrm{K}} \\ \mathrm{I}\subset\{1,\dots,n-1\}, \#\mathrm{I}=i}} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \mathscr{C}(\mathrm{I}, s_{i,\sigma})\big) \to 0. \end{aligned} \tag{3.12}$$

In particular, $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})) = n + (2^n - 1)d_{\mathrm{K}}$.

(2) The following map is surjective:

$$t_{\phi, \boldsymbol{h}} : \oplus_{w\in\mathrm{S}_n} \mathrm{Hom}\big(\mathrm{T}(\mathrm{K}), \mathrm{E}\big) \xrightarrow{(\zeta_w)} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})\big). \tag{3.13}$$

Proof. — We omit the subscript "$\mathrm{GL}_n(\mathrm{K})$" in the proof. The sequence follows by dévissage, and it suffices to prove the second last map in (3.12) is surjective. For $w \in \mathrm{S}_n$, using dévissage, we have an exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h})\big) &\to \mathrm{Ext}^1\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathrm{PS}_1\big(w(\phi), \mathbf{h}\big)\big) \\ &\to \oplus_{\substack{i=1,\dots,n-1 \\ \sigma\in\Sigma_{\mathrm{K}}}} \mathrm{Ext}^1\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathscr{C}(w, s_{i,\sigma})\big). \end{aligned} \tag{3.14}$$

By comparing dimensions (using Proposition 3.3 (1), Proposition 3.6 (1) and Lemma 3.5 (1)), the last map in (3.14) is surjective. The following diagram clearly commutes

$$\begin{array}{ccc} \mathrm{Ext}^1\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_1\big(w(\phi),\mathbf{h}\big)\big) & \twoheadrightarrow & \oplus_{\substack{i=1,\dots,n-1\\ \sigma\in\Sigma_{\mathrm{K}}}}\mathrm{Ext}^1\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(w,s_{i,\sigma})\big) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Ext}^1\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big) & \longrightarrow & \oplus_{\substack{i=1,\dots,n-1,\sigma\in\Sigma_{\mathrm{K}}\\ \mathrm{I}\subset\{1,\dots,n-1\},\#\mathrm{I}=i}}\mathrm{Ext}^1\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(\mathrm{I},s_{i,\sigma})\big). \end{array} \tag{3.15}$$

Varying w, the image of the right vertical map can "cover" the target. Together with the surjectivity of the top map, we see the bottom map is also surjective. (2) follows by the first statement in Proposition 3.6 (1) and (3.12). And the dimension part in (1) follows then from Lemma 3.5 (1) and Proposition 3.6 (1). □

Remark **3.9.** — By Proposition 3.6 (1) and [32, Lem. 2.26] (and using (3.2)), for $w\in\mathrm{S}_n$, we have $\zeta_w:\mathrm{Hom}(\mathrm{T}(\mathrm{K}),\mathrm{E})\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}(w(\phi),\mathbf{h}))$. Denote by $\pi(\phi,\mathbf{h})$ the unique quotient of $\oplus^{w\in\mathrm{S}_n}_{\pi_{\mathrm{alg}}(\phi,\mathbf{h})}\mathrm{PS}(w(\phi),\mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ (cf. [17, Def. 5.7], which is the representation $\pi(\rho)^{\mathrm{fs}}$ of *loc. cit.*). The representation $\pi_1(\phi,\mathbf{h})$ is in fact the first two layers in the socle filtration of $\pi(\phi,\mathbf{h})$. Moreover, using again [32, Lem. 2.26], we have

$$\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi(\phi,\mathbf{h})\big). \tag{3.16}$$

Proposition 3.8 (2) hence holds with $\pi_1(\phi,\mathbf{h})$ replaced by $\pi(\phi,\mathbf{h})$.

Denote by

$$\begin{aligned}\mathrm{Ext}^1_g\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)&\subset\mathrm{Ext}^1_{g'}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)\\&\subset\mathrm{Ext}^1_w\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)\end{aligned}$$

the respective image of $\mathrm{Ext}^1_{\mathrm{lalg}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$, $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$, and $\mathrm{Im}(\zeta_w)$ for $w\in\mathrm{S}_n$. We also use the notation $\mathrm{Ext}^1_{\mathscr{T}_w}$ for Ext^1_w whenever it is convenient for the context where $\mathscr{T}_w$ is the B-filtration of $\oplus_{i=1}^n\phi_i$ associated to w. So ζ_w (3.11) induces an isomorphism

$$\zeta_w:\mathrm{Hom}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big)\xrightarrow{\sim}\mathrm{Ext}^1_w\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big). \tag{3.17}$$

By Proposition 3.3 (2), for $w_1,w_2\in\mathrm{S}_n$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{g'}(\mathrm{T}(\mathrm{K}),\mathrm{E}) & \xrightarrow[\sim]{\zeta_{w_1}} & \mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \\ {\scriptstyle w_2w_1^{-1}}\big\downarrow\wr & & \big\| \\ \mathrm{Hom}_{g'}(\mathrm{T}(\mathrm{K}),\mathrm{E}) & \xrightarrow[\sim]{\zeta_{w_2}} & \mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})). \end{array} \tag{3.18}$$

3.1.3. *Parabolic inductions.* — Let $\mathrm{P}\supset\mathrm{B}$ be a standard parabolic subgroup of GL_n with $\mathrm{L_P}=\mathrm{diag}(\mathrm{GL}_{n_1},\dots,\mathrm{GL}_{n_r})$. Let $\mathscr{W}_\mathrm{P}$ be the Weyl group of $\mathrm{L_P}$. Let $\mathscr{F}_\mathrm{P}$ be a P-filtration of $\oplus_{i=1}^n\phi_i$ and $\phi_{\mathscr{F}_\mathrm{P},i}:=\otimes\phi_j$ for $\phi_j\in\mathrm{gr}_i\mathscr{F}_\mathrm{P}$ (where the order of these ϕ_j does not matter here). For $i=1,\dots,r$, let $\mathbf{h}^i:=(\mathbf{h}_{n_1+\cdots n_{i-1}+1},\dots,\mathbf{h}_{n_1+\cdots+n_i})$, $\lambda^i=(h_{n_1+\cdots+n_{i-1}+1,\sigma},\dots,h_{n_1+\cdots+n_i,\sigma}+n_i-1)_{\sigma\in\Sigma_\mathrm{K}}$. Applying the constructions in Section 3.1.2 to $(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i)$, we obtain $\mathrm{GL}_{n_i}(\mathrm{K})$-representations $\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i)$, $\pi_1(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i)$ etc. Note when $n_i=1$, we have $\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i)=\pi_1(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i)=\phi_{n_1+\cdots+n_{i-1}}z^{\mathbf{h}_{n_1+\cdots+n_{i-1}}}$. We fix an isomorphism $(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(\boxtimes_{i=1}^r\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i))\varepsilon^{-1}\circ\theta^\mathrm{P})^{\mathrm{lalg}}\cong\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ (where the supscript "lalg" means locally algebraic induction) and (then) fix isomorphisms $\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i)\cong\mathrm{I_{sm}}(w_i(\phi_{\mathscr{F}_p,i}))\otimes_\mathrm{E}\mathrm{L}(\lambda^i)$ for all i and $w_i\in\mathrm{S}_{n_i}$ such that the composition (noting the first isomorphism is obtained by the transitivity of parabolic induction) $\mathrm{I_{sm}}(w(\phi))\otimes_\mathrm{E}\mathrm{L}(\lambda)\cong(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(\boxtimes_{i=1}^r\mathrm{I_{sm}}(w_i(\phi_{\mathscr{F}_p,i}))\otimes_\mathrm{E}\mathrm{L}(\lambda^i))\varepsilon^{-1}\circ\theta^\mathrm{P})^{\mathrm{lalg}}\cong\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ coincides the fixed isomorphism (3.2) for all $w\in\mathscr{W}_\mathrm{P}$. Consider the parabolic induction

$$\begin{aligned}(\mathbf{3.19})\qquad&\Big(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\big(\widehat{\boxtimes}_{i=1}^r\pi_1\big(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i\big)\big)\varepsilon^{-1}\circ\theta^\mathrm{P}\Big)^{\mathbf{Q}_p-\mathrm{an}}\\&\longleftarrow\Big(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\big(\boxtimes_{i=1}^r\pi_{\mathrm{alg}}\big(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i\big)\big)\varepsilon^{-1}\circ\theta^\mathrm{P}\Big)^{\mathrm{lalg}}\cong\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\end{aligned}$$

where $\widehat{\boxtimes}$ denotes the completed (injective or equivalently projective) tensor product over E (cf. [38, Prop. 1.1.31]).

Lemma **3.10.** — *For $i=1,\dots,n-1$, $\sigma\in\Sigma_\mathrm{K}$ and $\mathrm{I}\subset\{1,\dots,n\}$, $\#\mathrm{I}=i$, $\mathscr{C}(\mathrm{I},s_{i,\sigma})$ appears as an irreducible constituent of $(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(\widehat{\boxtimes}_{i=1}^r\pi_1(\phi_{\mathscr{F}_\mathrm{P},i},\boldsymbol{h}^i))\varepsilon^{-1}\circ\theta^\mathrm{P})^{\mathbf{Q}_p-\mathrm{an}}$ if and only if one of the following conditions holds:*

(1) there exists $k\in\{1,\dots,r\}$ such that $(n_1+\cdots+n_{k-1})+1\leq i\leq(n_1+\cdots+n_k)-1$ and $\{j\mid\phi_j\in\mathrm{Fil}_{\mathscr{F}_\mathrm{P},k-1}\}\subset\mathrm{I}\subset\{j\mid\phi_j\in\mathrm{Fil}_{\mathscr{F}_\mathrm{P},k}\}$,

(2) $i=n_1+\cdots+n_k$ for some $k=1,\dots,r-1$, and $\mathrm{I}=\{j\mid\phi_j\in\mathrm{Fil}_k\mathscr{F}_\mathrm{P}\}$.

Moreover, each of such constituents has multiplicity one, and lies in the socle of

$$(\mathbf{3.20})\qquad\Big(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\big(\widehat{\boxtimes}_{i=1}^r\pi_1\big(\phi_{\mathscr{F}_\mathrm{P},i},\boldsymbol{h}^i\big)\big)\varepsilon^{-1}\circ\theta^\mathrm{P}\Big)^{\mathbf{Q}_p-\mathrm{an}}/\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}).$$

Proof. — Let $\mathrm{V}_0:=(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(\boxtimes_{i=1}^r\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i))\varepsilon^{-1}\circ\theta^\mathrm{P})^{\mathbf{Q}_p-\mathrm{an}}$. For all i as in (2), $\mathrm{L}^-(-s_{i,\sigma}\cdot\lambda)$ has multiplicity one in the parabolic Verma module $\mathrm{U}(\mathfrak{gl}_{n,\Sigma_\mathrm{K}})\otimes_{\mathrm{U}(\mathfrak{p}_{\Sigma_\mathrm{K}})}(-\lambda)$ and lies in the cosocle of $\mathrm{Ker}[\mathrm{U}(\mathfrak{gl}_{n,\Sigma_\mathrm{K}})\otimes_{\mathrm{U}(\mathfrak{p}_{\Sigma_\mathrm{K}})}(-\lambda)\to\mathrm{L}^-(-\lambda)]$. Using [55, Thm.], we deduce the constituents for i as in (2) appear with multiplicity one in V_0, and all lie in the socle of (3.20).

For i as in (1), let $\mathrm{V_I}:=(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}((\boxtimes_{\substack{i=1,\dots,r\\ i\neq k}}\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i)\widehat{\boxtimes}\mathscr{C}(\mathrm{I},s_{i,\sigma})_k)\varepsilon^{-1}\circ\theta^\mathrm{P})^{\mathbf{Q}_p-\mathrm{an}}$ where $\mathscr{C}(\mathrm{I},s_{i,\sigma})_k$ denotes the corresponding representation in the cosocle of $\pi_1(\phi_{\mathscr{F}_\mathrm{P},i},\mathbf{h}^i)$. By (3.1) for $\mathscr{C}(\mathrm{I},s_{i,\sigma})_k$ and the transitivity of parabolic inductions, $\mathrm{V_I}$ injects into $(\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}z^{-s_{i,\sigma}\cdot\lambda}w(\phi)\eta)^{\mathbf{Q}_p-\mathrm{an}}$ for any $w\in\mathrm{S}_n$ satisfying $w(\{1,\dots,i\})=\mathrm{I}$. Since the latter representation has socle $\mathscr{C}(\mathrm{I},s_{i,\sigma})$ with multiplicity one (cf. (3.1)), so does its subrepresentation $\mathrm{V_I}$. It is not difficult to see these give all the $\mathscr{C}(\mathrm{I},s_{i,\sigma})$ appearing in (1), and they all have multiplicity one. Let U be the closed subrepresentation of $\pi_1(\phi_{\mathscr{F}_{\mathrm{P}},k}, \mathbf{h}^k)$ of the form $[\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},k}, \mathbf{h}^k)\text{—}\mathscr{C}(\mathrm{I}, s_{i,\sigma})_k]$, which is clearly a closed subrepresentation of a certain principal series of $\mathrm{GL}_{n_k}(\mathrm{K})$. Using the transitivity of parabolic inductions, one sees $\mathrm{W} := (\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}((\boxtimes_{\substack{i=1,\dots,r\\ i\neq k}}\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i)\widehat{\boxtimes}\mathrm{U})\varepsilon^{-1}\circ\theta^{\mathrm{P}})^{\mathbf{Q}_p-\mathrm{an}}$ is a closed subrepresentation of $(\mathrm{Ind}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} z^{\lambda} w(\phi)\eta)^{\mathbf{Q}_p-\mathrm{an}}$ with $w\in \mathrm{S}_n$ satisfying $w(\{1,\dots,i\}) = \mathrm{I}$. For the latter representation, $\mathscr{C}(\mathrm{I}, s_{i,\sigma})$ has multiplicity one and lies in the socle of its quotient by $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$. We then deduce $\mathscr{C}(\mathrm{I}, s_{i,\sigma})$ lies in the socle of $\mathrm{W}/\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ hence in the socle of (3.20).

Finally, by [55, Thm.], one sees every $\mathscr{C}(\mathrm{I}, s_{i,\sigma})$ in the representation $(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(\widehat{\boxtimes}_{i=1}^r\pi_1(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i))\varepsilon^{-1}\circ\theta^{\mathrm{P}})^{\mathbf{Q}_p-\mathrm{an}}$ must come from either V_0 or V_{I} with I as in (1), and has multiplicity one. This completes the proof. □

Denote by $\mathrm{S}_{\mathscr{F}_{\mathrm{P}}}$ the subset of the constituents $\mathscr{C}(\mathrm{I}, s_{i,\sigma})$, which satisfy one of the conditions in Lemma 3.10. We have

$$\#\mathrm{S}_{\mathscr{F}_{\mathrm{P}}} = \Big(\sum_{i=1}^r \big(2^{n_i}-2\big) + (r-1)\Big) d_{\mathrm{K}}. \tag{3.21}$$

The representation $(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(\widehat{\boxtimes}_{i=1}^r\pi_1(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i))\varepsilon^{-1}\circ\theta^{\mathrm{P}})^{\mathbf{Q}_p-\mathrm{an}}$ contains a unique subrepresentation $\pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \mathbf{h})$ such that $\mathrm{soc}_{\mathrm{GL}_n(\mathrm{K})}\,\pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \mathbf{h}) \cong \pi_{\mathrm{alg}}(\phi, \mathbf{h})$ and $\pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \mathbf{h})/\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \cong \oplus_{\mathscr{C}\in \mathrm{S}_{\mathscr{F}_{\mathrm{P}}}}\mathscr{C}$. Note when $\mathrm{P}=\mathrm{B}$, $\mathscr{F}_{\mathrm{P}} = \mathscr{T}_w$, then $\pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \mathbf{h}) \cong \mathrm{PS}_1(w(\phi), \mathbf{h})$. It is easy to see the injection $\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \hookrightarrow \pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \mathbf{h})$ (cf. (3.19)) uniquely extends to $\pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \mathbf{h}) \hookrightarrow \pi_1(\phi, \mathbf{h})$.

Proposition **3.11.** — *We have* $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \boldsymbol{h})) = n + d_{\mathrm{K}} r + d_{\mathrm{K}}\sum_{i=1}^r(2^{n_i}-2)$. *And the following push-forward map is injective*

$$\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \boldsymbol{h})\big) \longhookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})\big). \tag{3.22}$$

Proof. — We have an exact sequence by dévissage

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \mathbf{h})\big) &\to \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big) \\ &\to \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \oplus_{\mathscr{C}\notin \mathrm{S}_{\mathscr{F}_{\mathrm{P}}}}\mathscr{C}\big). \end{aligned} \tag{3.23}$$

The injectivity of (3.22) follows. By Proposition 3.8 (1), the last map in (3.23) is surjective. The first part follows then by a direct calculation using Proposition 3.8 (1), Lemma 3.5 (1) and (3.21). □

Set $\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$ to be the image of (3.22). The injection $\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \hookrightarrow \pi_{\mathscr{F}_{\mathrm{P}}}(\phi, \mathbf{h})$ induces a natural injection

$$\mathrm{Ext}^1_{g'}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big) \longhookrightarrow \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big). \tag{3.24}$$

We have natural maps

$$(\mathbf{3.25})\qquad \mathrm{Ext}^1_{\mathrm{L_P(K)}}\Big(\boxtimes_{i=1}^r \pi_{\mathrm{alg}}\big(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i\big), \widehat{\boxtimes}_{i=1}^r \pi_1\big(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i\big)\Big) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\Big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \big(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\big(\widehat{\boxtimes}_{i=1}^r \pi_1\big(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i\big)\big)\varepsilon^{-1}\circ\theta^{\mathrm{P}}\big)^{\mathbf{Q}_p-\mathrm{an}}\Big) \xleftarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_{\mathscr{F}_{\mathrm{P}}}(\phi,\mathbf{h})\big),$$

where the first map is obtained by taking $(\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})} - \otimes_{\mathrm{E}} \varepsilon^{-1}\circ\theta^{\mathrm{P}})^{\mathbf{Q}_p-\mathrm{an}}$ and using the pull-back via $\pi_{\mathrm{alg}}(\phi,\mathbf{h}) \hookrightarrow (\mathrm{Ind}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(\boxtimes_{i=1}^r \pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i))\varepsilon^{-1}\circ\theta^{\mathrm{P}})^{\mathbf{Q}_p-\mathrm{an}}$ (cf. (3.19)), and where the second map is the natural push-forward map, which is bijective by [32, Lem. 2.26] (and Lemma 3.10).

For $(\widetilde{\pi}_i) \in \prod_{i=1}^r \mathrm{Ext}^1_{\mathrm{GL}_{n_i}}(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i), \pi_1(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i))$, consider the $\mathrm{L_P(K)}$-representation $\widehat{\boxtimes}_{i=1}^r \widetilde{\pi}_i$ (where the completed tensor product is taken over E). It is clear that $\widehat{\boxtimes}_{i=1}^r \widetilde{\pi}_i$ admits a quotient V given by an extension of $\boxtimes_{i=1}^r \pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i)$ by $\mathrm{W} := \oplus_{i=1}^r (\widehat{\boxtimes}_{\substack{j=1,\dots,r\\ j\neq i}} \pi_1(\phi_{\mathscr{F}_{\mathrm{P}},j}, \mathbf{h}^j) \boxtimes_{\mathrm{E}} \pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}\mathbf{h}^i))$. The push-forward of V via the natural map $\mathrm{W} \to \widehat{\boxtimes}_{i=1}^r \pi_1(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i)$ (induced by $\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i) \hookrightarrow \pi_1(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i)$) gives an element in $\mathrm{Ext}^1_{\mathrm{L_P(K)}}(\boxtimes_{i=1}^r \pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i), \widehat{\boxtimes}_{i=1}^r \pi_1(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i))$. In this way, we obtain a map

$$(\mathbf{3.26})\qquad \prod_{i=1}^r \mathrm{Ext}^1_{\mathrm{GL}_{n_i}(\mathrm{K})}\Big(\pi_{\mathrm{alg}}\big(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i\big), \pi_1\big(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i\big)\Big) \longrightarrow \mathrm{Ext}^1_{\mathrm{L_P(K)}}\Big(\boxtimes_{i=1}^r \pi_{\mathrm{alg}}\big(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i\big), \widehat{\boxtimes}_{i=1}^r \pi_1\big(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i\big)\Big).$$

Together with (3.25), we finally get a map:

$$(\mathbf{3.27})\qquad \zeta_{\mathscr{F}_{\mathrm{P}}} : \prod_{i=1}^r \mathrm{Ext}^1_{\mathrm{GL}_{n_i}(\mathrm{K})}\Big(\pi_{\mathrm{alg}}\big(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i\big), \pi_1\big(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i\big)\Big) \longrightarrow \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h})\big).$$

For $w \in \mathrm{S}_n$, let $\mathscr{T}_w$ be the B-filtration of $\oplus_{i=1}^n \phi_i$ associated to w. Suppose $\mathscr{T}_w$ is compatible with $\mathscr{F}_{\mathrm{P}}$. It is clear that $\mathrm{PS}_1(w(\phi),\mathbf{h})$ is a subrepresentation of $\pi_{\mathscr{F}_{\mathrm{P}}}(\phi,\mathbf{h})$ (e.g. by comparing constituents and using Lemma 3.5 (1)), hence (by dévissage) $\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h})) \hookrightarrow \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \pi_1(\phi,\mathbf{h}))$.

Proposition **3.12.** — *(1) The map $\zeta_{\mathscr{F}_{\mathrm{P}}}$ is bijective.*

(2) For any w such that the associated B*-filtration $\mathscr{T}_w$ is compatible with $\mathscr{F}_{\mathrm{P}}$, the following diagram commutes*

$$\begin{array}{ccc} \prod_{i=1}^r \mathrm{Hom}(\mathrm{T}(\mathrm{K}) \cap \mathrm{L}_{\mathrm{P},i}(\mathrm{K}), \mathrm{E}) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{T}(\mathrm{K}), \mathrm{E}) \\ \sim\downarrow(3.17) & & \sim\downarrow(3.17) \\ \prod_{i=1}^r \mathrm{Ext}^1_{\mathscr{T}_{w,i}}(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}, \boldsymbol{h}^i), \pi_1(\phi_{\mathscr{F}_{\mathrm{P}},i}, \boldsymbol{h}^i)) & \xrightarrow{\sim} & \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})) \end{array} \tag{3.28}$$

where $\mathscr{T}_{w,i}$ is the induced $\mathrm{B} \cap \mathrm{L}_{\mathrm{P}}$-filtration on $\mathrm{gr}_i \mathscr{F}_{\mathrm{P}}$, and the bottom map is induced by $\zeta_{\mathscr{F}_{\mathrm{P}}}$.

Proof. — Given $\psi = (\psi_i) \in \prod_{i=1}^r \mathrm{Hom}(\mathrm{T}(\mathrm{K}) \cap \mathrm{L}_{\mathrm{P},i}(\mathrm{K}), \mathrm{E}) \cong \mathrm{Hom}(\mathrm{T}(\mathrm{K}), \mathrm{E})$, we have $(1 + \psi\epsilon) \cong \boxtimes^{i=1,\ldots,r}_{\mathrm{E}[\epsilon]/\epsilon^2}(1 + \psi_i\epsilon)$ as character of $\mathrm{T}(\mathrm{K})$ over $\mathrm{E}[\epsilon]/\epsilon^2$ hence as element in $\mathrm{Ext}^1_{\mathrm{T}(\mathrm{K})}(1, 1)$. Note $\boxtimes^{i=1,\ldots,r}_{\mathrm{E}[\epsilon]/\epsilon^2}(1 + \psi_i\epsilon)$ admits an extension construction in a similar way as given above (3.26). The commutativity of (3.28) then follows by definition and the transitivity of parabolic inductions (see also the discussion above (3.19)). In particular, we deduce the bottom map of (3.28) is bijective. Note that any $\mathscr{C}(\mathrm{I}, s_{i,\sigma}) \in \mathrm{S}_{\mathscr{F}_{\mathrm{P}}}$ is a constituent of some $\mathrm{PS}_1(w(\phi), \mathbf{h})$ with $\mathscr{T}_w$ compatible with $\mathscr{F}_{\mathrm{P}}$. By similar arguments as in the proof of Proposition 3.8, one sees the natural map ("$\subset$" means compatible)

$$\bigoplus_{\mathscr{T}_w \subset \mathscr{F}_{\mathrm{P}}} \mathrm{Ext}^1_w\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big) \longrightarrow \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big) \tag{3.29}$$

is surjective, hence so is $\zeta_{\mathscr{F}_{\mathrm{P}}}$. By Proposition 3.8 (1) and Proposition 3.11, both sides of (3.27) have the same dimension over E hence $\zeta_{\mathscr{F}_{\mathrm{P}}}$ is bijective. □

Let $\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}},g'}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) := \zeta_{\mathscr{F}_{\mathrm{P}}}(\prod_{i=1}^r \mathrm{Ext}^1_{g'}(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}, \mathbf{h}^i), \pi_1(\phi_{\mathscr{F}_p,i}, \mathbf{h}^i)))$. By Proposition 3.12 (2), and (3.18), assuming $\mathscr{T}_{w_i}$ compatible with $\mathscr{F}_{\mathrm{P}}$, the following diagram commutes (cf. (2.21)):

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{P},g'}(\mathrm{T}(\mathrm{K}), \mathrm{E}) & \xrightarrow[\sim]{\zeta_{w_1}} & \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}},g'}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) \\ w_2 w_1^{-1}\downarrow\sim & & \Vert \\ \mathrm{Hom}_{\mathrm{P},g'}(\mathrm{T}(\mathrm{K}), \mathrm{E}) & \xrightarrow[\sim]{\zeta_{w_2}} & \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}},g'}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})). \end{array} \tag{3.30}$$

We finally discuss some intertwining properties related to Section 2.4. Let $\phi^1 := \phi_1 \boxtimes \cdots \boxtimes \phi_{n-1} : \mathrm{T}_{n-1}(\mathrm{K}) \to \mathrm{E}^\times$, $\mathbf{h}^1 := (\mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ and $\mathbf{h}^2 := (\mathbf{h}_2, \ldots, \mathbf{h}_n)$ which are dominant weights of $\mathfrak{t}_{n-1,\Sigma_{\mathrm{K}}}$. We have locally $\mathbf{Q}_p$-analytic $\mathrm{GL}_{n-1}(\mathrm{K})$-representations $\pi_{\mathrm{alg}}(\phi^1, \mathbf{h}^i) \subset \pi_1(\phi^1, \mathbf{h}^i)$ for $i = 1, 2$, and parabolic inductions $(\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{P}_1^-(\mathrm{K})}(\pi_1(\phi^1, \mathbf{h}^1) \otimes \varepsilon) \boxtimes \phi_n z^{\mathbf{h}_n})^{\mathbf{Q}_p-\mathrm{an}}$ and $(\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{P}_2^-(\mathrm{K})} \phi_n z^{\mathbf{h}_1} \varepsilon^{n-1} \boxtimes \pi_1(\phi^1, \mathbf{h}^2))^{\mathbf{Q}_p-\mathrm{an}}$. Let $\mathscr{F}$ be the filtration $\oplus_{i=1}^{n-1}\phi_i \subset \oplus_{i=1}^n \phi_i$ and $\mathscr{G}$ be the filtration $\phi_n \subset \oplus_{i=1}^n \phi_i$. By Lemma 3.10, $\mathscr{C}(\mathrm{I}, s_{i,\sigma})$ appears in $\pi_{\mathscr{F}}(\phi, \mathbf{h})$ (resp. in $\pi_{\mathscr{G}}(\phi, \mathbf{h})$) if and only if $i = 1, \ldots, n-1$, $\sigma \in \Sigma_{\mathrm{K}}$ and

$\mathrm{I}\subset\{1,\dots,n-1\}$, $\#\mathrm{I}=i$ (resp. $\mathrm{I}=\mathrm{I}_1\cup\{n\}$ with $\mathrm{I}_1\subset\{1,\dots,n-1\}$ and $\#\mathrm{I}_1=i-1$). In particular, $\pi_1(\phi,\mathbf{h})/\pi_{\mathrm{alg}}(\phi,\mathbf{h})\cong(\pi_{\mathscr{F}}(\phi,\mathbf{h})/\pi_{\mathrm{alg}}(\phi,\mathbf{h}))\oplus(\pi_{\mathscr{G}}(\phi,\mathbf{h})/\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$. The following proposition is straightforward (where the right exactness of the last sequence follows by comparing dimensions, using Proposition 3.3 (1), Proposition 3.8 (1) and Proposition 3.11):

Proposition **3.13.** — *There is a natural exact sequence of* $\mathrm{GL}_n(\mathrm{K})$*-representations*

$$0\longrightarrow\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\longrightarrow\pi_{\mathscr{F}}(\phi,\boldsymbol{h})\oplus\pi_{\mathscr{G}}(\phi,\boldsymbol{h})\longrightarrow\pi_1(\phi,\boldsymbol{h})\longrightarrow 0.$$

Consequently, we have a natural exact sequence

$$\begin{aligned}0&\longrightarrow\mathrm{Ext}^1_{g'}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\\&\longrightarrow\mathrm{Ext}^1_{\mathscr{F}}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\oplus\mathrm{Ext}^1_{\mathscr{G}}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\\&\longrightarrow\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\longrightarrow 0.\end{aligned}$$

Remark **3.14.** — By Proposition 3.12 (1), we have a bijection

$$\begin{aligned}\zeta_{\mathscr{F}}:\mathrm{Ext}^1_{\mathrm{GL}_{n-1}}\big(\pi_{\mathrm{alg}}(\phi^1,\mathbf{h}^1),\pi_1(\phi^1,\mathbf{h}^1)\big)\times\mathrm{Hom}(\mathrm{K}^\times,\mathrm{E})\\ \xrightarrow{\sim}\mathrm{Ext}^1_{\mathscr{F}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big),\end{aligned}$$

and a similar bijection $\zeta_{\mathscr{G}}$.

3.1.4. *Locally* σ*-analytic parabolic inductions.* — Let $\sigma\in\Sigma_{\mathrm{K}}$. Recall a locally $\mathbf{Q}_p$-analytic representation V of $\mathrm{GL}_n(\mathrm{K})$ over E is called locally σ*-analytic* if the $\mathfrak{gl}_n(\mathrm{K})\otimes_{\mathbf{Q}_p}\mathrm{E}$-action (obtained by derivation) on V factors through $\mathfrak{gl}_n(\mathrm{K})\otimes_{\mathrm{K},\sigma}\mathrm{E}$ (cf. [57, Section 2]). And V is called $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$*-algebraic* if $\mathrm{U}(\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}})v$ is a finite dimensional algebraic representation of $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$ over E for all $v\in\mathrm{V}$. Let λ_σ be the σ-component of λ, and $\lambda^\sigma:=(\lambda_\tau)_{\tau\neq\sigma}$. We also view them as weights of $\mathfrak{t}_{\Sigma_{\mathrm{K}}}$ in the obvious way. For $i=1,\dots,n-1$, $\mathrm{I}\subset\{1,\dots,n\}$, $\#\mathrm{I}=i$, let $w\in S_n$ such that $w(\{1,\dots,i\})=\mathrm{I}$. We have

$$\begin{aligned}\mathscr{C}(\mathrm{I},s_{i,\sigma})&\cong\mathcal{F}^{\mathrm{GL}_n}_{\mathrm{B}^-}\big(\mathrm{L}^-(-s_{i,\sigma}\cdot\lambda),w(\phi)\eta\big)\\&\cong\mathcal{F}^{\mathrm{GL}_n}_{\mathrm{B}^-}\big(\mathrm{L}^-(-s_{i,\sigma}\cdot\lambda_\sigma),w(\phi)\eta\big)\otimes_{\mathrm{E}}\mathrm{L}\big(\lambda^\sigma\big).\end{aligned}$$

Note we have $\mathcal{F}^{\mathrm{GL}_n}_{\mathrm{B}^-}(\mathrm{L}^-_\sigma(-s_{i,\sigma}\cdot\lambda_\sigma),w(\phi)\eta)\hookrightarrow(\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{B}^-(\mathrm{K})}w(\phi)\eta z^{s_{i,\sigma}\cdot\lambda_\sigma})^{\sigma-\mathrm{an}}$, where the sup-script "$\sigma-\mathrm{an}$" means the locally σ-analytic induction. So the both are locally σ-analytic. Let $\pi_{1,\sigma}(\phi,\mathbf{h})$ be the subrepresentation of $\pi_1(\phi,\mathbf{h})$ given by the extension of $\bigoplus_{\substack{i=1,\dots,n-1\\ \mathrm{I}\subset\{1,\dots,n\},\#\mathrm{I}=i}}\mathscr{C}(\mathrm{I},s_{i,\sigma})$ by $\pi_{\mathrm{alg}}(\phi,\lambda)$. Similarly, for $w\in S_n$, let $\mathrm{PS}_{1,\sigma}(w(\phi),\mathbf{h})\subset\mathrm{PS}_1(w(\phi),\mathbf{h})$ be the subrepresentation with irreducible constituents $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ and $\mathscr{C}(w,s_{i,\sigma})$ for $i=1,\dots,n-1$. It is easy to see

$$\mathrm{PS}_{1,\sigma}\big(w(\phi),\mathbf{h}\big)=\mathrm{PS}_1\big(w(\phi),\mathbf{h}\big)\cap\Big(\big(\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{B}^-(\mathrm{K})}w(\phi)\eta z^{\lambda_\sigma}\big)^{\sigma-\mathrm{an}}\otimes_{\mathrm{E}}\mathrm{L}(\lambda^\sigma)\Big)$$

$$\hookrightarrow \mathrm{PS}\big(w(\phi), \mathbf{h}\big).$$

Moreover, $\pi_{1,\sigma}(\phi, \mathbf{h})$ is the unique quotient of $\oplus^{w\in \mathrm{S}_n}_{\pi_{\mathrm{alg}}(\phi,\lambda)} \mathrm{PS}_{1,\sigma}(w(\phi), \mathbf{h})$ of socle $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$. In particular, $\pi_{1,\sigma}(\phi, \mathbf{h})$ is $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$-algebraic. In fact, $\pi_{1,\sigma}(\phi, \mathbf{h})$ is the maximal $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$-algebraic subrepresentation of $\pi_1(\phi, \mathbf{h})$.

For $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$-algebraic representations V and W, we denote by $\mathrm{Ext}^1_\sigma(\mathrm{V}, \mathrm{W}) \subset \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\mathrm{V}, \mathrm{W})$ the subspace of extensions, which are $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$-algebraic. Let $\mathrm{Hom}_{\sigma,g'}(\mathrm{T}(\mathrm{K}), \mathrm{E}) := \mathrm{Hom}_{g'}(\mathrm{T}(\mathrm{K}), \mathrm{E}) \cap \mathrm{Hom}_\sigma(\mathrm{T}(\mathrm{K}), \mathrm{E})$ (recalling $\mathrm{Hom}_\sigma(\mathrm{T}(\mathrm{K}), \mathrm{E})$ is the subspace of locally σ-analytic characters).

Lemma **3.15.** — *We have* $\dim_{\mathrm{E}} \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\mathrm{alg}}(\phi, \boldsymbol{h})) = n+1$, *and (3.4) induces an isomorphism* $\mathrm{Hom}_{\sigma,g'}(\mathrm{T}(\mathrm{K}), \mathrm{E}) \xrightarrow{\sim} \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\mathrm{alg}}(\phi, \boldsymbol{h}))$.

Proof. — As $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K}),\mathrm{Z}}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h})) \subset \mathrm{Ext}^1_{\mathrm{lalg}}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h}))$ (by (3.6)) hence is contained in $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h}))$, we have an exact sequence (similarly as in [14, Lem. 3.16])

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K}),\mathrm{Z}}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h})\big) \longrightarrow \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h})\big) \longrightarrow \mathrm{Hom}_\sigma\big(\mathrm{Z}(\mathrm{K}), \mathrm{E}\big) \longrightarrow 0.$$

The first part follows. It is clear that (3.4) induces the map in the lemma by restriction, which is hence injective. However, both the source and target spaces have the same dimension $n+1$, so the map is bijective. □

Proposition **3.16.** — *Let* $w \in \mathrm{S}_n$, *the map (3.10) induces an isomorphism*

$$\mathrm{Hom}_\sigma\big(\mathrm{T}(\mathrm{K}), \mathrm{E}\big) \xrightarrow{\sim} \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \mathrm{PS}_{1,\sigma}\big(w(\phi), \boldsymbol{h}\big)\big). \tag{3.31}$$

Proof. — For $\psi \in \mathrm{Hom}_\sigma(\mathrm{T}(\mathrm{K}), \mathrm{E})$, by similar arguments as in the proof of Proposition 3.3, $\mathrm{I}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{B}^-(\mathrm{K})}(w(\phi)\eta z^\lambda(1+\psi\epsilon))$ is a subrepresentation of $(\mathrm{Ind}^{\mathrm{GL}_n(\mathrm{K})}_{\mathrm{B}^-(\mathrm{K})} w(\phi)\eta z^{\lambda_\sigma}(1+\psi\epsilon))^{\sigma-\mathrm{an}} \otimes_{\mathrm{E}} \mathrm{L}(\lambda^\sigma)$, hence is $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$-algebraic. Together with the description of (3.10) in Remark 3.7, we deduce (3.10) induces the injective map in (3.31) by restriction. We have an exact sequence by dévissage

$$0 \longrightarrow \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\mathrm{alg}}(\phi, \mathbf{h})\big) \longrightarrow \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathrm{PS}_{1,\sigma}\big(w(\phi), \mathbf{h}\big)\big) \longrightarrow \oplus_{i=1}^{n-1} \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathscr{C}\big(w(\phi), s_{i,\sigma}\big)\big). \tag{3.32}$$

By Lemma 3.15 and Lemma 3.5 (1), $\dim_{\mathrm{E}} \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathrm{PS}_{1,\sigma}(w(\phi), \mathbf{h})) \leq (n+1) + (n-1) = 2n$. However, the source of (the injective) (3.31) has dimension $2n$, so (3.31) must be bijective. □

Remark **3.17.** — By the above proof, we see the last map in (3.32) is surjective and $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(w(\phi),s_{i,\sigma}))\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(w(\phi),s_{i,\sigma}))$.

Denote by $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ (resp. $\mathrm{Ext}^1_{\sigma,g'}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$) the image of $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{1,\sigma}(\phi,\mathbf{h}))$ (resp. $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$) via the (injective) push-forward map. It is easy to see

$$\begin{aligned}&\mathrm{Ext}^1_{\sigma,g'}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)\\&\quad=\mathrm{Ext}^1_{g'}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)\cap\mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big).\end{aligned}$$

Proposition **3.18.** — *(1) We have an exact sequence*

$$\begin{aligned}0\longrightarrow\mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\big)\longrightarrow\mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\\\longrightarrow\bigoplus_{\substack{i=1,\dots,n-1\\ \mathrm{I}\subset\{1,\dots,n-1\},\#\mathrm{I}=i}}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\mathscr{C}(\mathrm{I},s_{i,\sigma})\big)\longrightarrow 0.\end{aligned}\tag{3.33}$$

And $\dim_{\mathrm{E}}\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))=n+2^n-1$.

(2) The map (3.13) induces a surjection $t_{\phi,\boldsymbol{h}}:\oplus_{w\in\mathrm{S}_n}\mathrm{Hom}_\sigma(\mathrm{T}(\mathrm{K}),\mathrm{E})\twoheadrightarrow\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))$.

(3) The following map is surjective

$$\oplus_{\tau\in\Sigma_{\mathrm{K}}}\mathrm{Ext}^1_\tau\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\longrightarrow\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big),\tag{3.34}$$

and induces an isomorphism

$$\begin{aligned}&\oplus_{\tau\in\Sigma_{\mathrm{K}}}\Big(\mathrm{Ext}^1_\tau\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)/\mathrm{Ext}^1_g\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\Big)\\&\quad\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)/\mathrm{Ext}^1_g\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big).\end{aligned}\tag{3.35}$$

Proof. — (1) follows by the same argument as in the proof of Proposition 3.8 (1). (2) follows from Proposition 3.16 and Remark 3.17 by the same argument as in the proof of Proposition 3.8 (2). The first part of (3) follows easily by comparing the exact sequences (3.12) and (3.33). It is clear that (3.34) induces (3.35), which is hence surjective. However, both the sources and target spaces have the same dimension $(2^n-1)d_{\mathrm{K}}$ by (1) and Proposition 3.3 (1), Proposition 3.8 (1). So (3.35) is bijective. □

Now let P be a standard parabolic subgroup of GL_n, and $\mathscr{F}_{\mathrm{P}}$ be a P-filtration on ϕ. We use the notation in Section 3.1.3. Let $\pi_{\mathscr{F}_{\mathrm{P}},\sigma}(\phi,\mathbf{h}):=\pi_{\mathscr{F}_{\mathrm{P}}}(\phi,\mathbf{h})\cap\pi_{1,\sigma}(\phi,\mathbf{h})$, which is the maximal $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$-algebraic subrepresentation of $\pi_{\mathscr{F}_{\mathrm{P}}}(\phi,\mathbf{h})$. Then $\pi_{\mathscr{F}_{\mathrm{P}},\sigma}(\phi,\mathbf{h})$ is an extension of the direct sum of $\mathscr{C}(\mathrm{I},s_{i,\sigma})\in\mathrm{S}_{\mathscr{F}_{\mathrm{P}}}$ (for the fixed σ) by $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$. We denote by $\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ the image of $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathscr{F}_{\mathrm{P}},\sigma}(\phi,\mathbf{h}))$ via the (injective) push-forward map. As previously, we also write $\mathrm{Ext}^1_{\sigma,w}$ for $\mathrm{Ext}^1_{\sigma,\mathscr{F}_w}$. One easily sees $\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))=\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\cap\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$.

Proposition **3.19.** — *(1) We have* $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) = n+r+\sum_{i=1}^r(2^{n_i}-2)$.

(2) The isomorphism (3.27) induces an isomorphism

$$\prod_{i=1}^r \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}\big(\phi_{\mathscr{F}_{\mathrm{P}},i},\boldsymbol{h}^i\big),\pi_1\big(\phi_{\mathscr{F}_{\mathrm{P}},i},\boldsymbol{h}^i\big)\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\lambda)\big). \tag{3.36}$$

Moreover, for any w *such that the associated* B*-filtration* $\mathscr{T}_w$ *is compatible with* $\mathscr{F}_{\mathrm{P}}$*, the following diagram commutes*

$$\begin{array}{ccc}
\prod_{i=1}^r \mathrm{Hom}_\sigma(\mathrm{T}(\mathrm{K})\cap \mathrm{L}_{\mathrm{P},i}(\mathrm{K}),\mathrm{E}) & \xrightarrow{\sim} & \mathrm{Hom}_\sigma(\mathrm{T}(\mathrm{K}),\mathrm{E}) \\
\sim\big\downarrow (3.31) & & \sim\big\downarrow (3.31) \\
\prod_{i=1}^r \mathrm{Ext}^1_{\sigma,\mathscr{T}_{w,i}}(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i},\boldsymbol{h}^i),\pi_{1,\sigma}(\phi_{\mathscr{F}_{\mathrm{P}},i},\boldsymbol{h}^i)) & \xrightarrow{\sim} & \mathrm{Ext}^1_{\sigma,w}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_{1,\sigma}(\phi,\boldsymbol{h}))
\end{array}$$

where $\mathscr{T}_{w,i}$ *is the induced* $\mathrm{B}\cap\mathrm{L}_{\mathrm{P}}$*-filtration on* $\mathrm{gr}_i\,\mathscr{F}_{\mathrm{P}}$.

Proof. — By dévissage, Lemma 3.10 and a similar argument as in the proof of Proposition 3.16 (noting for a fixed σ, $\#\{\mathscr{C}(\mathrm{I},s_{i,\sigma})\in \mathrm{S}_{\mathscr{F}_{\mathrm{P}}}\} = \sum_{i=1}^r(2^{n_i}-2)+(r-1)$), we see $\dim_{\mathrm{E}}\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\leq n+r+\sum_{i=1}^r(2^{n_i}-2)$ and that (3.27) restricts to an injective map as in (3.36). Its source space has dimension $n+r+\sum_{i=1}^r(2^{n_i}-2)$ by Proposition 3.18 (1). We deduce (3.36) is bijective and (1) follows. The second part of (2) follows from (3.28). ☐

Finally, we have similarly as in Proposition 3.13:

Proposition **3.20.** — *Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be as in Proposition 3.13, there is a natural exact sequence* $0\to\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})\to\pi_{\mathscr{F},\sigma}(\phi,\boldsymbol{h})\oplus\pi_{\mathscr{G},\sigma}(\phi,\boldsymbol{h})\to\pi_{1,\sigma}(\phi,\boldsymbol{h})\to 0$*. Consequently, we have a natural exact sequence*

$$\begin{aligned}
0 &\longrightarrow \mathrm{Ext}^1_{\sigma,\mathscr{G}'}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\\
&\longrightarrow \mathrm{Ext}^1_{\sigma,\mathscr{F}}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\oplus \mathrm{Ext}^1_{\sigma,\mathscr{G}}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\\
&\longrightarrow \mathrm{Ext}^1_{\sigma}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\longrightarrow 0.
\end{aligned}\tag{3.37}$$

3.2. *Hodge parameters in* $\mathrm{GL}_n(\mathrm{K})$*-representations.*

3.2.1. *Construction and properties.* — In this section, we associate to $\mathrm{D}\in\Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$ a locally $\mathbf{Q}_p$-analytic representation $\pi_{\min}(\mathrm{D})$ of $\mathrm{GL}_n(\mathrm{K})$ over E, which determines those Hodge parameters of D reinterpreted in Section 2.2 (hence determines D when $\mathrm{K}=\mathbf{Q}_p$).

Consider the following composition (see (2.15) and (3.13) for the maps)

$$\oplus_{w\in \mathrm{S}_n}\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D}) \xrightarrow[\sim]{(\kappa_w)} \oplus_{w\in\mathrm{S}_n}\mathrm{Hom}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big)\overset{t_{\phi,\mathbf{h}}}{\twoheadrightarrow}\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big). \tag{3.38}$$

The following theorem is crucial for the paper.

Theorem **3.21.** — *The natural surjection* $\oplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D}) \twoheadrightarrow \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})$ *(cf. Proposition 2.12) factors through (3.38), i.e. there exists a unique map*

$$t_{\mathrm{D}}:\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\longrightarrow\!\!\!\!\rightarrow \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D}) \tag{3.39}$$

such that $\oplus_{w\in S_n}\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D}) \twoheadrightarrow \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})$ *is equal to* t_{D} *composed with (3.38).*

Proof. — We prove the theorem by induction on n. It is trivial for $n=1$. Suppose it holds for $n-1$. As in Section 2.4, let $\mathrm{D}_1\in\Phi\Gamma_{\mathrm{nc}}(\phi^1,\mathbf{h}^1)$ (resp. $\mathrm{C}_1\in\Phi\Gamma_{\mathrm{nc}}(\phi^1,\mathbf{h}^2)$) be the corresponding saturate (φ,Γ)-submodule (resp. quotient) of D (where $\phi^1:=\phi_1\boxtimes\cdots\boxtimes\phi_{n-1}$, $\mathbf{h}^1:=(\mathbf{h}_1,\dots,\mathbf{h}_{n-1})$, and $\mathbf{h}^2:=(\mathbf{h}_2,\dots,\mathbf{h}_n)$), and $\mathscr{F}$, $\mathscr{G}$ be the associated filtrations on D. For $w\in S_{n-1}$, the following diagram commutes (cf. (2.20)):

$$\begin{array}{ccc}
\mathrm{Hom}(\mathrm{T}(\mathrm{K}),\mathrm{E}) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{T}_1(\mathrm{K}),\mathrm{E})\times\mathrm{Hom}(\mathrm{K}^\times,\mathrm{E}) \\
{\scriptstyle\kappa_w}\uparrow{\scriptstyle\sim} & & {\scriptstyle\kappa_w}\uparrow{\scriptstyle\sim} \\
\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D}) & \xrightarrow{\sim} & \overline{\mathrm{Ext}}^1_w(\mathrm{D}_1,\mathrm{D}_1)\times\mathrm{Hom}(\mathrm{K}^\times,\mathrm{E}) \\
\downarrow & & \downarrow \\
\overline{\mathrm{Ext}}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D}) & \xrightarrow[\kappa_{\mathscr{F}}]{\sim} & \overline{\mathrm{Ext}}^1(\mathrm{D}_1,\mathrm{D}_1)\times\mathrm{Hom}(\mathrm{K}^\times,\mathrm{E}).
\end{array}$$

By induction hypothesis, the map $\oplus_{w\in S_{n-1}}\overline{\mathrm{Ext}}^1_w(\mathrm{D}_1,\mathrm{D}_1)\twoheadrightarrow\overline{\mathrm{Ext}}^1(\mathrm{D}_1,\mathrm{D}_1)$ factors through the following map (defined similarly as in (3.38))

$$t_{\phi^1,\mathbf{h}^1}:\oplus_{w\in S_{n-1}}\overline{\mathrm{Ext}}^1_w(\mathrm{D}_1,\mathrm{D}_1)\longrightarrow\!\!\!\!\rightarrow\mathrm{Ext}^1_{\mathrm{GL}_{n-1}(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi^1,\mathbf{h}^1),\pi_1(\phi^1,\mathbf{h}^1)\big).$$

This together with Proposition 3.12 and (2.19) imply $\oplus_{w\in S_{n-1}}\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})\twoheadrightarrow\overline{\mathrm{Ext}}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})$ $(\hookrightarrow\overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D}))$ factors through

$$t_{\mathscr{F},\mathrm{D}}:\mathrm{Ext}^1_{\mathscr{F}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)\longrightarrow\!\!\!\!\rightarrow\overline{\mathrm{Ext}}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})\hookrightarrow\overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D}). \tag{3.40}$$

Let $S'_{n-1}:=\{w\in S_n\mid w(n)=1\}$, that is a subset of S_n of cardinality $(n-1)!$. By a similar discussion with D_1 replaced by C_1, the map $\oplus_{w\in S'_{n-1}}\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})\twoheadrightarrow\overline{\mathrm{Ext}}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D})\hookrightarrow\overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})$ factors through $t_{\mathscr{G},\mathrm{D}}:\mathrm{Ext}^1_{\mathscr{G}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\twoheadrightarrow\overline{\mathrm{Ext}}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D})\hookrightarrow\overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})$. By (3.18) and (2.13) (with g replaced by g'), the following diagram commutes (see

(3.24) for the injections from $\mathrm{Ext}^1_{g'}$)

$$\begin{array}{ccc}\mathrm{Ext}^1_{g'}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big) & \hookrightarrow & \mathrm{Ext}^1_{\mathscr{F}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big) \\ \big\downarrow & & \big\downarrow t_{\mathscr{F},\mathrm{D}} \\ \mathrm{Ext}^1_{\mathscr{G}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big) & \xrightarrow{t_{\mathscr{G},\mathrm{D}}} & \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D}).\end{array}$$

Hence by the second exact sequence in Proposition 3.13, the composition

$$\mathrm{Ext}^1_{\mathscr{F}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)\oplus \mathrm{Ext}^1_{\mathscr{G}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)$$
$$\twoheadrightarrow \overline{\mathrm{Ext}}^1_{\mathscr{F}}(\mathrm{D},\mathrm{D})\oplus \overline{\mathrm{Ext}}^1_{\mathscr{G}}(\mathrm{D},\mathrm{D}) \longrightarrow \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})$$

factors though a map

$$t_{\mathrm{D}}:\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)\twoheadrightarrow \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D}). \tag{3.41}$$

Next, we show t_{D} satisfies the property in the theorem. By construction, the map $\oplus_{w\in \mathrm{S}_{n-1}\cup \mathrm{S}'_{n-1}}\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})\to \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})$ factors through t_{D}. It suffices to show for the other $w\in \mathrm{S}_n$, $\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})\to \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})$ also factors as

$$\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})\xrightarrow[\sim]{\kappa_w}\mathrm{Hom}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big)\xrightarrow{\zeta_w}\mathrm{Ext}^1_{\mathrm{GL}_n}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big)\xrightarrow{(3.41)}\overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D}). \tag{3.42}$$

Suppose hence $w(n)=i$ with $1<i<n$. We have

$$\mathrm{Hom}\big(\mathrm{T}(\mathrm{K}),\mathrm{E}\big)\cong \oplus_{j=1}^{n-1}\mathrm{Hom}\big(\mathrm{Z}_j(\mathrm{K}),\mathrm{E}\big)\oplus \mathrm{Hom}\big(\mathrm{Z}(\mathrm{K}),\mathrm{E}\big), \tag{3.43}$$

where $\mathrm{Z}_j\subset \mathrm{T}$ is the centre of the Levi subgroup L_j (containing T) of the maximal parabolic subgroup P_j (with $j\notin \mathscr{W}_{\mathrm{P}_j}$). For any $j=1,\dots,n-1$, $\kappa_w^{-1}(\mathrm{Hom}(\mathrm{Z}_j(\mathrm{K}),\mathrm{E}))\subset \overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}_j},g'}(\mathrm{D},\mathrm{D})$ (cf. Corollary 2.15), where $\mathscr{F}_{\mathrm{P}_j}$ is the P_j-filtration associated to the B-filtration $\mathscr{T}_w$ (such that $\mathscr{T}_w$ is compatible with $\mathscr{F}_{\mathrm{P}_j}$). Let w_j be an element in the Weyl group of L_j such that $w_j(i)=1$ or $w_j(i)=n$ (whose existence is clear). By Corollary 2.15 (2) and (3.30), we have a commutative diagram

$$\begin{array}{ccccc}\overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}_j},g'}(\mathrm{D},\mathrm{D}) & \xrightarrow[\sim]{\kappa_w} & \mathrm{Hom}_{\mathrm{P}_j,g'}(\mathrm{T}(\mathrm{K}),\mathrm{E}) & \xhookrightarrow{\zeta_w} & \mathrm{Ext}^1_{\mathrm{GL}_n}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \\ \| & & w_j\big\downarrow\wr & & \| \\ \overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}_j},g'}(\mathrm{D},\mathrm{D}) & \xrightarrow[\sim]{\kappa_{w_jw}} & \mathrm{Hom}_{\mathrm{P}_j,g'}(\mathrm{T}(\mathrm{K}),\mathrm{E}) & \xhookrightarrow{\zeta_{w_jw}} & \mathrm{Ext}^1_{\mathrm{GL}_n}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})).\end{array}$$

It is clear that $w_j w \in \mathrm{S}_{n-1} \cup \mathrm{S}'_{n-1}$, hence the map $\overline{\mathrm{Ext}}^1_{w_j w}(\mathrm{D}, \mathrm{D}) \to \overline{\mathrm{Ext}}^1(\mathrm{D}, \mathrm{D})$ is equal to $t_{\mathrm{D}} \circ (\zeta_{w_j w} \circ \kappa_{w_j w})$. In particular, its restriction to $\overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}_j}, g'}(\mathrm{D}, \mathrm{D})$ is equal to $t_{\mathrm{D}} \circ (\zeta_{w_j w} \circ \kappa_{w_j w}) = t_{\mathrm{D}} \circ (\zeta_w \circ \kappa_w)$ by the above commutative diagram. As $\overline{\mathrm{Ext}}^1_w(\mathrm{D}, \mathrm{D})$ is spanned by $\overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}_j}, g'}(\mathrm{D}, \mathrm{D})$ and $\mathrm{Hom}(\mathrm{Z}(\mathrm{K}), \mathrm{E})$ (e.g. using (3.43)), we obtain the factorisation as in (3.42). This concludes the proof. □

Remark **3.22.** — (1) By comparing dimensions (using Proposition 2.10, Proposition 3.8 (1)), we have $\dim_{\mathrm{E}} \mathrm{Ker}(t_{\mathrm{D}}) = (2^n - \frac{n(n+1)}{2} - 1) d_{\mathrm{K}}$.

(2) The same argument holds with $\pi_1(\phi, \mathbf{h})$ replaced by $\pi(\phi, \mathbf{h})$ (with the same t_{D} under the isomorphism (3.16)).

The following lemma is clear.

Lemma **3.23.** — *For any* $w \in \mathrm{S}_n$, $\mathrm{Ker}(t_{\mathrm{D}}) \cap \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})) = 0$.

Let $\pi_{\min}(\mathrm{D})$ (resp. $\pi_{\mathrm{fs}}(\mathrm{D})$) be the extension of $\mathrm{Ker}(t_{\mathrm{D}}) \otimes_{\mathrm{E}} \pi_{\mathrm{alg}}(\phi, \mathbf{h})$ ($\cong \pi_{\mathrm{alg}}(\phi, \mathbf{h})^{\oplus (2^n - \frac{n(n+1)}{2} - 1) d_{\mathrm{K}}}$) by $\pi_1(\phi, \mathbf{h})$ (resp. $\pi(\phi, \mathbf{h})$) associated to $\mathrm{Ker}(t_{\mathrm{D}})$ (cf. Section 3.1.1, see also Remark 3.22 (2)). Note that as $\mathrm{End}_{\mathrm{GL}_n(\mathrm{K})}(\pi(\phi, \mathbf{h})) \xrightarrow{\sim} \mathrm{End}_{\mathrm{GL}_n(\mathrm{K})}(\pi_1(\phi, \mathbf{h})) \xrightarrow{\sim} \mathrm{End}_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}) \cong \mathrm{E}$, either $\pi_{\min}(\mathrm{D})$ or $\pi_{\mathrm{fs}}(\mathrm{D})$ determines $\mathrm{Ker}(t_{\mathrm{D}})$. We have

$$\pi_{\mathrm{fs}}(\mathrm{D}) \cong \pi_{\min}(\mathrm{D}) \oplus_{\pi_1(\phi, \mathbf{h})} \pi(\phi, \mathbf{h}). \tag{3.44}$$

In the sequel, we will mainly work with $\pi_{\min}(\mathrm{D})$, noting that most of the statements generalize to $\pi_{\mathrm{fs}}(\mathrm{D})$ without effort. We have an exact sequence

$$\begin{aligned}(3.45)\qquad & 0 \longrightarrow \mathrm{Hom}_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \mathrm{Ker}(t_{\mathrm{D}}) \otimes_{\mathrm{E}} \pi_{\mathrm{alg}}(\phi, \mathbf{h})\big) \\ & \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big) \xrightarrow{f_{\mathrm{D}}} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\min}(\mathrm{D})\big).\end{aligned}$$

By Lemma 3.5 (2), one sees the last map f_{D} is surjective. For a P-filtration $\mathscr{F}_{\mathrm{P}}$ on D, we denote by $\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\min}(\mathrm{D}))$ the image of $\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$ under f_{D}, and write Ext^1_w for $\mathrm{Ext}^1_{\mathscr{T}_w}$. Denote by $\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\min}(\mathrm{D}))$ the image of $\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$ under f_{D}.

Corollary **3.24.** — *The map* t_{D} *induces* $\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\min}(\mathrm{D})) \xrightarrow{\sim} \overline{\mathrm{Ext}}^1_g(\mathrm{D}, \mathrm{D})$ *and* $\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\min}(\mathrm{D})) \xrightarrow{\sim} \overline{\mathrm{Ext}}^1_w(\mathrm{D}, \mathrm{D})$ *for all* $w \in \mathrm{S}_n$.

Proof. — By Lemma 3.23, $\mathrm{Ext}^1_*(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) \xrightarrow{\sim} \mathrm{Ext}^1_*(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{\min}(\mathrm{D}))$ for $* \in \{w, g\}$. The corollary then follows from the definition of t_{D}, (2.15), (3.17) and Proposition 3.3 (1). □

The following corollary is a direct consequence of Theorem 3.21.

Corollary **3.25.** — *Let* $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\phi, \boldsymbol{h})$. *The representation* $\pi_{\min}(\mathrm{D})$ *is the unique extension of* $\pi_{\mathrm{alg}}(\phi, \boldsymbol{h})^{\oplus(2^n - \frac{n(n+1)}{2} - 1)d_{\mathrm{K}}}$ *by* $\pi_1(\phi, \boldsymbol{h})$ *satisfying the following properties:*

(1) $\mathrm{soc}_{\mathrm{GL}_n(\mathrm{K})}\pi_{\min}(\mathrm{D}) \cong \pi_{\mathrm{alg}}(\phi, \boldsymbol{h})$, *and* $\mathrm{soc}_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\min}(\mathrm{D})/\pi_{\mathrm{alg}}(\phi, \boldsymbol{h})) \cong \mathrm{soc}_{\mathrm{GL}_n(\mathrm{K})}(\pi_1(\phi, \boldsymbol{h})/\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}))$.

(2) There is a bijection

$$t_{\mathrm{D}} : \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\min}(\mathrm{D})\big) \xrightarrow{\sim} \overline{\mathrm{Ext}}^1(\mathrm{D}, \mathrm{D})$$

which is compatible with trianguline deformations, i.e. for $w \in \mathrm{S}_n$, *the composition* $\mathrm{Hom}(\mathrm{T}(\mathrm{K}), \mathrm{E}) \xrightarrow[\sim]{\zeta_w} \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\min}(\mathrm{D})) \xhookrightarrow{t_{\mathrm{D}}} \overline{\mathrm{Ext}}^1(\mathrm{D}, \mathrm{D})$ *coincides with* $\mathrm{Hom}(\mathrm{T}(\mathrm{K}), \mathrm{E}) \xrightarrow[\sim]{\kappa_w^{-1}} \overline{\mathrm{Ext}}^1_w(\mathrm{D}, \mathrm{D}) \hookrightarrow \overline{\mathrm{Ext}}^1(\mathrm{D}, \mathrm{D})$.

Let χ_{D} be the character $z^{|\lambda|}|\cdot|_{\mathrm{K}}^{\frac{n(n-1)}{2}} \prod_{i=1}^n \phi_i$ of $\mathrm{K}^\times$ with $|\lambda| = \sum_{\substack{i=1,\dots,n \\ \sigma\in\Sigma_{\mathrm{K}}}} \lambda_{i,\sigma}$. We have $\wedge^n \mathrm{D} \cong \mathcal{R}_{\mathrm{K},\mathrm{E}}(\chi_{\mathrm{D}}\varepsilon^{-\frac{n(n-1)}{2}})$. For an integral weight μ of $\mathfrak{t}_{\Sigma_{\mathrm{K}}}$, let ξ_μ be the central character of $\mathrm{U}(\mathfrak{gl}_{n,\Sigma_{\mathrm{K}}})$ acting on $\mathrm{L}(\mu)$.

Proposition **3.26.** — *The representation* $\pi_{\min}(\mathrm{D})$ *has central character* χ_{D} *and infinitesimal character* ξ_λ.

Proof. — We only prove the statement for the infinitesimal character, the central character being similar. Let $\mathcal{Z}_{\mathrm{K}}$ be the centre of $\mathrm{U}(\mathfrak{gl}_{n,\Sigma_{\mathrm{K}}})$. Recall we have the Harish-Chandra isomorphism $\mathrm{HC} : \mathcal{Z}_{\mathrm{K}} \xrightarrow{\sim} \mathrm{U}(\mathfrak{t}_{\Sigma_{\mathrm{K}}})^{\mathscr{W}_{n,\mathrm{K}}}$, where $\mathscr{W}_{n,\mathrm{K}}$ is the Weyl group of $\mathrm{Res}^{\mathrm{K}}_{\mathbf{Q}_p}\mathrm{GL}_n$, isomorphic to $\mathrm{S}_n^{d_{\mathrm{K}}}$, and where we normalize the map such that a weight μ of $\mathfrak{t}_{\Sigma_{\mathrm{K}}}$, seen as a character of $\mathrm{U}(\mathfrak{t}_{\Sigma_{\mathrm{K}}})^{\mathscr{W}_{n,\mathrm{K}}}$, corresponds to $\xi_{\mu-\theta^{[\mathrm{K}:\mathbf{Q}_p]}}$ of $\mathcal{Z}_{\mathrm{K}}$ (recalling $\theta^{[\mathrm{K}:\mathbf{Q}_p]} = (0, \dots, 1-n)_{\sigma\in\Sigma_{\mathrm{K}}}$). In particular, the weight $\mathbf{h}$ corresponds to ξ_λ. Let X_{ξ_λ} (resp. $\mathrm{X}_{\mathbf{h}}$) be the tangent space of $\mathcal{Z}_{\mathrm{K}}$ (resp. $\mathrm{U}(\mathfrak{t}_{\Sigma_{\mathrm{K}}})$) at ξ_λ (resp. at $\mathbf{h}$), i.e. $\mathrm{X}_{\xi_\lambda} = \{f : \mathcal{Z}_{\mathrm{K}} \twoheadrightarrow \mathrm{E}[\epsilon]/\epsilon^2 \mid f \equiv \xi_\lambda \pmod{\epsilon}\}$ and similarly for $\mathrm{X}_{\mathbf{h}}$. The map HC induces a bijection $\mathrm{HC} : \mathrm{X}_{\mathbf{h}} \xrightarrow{\sim} \mathrm{X}_{\xi_\lambda}$ (noting the injection $\mathrm{U}(\mathfrak{t}_{\Sigma_{\mathrm{K}}})^{\mathscr{W}_{n,\mathrm{K}}} \hookrightarrow \mathrm{U}(\mathfrak{t}_{\Sigma_{\mathrm{K}}})$ induces bijections on tangent spaces, e.g. by the explicit description of the invariants $\mathrm{U}(\mathfrak{t}_{\Sigma_{\mathrm{K}}})^{\mathscr{W}_{n,\mathrm{K}}}$ as a polynomial algebra).

For $\widetilde{\mathrm{D}} \in \mathrm{Ext}^1(\mathrm{D}, \mathrm{D})$, the Sen weights of $\widetilde{\mathrm{D}}$ (over $\mathrm{E}[\epsilon]/\epsilon^2$) have the form $(h_{i,\sigma} + a_{i,\sigma}\epsilon)_{\substack{\sigma\in\Sigma_{\mathrm{K}} \\ i=1,\dots,n}}$. We obtain hence an E-linear map $\mathrm{Ext}^1(\mathrm{D}, \mathrm{D}) \to \mathrm{X}_{\mathbf{h}}$, $\widetilde{\mathrm{D}} \mapsto (a_{i,\sigma})$. The map sends $\mathrm{Ext}^1_g(\mathrm{D}, \mathrm{D})$ hence $\mathrm{Ext}^1_0(\mathrm{D}, \mathrm{D})$ to zero, thus induces an E-linear map $d_{\mathrm{Sen}} : \overline{\mathrm{Ext}}^1(\mathrm{D}, \mathrm{D}) \to \mathrm{X}_{\mathbf{h}}$. For $\widetilde{\pi} \in \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$ equipped with the natural $\mathrm{E}[\epsilon]/\epsilon^2$-action, and $a \in \mathcal{Z}_{\mathrm{K}}$, the operator $(a - \xi_\lambda(a))$ (on $\widetilde{\pi}$) annihilates $\pi_1(\phi, \mathbf{h})$ hence induces a $\mathrm{GL}_n(\mathrm{K})$-equivariant map $\widetilde{\pi} \twoheadrightarrow \pi_{\mathrm{alg}}(\phi, \mathbf{h}) \to \widetilde{\pi}$. As $\mathrm{Hom}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) \cong$

E, this map is equal to $\alpha\epsilon$ for some $\alpha \in \mathrm{E}$ (depending on a). We deduce $\mathcal{Z}_{\mathrm{K}}$ acts on $\widetilde{\pi}$ via a character over $\mathrm{E}[\epsilon]/\epsilon^2$, which corresponds to an element in X_{ξ_λ}. We obtain hence an E-linear map: $d_{\inf}: \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) \to \mathrm{X}_{\xi_\lambda}$. The proposition (for the infinitesimal character) will be a direct consequence of the commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) & \xrightarrow{d_{\inf}} & \mathrm{X}_{\xi_\lambda} \\ \downarrow t_{\mathrm{D}} & & \downarrow \mathrm{HC}^{-1} \\ \overline{\mathrm{Ext}}^1(\mathrm{D}, \mathrm{D}) & \xrightarrow{d_{\mathrm{Sen}}} & \mathrm{X}_{\mathbf{h}}. \end{array} \tag{3.46}$$

By the construction of t_{D}, it suffices to show for all $w \in \mathrm{S}_n$, the following diagram commutes

$$\begin{array}{ccc} \overline{\mathrm{Ext}}^1_w(\mathrm{D}, \mathrm{D}) & \xrightarrow{d_{\mathrm{Sen}}} & \mathrm{X}_{\mathbf{h}} \\ \sim\downarrow & & \downarrow \mathrm{HC} \\ \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) & \xrightarrow{d_{\inf}} & \mathrm{X}_{\xi_\lambda}. \end{array} \tag{3.47}$$

Let $\psi \in \mathrm{Hom}(\mathrm{T}(\mathrm{K}), \mathrm{E})$, and $\widetilde{\mathrm{D}} \in \mathrm{Ext}^1_w(\mathrm{D}, \mathrm{D})$ be a (φ, Γ)-module over $\mathcal{R}_{\mathrm{K}, \mathrm{E}[\epsilon]/\epsilon^2}$ of trianguline parameter $\widetilde{\delta} := w(\phi) z^{\mathbf{h}}(1 + \psi\epsilon)$. Let $d\widetilde{\delta}: \mathrm{U}(\mathfrak{t}_{\Sigma_{\mathrm{K}}}) \to \mathrm{E}[\epsilon]/\epsilon^2$ be the morphism induced by $\widetilde{\delta}$ by derivation. Then $d_{\mathrm{Sen}}(\widetilde{\mathrm{D}}) = d\widetilde{\delta}$. By (2.12) (3.11) and Remark 3.7, the image $\widetilde{\pi}$ of $\widetilde{\mathrm{D}}$ under the left vertical map in (3.47) satisfies $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}(\widetilde{\delta}\eta z^{-\theta^{[\mathrm{K}:\mathbf{Q}_p]}}) \hookrightarrow \widetilde{\pi}$. By [37, (0.4)], we have $\widetilde{\delta}\eta\delta_{\mathrm{B}} z^{-\theta^{[\mathrm{K}:\mathbf{Q}_p]}} \hookrightarrow \mathrm{J}_{\mathrm{B}}(\widetilde{\pi}) \hookrightarrow \widetilde{\pi}^{\mathfrak{n}}$, where $\mathfrak{n}$ is the Lie algebra of N. We see $\mathcal{Z}_{\mathrm{K}}$ acts on the image of the injection via $d\widetilde{\delta} \circ \mathrm{HC}$. Moreover, the composition $\widetilde{\delta}\eta\delta_{\mathrm{B}} z^{-\theta^{[\mathrm{K}:\mathbf{Q}_p]}} \hookrightarrow \mathrm{J}_{\mathrm{B}}(\widetilde{\pi}) \hookrightarrow \widetilde{\pi} \twoheadrightarrow \pi_{\mathrm{alg}}(\phi, \mathbf{h})$ is non-zero, since $\dim_{\mathrm{E}} \mathrm{Hom}_{\mathrm{T}(\mathrm{K})}(w(\phi) z^{\mathbf{h}}\eta z^{-\theta^{[\mathrm{K}:\mathbf{Q}_p]}}\delta_{\mathrm{B}}, \mathrm{J}_{\mathrm{B}}(\pi_1(\phi, \mathbf{h}))) \cong \mathrm{E}$. We deduce the subrepresentation $\widetilde{\pi}[\mathcal{Z}_{\mathrm{K}} = d\widetilde{\delta} \circ \mathrm{HC}]$ strictly contains $\pi_1(\phi, \mathbf{h})$ hence is equal to $\widetilde{\pi}$ itself. So $d_{\inf}(\widetilde{\pi}) = d\widetilde{\delta} \circ \mathrm{HC}$ and (3.47) commutes. This concludes the proof. □

We next discuss the compatibility of t_{D} (and $\pi_{\min}(\mathrm{D})$) with parabolic inductions. Let $\mathrm{P} \supset \mathrm{B}$ be a standard parabolic subgroup of GL_n with L_{P} equal to $\mathrm{diag}(\mathrm{GL}_{n_1}, \dots, \mathrm{GL}_{n_r})$. Let $\mathscr{F}_{\mathrm{P}}$ be a P-filtration of D, $\mathrm{M}_i := \mathrm{gr}_i \mathscr{F}_{\mathrm{P}}$, which is a (φ, Γ)-module of rank n_i, for $i = 1, \dots, r$. Recall we have defined $\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}$ for both (φ, Γ)-modules (cf. the discussion above Proposition 2.13), and $\mathrm{GL}_n(\mathrm{K})$-representations (cf. Section 3.1.3).

Proposition **3.27.** — *The map t_{D} restricts to a surjection*

$$t_{\mathrm{D}, \mathscr{F}_{\mathrm{P}}}: \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})\big) \twoheadrightarrow \overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D}, \mathrm{D}).$$

Moreover, the following diagram commutes

$$(\mathbf{3.48})\qquad \begin{array}{ccc} \prod_{i=1}^r \mathrm{Ext}^1_{\mathrm{GL}_{n_i}(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i},\boldsymbol{h}^i),\pi_1(\phi_{\mathscr{F}_{\mathrm{P}},i},\boldsymbol{h}^i)) & \xrightarrow{(t_{\mathrm{M}_i})} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{M}_i,\mathrm{M}_i) \\ \sim\downarrow (3.27) & & \sim\downarrow (2.19) \\ \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) & \xrightarrow{t_{\mathrm{D},\mathscr{F}_{\mathrm{P}}}} & \overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}). \end{array}$$

In particular, the parabolic induction (3.27) induces a natural isomorphism

$$(\mathbf{3.49})\qquad \oplus_{i=1}^r \mathrm{Ker}(t_{\mathrm{M}_i}) \xrightarrow{\sim} \mathrm{Ker}(t_{\mathrm{D},\mathscr{F}_{\mathrm{P}}}) = \mathrm{Ker}(t_{\mathrm{D}}) \cap \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big).$$

Proof. — By Corollary 2.14 (2), Proposition 2.12, $\overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D})$ can be spanned by $\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D})$ for $\mathscr{T}_w$ compatible with $\mathscr{F}_{\mathrm{P}}$. Together with (3.29), the first part follows. The commutativity of the diagram follows from (2.20) and (3.28). □

Remark **3.28.** — Let $\pi_{\mathrm{min},\mathscr{F}_{\mathrm{P}}}(\mathrm{D}) \subset \pi_{\mathrm{min}}(\mathrm{D})$ be the extension of $\mathrm{Ker}(t_{\mathrm{D},\mathscr{F}_{\mathrm{P}}}) \otimes_{\mathrm{E}} \pi_{\mathrm{alg}}(\phi,\mathbf{h}) \cong \pi_{\mathrm{alg}}(\phi,\mathbf{h})^{\oplus \sum_{i=1}^r (d_{\mathrm{K}}(2^{n_i}-\frac{n_i(n_i+1)}{2}-1))}$ by $\pi_1(\phi,\mathbf{h})$ associated to $\mathrm{Ker}(t_{\mathrm{D},\mathscr{F}_{\mathrm{P}}})$. By Proposition 3.27, $\pi_{\mathrm{min},\mathscr{F}_{\mathrm{P}}}(\mathrm{D})$ is the maximal subrepresentation of $\pi_{\mathrm{min}}(\mathrm{D})$ which comes from the push-forward of extensions of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ by $\pi_{\mathscr{F}_{\mathrm{P}}}(\phi,\mathbf{h})$ via $\pi_{\mathscr{F}_{\mathrm{P}}}(\phi,\mathbf{h}) \hookrightarrow \pi_1(\phi,\mathbf{h})$. We have $\mathrm{I}_{\mathrm{P}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}((\widehat{\boxtimes}_{i=1}^r \pi_{\mathrm{min}}(\mathrm{M}_i)) \otimes_{\mathrm{E}} \varepsilon^{-1} \circ \theta^{\mathrm{P}}) \hookrightarrow \pi_{\mathrm{min},\mathscr{F}_{\mathrm{P}}}(\mathrm{D})$. Moreover, (3.48) induces a commutative diagram

$$\begin{array}{ccc} \prod_{i=1}^r \mathrm{Ext}^1_{\mathrm{GL}_{n_i}(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i},\mathbf{h}^i),\pi_{\mathrm{min}}(\mathrm{M}_i)) & \xrightarrow[\sim]{(t_{\mathrm{M}_i})} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_{(\varphi,\Gamma)}(\mathrm{M}_i,\mathrm{M}_i) \\ \sim\downarrow & & \sim\downarrow \\ \mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{min},\mathscr{F}_{\mathrm{P}}}(\mathrm{D})) & \xrightarrow[\sim]{t_{\mathrm{D},\mathscr{F}_{\mathrm{P}}}} & \overline{\mathrm{Ext}}^1_{\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}), \end{array}$$

where $\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{min},\mathscr{F}_{\mathrm{P}}}(\mathrm{D}))$ is the image of $\mathrm{Ext}^1_{\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ via the push-forward map, and where the left vertical map is obtained in a similar way as (3.27).

Let $\sigma \in \Sigma_{\mathrm{K}}$. By Proposition 3.18 and Corollary 2.19, (3.38) restricts to a surjection

$$(\mathbf{3.50})\qquad \oplus_{w\in S_n} \overline{\mathrm{Ext}}^1_{\sigma,w}(\mathrm{D},\mathrm{D}) \twoheadrightarrow \mathrm{Ext}^1_{\sigma}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big).$$

Corollary **3.29.** — *Let* $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\phi,\boldsymbol{h})$.

(1) The map (2.26), quotienting by $\mathrm{Ext}^1_0(\mathrm{D},\mathrm{D})$*, factors through (3.50) and the restriction of* t_{D}*:*

$$t_{\mathrm{D},\sigma} : \mathrm{Ext}^1_{\sigma}\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big) \twoheadrightarrow \overline{\mathrm{Ext}}^1_{\sigma}(\mathrm{D},\mathrm{D}).$$

(2) Let P *be a standard parabolic subgroup and* $\mathscr{F}_{\mathrm{P}}$ *be a* P*-filtration on* D*. Let* $t_{\mathrm{D},\mathscr{F}_{\mathrm{P}},\sigma}$ *be the restriction of* $t_{\mathrm{D},\sigma}$ *to* $\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))$*, we have a commutative diagram*

$$\begin{array}{ccc} \prod_{i=1}^r \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i},\boldsymbol{h}^i),\pi_1(\phi_{\mathscr{F}_{\mathrm{P}},i},\boldsymbol{h}^i)) & \xrightarrow{(t_{\mathrm{M}_i,\sigma})} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(\mathrm{M}_i,\mathrm{M}_i) \\ \sim\downarrow(3.36) & & \sim\downarrow(2.24) \\ \mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})) & \xrightarrow{t_{\mathrm{D},\mathscr{F}_{\mathrm{P}},\sigma}} & \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D},\mathrm{D}). \end{array}$$

In particular, (3.36) induces $\oplus_{i=1}^r \mathrm{Ker}\, t_{\mathrm{M}_{i,\sigma}} \xrightarrow{\sim} \mathrm{Ker}\, t_{\mathrm{D},\mathscr{F}_{\mathrm{P}},\sigma}$*.*

Proof. — (1) follows by Theorem 3.21 and Corollary 2.40. (2) follows from (3.48). □

Remark **3.30.** — (1) It is clear that $\mathrm{Ker}(t_{\mathrm{D},\sigma}) = \mathrm{Ker}(t_{\mathrm{D}}) \cap \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$. By Lemma 2.16 and (2.14), $\dim_{\mathrm{E}} \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D},\mathrm{D}) = n + \frac{n(n+1)}{2}$. Together with Proposition 3.18 (1), we then deduce $\dim_{\mathrm{E}} \mathrm{Ker}(t_{\mathrm{D},\sigma}) = 2^n - \frac{n(n+1)}{2} - 1$.

(2) Recall we have $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{1,\sigma}(\phi,\mathbf{h})) \xrightarrow{\sim} \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ (see the discussion below Remark 3.17). We view hence $\mathrm{Ker}(t_{\mathrm{D},\sigma})$ as subspace of $\mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{1,\sigma}(\phi,\mathbf{h}))$. Set $\pi_{\min}(\mathrm{D})_\sigma$ to be the extension of $\mathrm{Ker}(t_{\mathrm{D},\sigma}) \otimes_{\mathrm{E}} \pi_{\mathrm{alg}}(\phi,\mathbf{h})$ by $\pi_{1,\sigma}(\phi,\mathbf{h})$. Then $\pi_{\min}(\mathrm{D})_\sigma$ is just the maximal $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$-algebraic subrepresentation of $\pi_{\min}(\mathrm{D})$.

Corollary **3.31.** — *We have* $\oplus_{\sigma\in\Sigma_{\mathrm{K}}} \mathrm{Ker}(t_{\mathrm{D},\sigma}) \xrightarrow{\sim} \mathrm{Ker}(t_{\mathrm{D}})$*. Consequently,* $\pi_{\min}(\mathrm{D}) \cong \bigoplus^{\sigma\in\Sigma_{\mathrm{K}}}_{\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})} \pi_{\min}(\mathrm{D})_\sigma$*.*

Proof. — The second part follows from the first one (see also Remark 3.30 (2)). Consider the induced maps (where we omit "$(\phi,\mathbf{h})$" for short)

$$\mathrm{Ext}^1_{\mathrm{GL}_n}\big(\pi_{\mathrm{alg}},\pi_1\big)/\,\mathrm{Ext}^1_g\big(\pi_{\mathrm{alg}},\pi_1\big) \xrightarrow{\bar{t}_{\mathrm{D}}} \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})/\overline{\mathrm{Ext}}^1_g(\mathrm{D},\mathrm{D}),$$

$$\mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}},\pi_1\big)/\,\mathrm{Ext}^1_g\big(\pi_{\mathrm{alg}},\pi_1\big) \xrightarrow{\bar{t}_{\mathrm{D},\sigma}} \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D},\mathrm{D})/\overline{\mathrm{Ext}}^1_g(\mathrm{D},\mathrm{D}).$$

As $\mathrm{Ext}^1_g(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \cap \mathrm{Ker}(t_{\mathrm{D}}) = 0$ (cf. Lemma 3.23), we have isomorphisms $\mathrm{Ker}(t_{\mathrm{D}}) \xrightarrow{\sim} \mathrm{Ker}\,\bar{t}_{\mathrm{D}}$ and $\mathrm{Ker}(t_{\mathrm{D},\sigma}) \xrightarrow{\sim} \mathrm{Ker}\,\bar{t}_{\mathrm{D},\sigma}$. Using Proposition 3.18 (3), $\oplus_{\sigma\in\Sigma_{\mathrm{K}}} \mathrm{Ker}\,\bar{t}_{\mathrm{D},\sigma} \to \mathrm{Ker}\,\bar{t}_{\mathrm{D}}$ is injective. We deduce the natural map $\oplus_{\sigma\in\Sigma_{\mathrm{K}}} \mathrm{Ker}(t_{\mathrm{D},\sigma}) \to \mathrm{Ker}(t_{\mathrm{D}})$ is injective. As the both sides have dimension $(2^n - \frac{n(n+1)}{2} - 1)d_{\mathrm{K}}$ by Remark 3.22 (1) and Remark 3.30 (1), the map is actually bijective. □

Let $\mathrm{D}_\sigma = \mathfrak{T}_\sigma(\mathrm{D})$ (cf. (2.7)), and consider (see Corollary 2.27 (1) for the last isomorphism)

$$t_{\mathrm{D}_\sigma} := \mathfrak{T}_\sigma \circ t_{\mathrm{D},\sigma} : \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big) \twoheadrightarrow \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}, \mathrm{D}) \xrightarrow[\sim]{\mathfrak{T}_\sigma} \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}_\sigma, \mathrm{D}_\sigma).$$

For a P-filtration $\mathscr{F}_{\mathrm{P}}$ on D_σ (which corresponds to a P-filtration on D, still denoted by $\mathscr{F}_{\mathrm{P}}$), let $t_{\mathrm{D}_\sigma,\mathscr{F}_{\mathrm{P}}} := \mathfrak{T}_\sigma \circ t_{\mathrm{D},\mathscr{F}_{\mathrm{P}},\sigma}$, which is equal to the restriction of t_{D_σ} to $\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi_{1,\sigma}(\phi, \mathbf{h}))$. It is clear $\mathrm{Ker}\, t_{\mathrm{D}_\sigma} = \mathrm{Ker}\, t_{\mathrm{D},\sigma}$, and $\mathrm{Ker}\, t_{\mathrm{D}_\sigma,\mathscr{F}_{\mathrm{P}}} = \mathrm{Ker}\, t_{\mathrm{D},\mathscr{F}_{\mathrm{P}},\sigma}$. The following corollary is clear.

Corollary **3.32.** — *(1) The surjective map* $\oplus_w \overline{\mathrm{Ext}}^1_{\sigma,w}(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \twoheadrightarrow \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}_\sigma, \mathrm{D}_\sigma)$ *(cf. Corollary 2.40) factors through* t_{D_σ} *composed with the following composition (which is compatible with (3.50), by (2.32))*

$$\oplus_{w\in S_n} \overline{\mathrm{Ext}}^1_{\sigma,w}(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \xrightarrow[\sim]{(\kappa_w)} \oplus_{w\in S_n} \mathrm{Hom}_\sigma\big(\mathrm{T}(\mathrm{K}), \mathrm{E}\big) \xrightarrow{(\zeta_w)} \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{1,\sigma}(\phi, \boldsymbol{h})\big) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_1(\phi, \boldsymbol{h})\big).$$

Consequently, $\mathrm{Ker}\, t_{\mathrm{D}_\sigma}$ *depends only on* D_σ.

(2) The statements in Corollary 3.29 (2) hold with D, M_i *replace by* D_σ, $\mathrm{M}_{i,\sigma} = \mathfrak{T}_\sigma(\mathrm{M}_i)$. *In particular, we have*

$$\oplus_{i=1}^r \mathrm{Ker}\, t_{\mathrm{M}_{i,\sigma}} \xrightarrow{\sim} \mathrm{Ker}\, t_{\mathrm{D}_\sigma,\mathscr{F}_{\mathrm{P}}} = \mathrm{Ker}\, t_{\mathrm{D}_\sigma} \cap \mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{1,\sigma}(\phi, \boldsymbol{h})\big). \tag{3.51}$$

Similarly as in Corollary 3.25 and Remark 3.28, we have

Corollary **3.33.** — *We have natural isomorphisms (cf. Remark 3.30 (2))*

$$t_{\mathrm{D}_\sigma} : \mathrm{Ext}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\min}(\mathrm{D})_\sigma\big) \xrightarrow[\sim]{t_{\mathrm{D},\sigma}} \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}, \mathrm{D}) \xrightarrow[\sim]{\mathfrak{T}_\sigma} \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}_\sigma, \mathrm{D}_\sigma).$$

Moreover, for a P*-filtration* $\mathscr{F}_{\mathrm{P}}$ *on* D *as in Corollary 3.29 (2), we have a commutative diagram (see Corollary 2.27 for the right square):*

$$\begin{array}{ccccc}
\prod_{i=1}^r \mathrm{Ext}^1_\sigma(\pi_{\mathrm{alg}}(\phi_{\mathscr{F}_{\mathrm{P}},i}, \boldsymbol{h}^i), \pi_{\min}(\mathrm{M}_i)_\sigma) & \xrightarrow[\sim]{(t_{\mathrm{M}_i,\sigma})} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(\mathrm{M}_i, \mathrm{M}_i) & \xrightarrow[\sim]{\mathfrak{T}_\sigma} & \prod_{i=1}^r \overline{\mathrm{Ext}}^1_\sigma(\mathrm{M}_{i,\sigma}, \mathrm{M}_{i,\sigma}) \\
\downarrow\sim & & \downarrow\sim & & \downarrow\sim \\
\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\min,\mathscr{F}_{\mathrm{P}}}(\mathrm{D})_\sigma) & \xrightarrow[\sim]{t_{\mathrm{D},\mathscr{F}_{\mathrm{P}},\sigma}} & \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D}, \mathrm{D}) & \xrightarrow[\sim]{\mathfrak{T}_\sigma} & \overline{\mathrm{Ext}}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\mathrm{D}_\sigma, \mathrm{D}_\sigma)
\end{array} \tag{3.52}$$

where $\mathrm{M}_{i,\sigma} = \mathfrak{T}_\sigma(\mathrm{M}_i)$, $\pi_{\min,\mathscr{F}_{\mathrm{P}}}(\mathrm{D})_\sigma \supset \pi_{1,\sigma}(\phi, \boldsymbol{h})$ *is the maximal* $\mathfrak{g}_{\Sigma_{\mathrm{K}}\setminus\{\sigma\}}$*-algebraic subrepresentation of* $\pi_{\min,\mathscr{F}_{\mathrm{P}}}(\mathrm{D})$ *(Rk. 3.28), and* $\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{\min,\mathscr{F}_{\mathrm{P}}}(\mathrm{D})_\sigma)$ *is the image of* $\mathrm{Ext}^1_{\sigma,\mathscr{F}_{\mathrm{P}}}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \pi_{1,\sigma}(\phi, \boldsymbol{h}))$ *via the natural push-forward map.*

Theorem **3.34.** — *Let* $\mathrm{D}, \mathrm{D}' \in \Phi\Gamma_{\mathrm{nc}}(\phi, \boldsymbol{h})$, *and* $\sigma \in \Sigma_{\mathrm{K}}$. *Then* $\pi_{\min}(\mathrm{D})_\sigma \cong \pi_{\min}(\mathrm{D}')_\sigma$ *if and only if* $\mathrm{D}_\sigma \cong \mathrm{D}'_\sigma$. *Consequently,* $\pi_{\min}(\mathrm{D}) \cong \pi_{\min}(\mathrm{D}')$ *if and only if* $\mathrm{D}_\sigma \cong \mathrm{D}'_\sigma$ *for all* $\sigma \in \Sigma_{\mathrm{K}}$.

Proof. — The second part follows from the first part by Corollary 3.31 and Remark 3.30 (2). As $\mathrm{End}_{\mathrm{GL}_n(\mathrm{K})}(\pi_1(\phi, \mathbf{h})) \xrightarrow{\sim} \mathrm{End}_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi, \mathbf{h})) \cong \mathrm{E}$, $\pi_{\min}(\mathrm{D})_\sigma \cong \pi_{\min}(\mathrm{D}')_\sigma$ if and only if $\mathrm{Ker}(t_{\mathrm{D}_\sigma}) = \mathrm{Ker}(t_{\mathrm{D}'_\sigma})$. The "only if" in the first part follows by Corollary 3.32 (1). We prove "if" in the first statement by induction on n. The case where $n \leq 2$ is trivial. Indeed, in this case, $\pi_{\min}(\mathrm{D})_\sigma$ are all isomorphic, and D_σ are all isomorphic as well, for $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$. Suppose it holds for $n-1$. Let D_1 (resp. D'_1) be the saturated (φ, Γ)-submodule of D (resp. of D′) of rank $n-1$, and C_1 (resp. C'_1) be the quotient of D (resp. of D′), both with the refinement $\phi^1 = (\phi_1, \dots, \phi_{n-1})$. Let $\mathscr{F}$ (resp. $\mathscr{F}'$, resp. $\mathscr{G}$, resp. $\mathscr{G}'$) be the filtration $\mathrm{D}_1 \subset \mathrm{D}$ (resp. $\mathrm{D}'_1 \subset \mathrm{D}'$, resp. $\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_1}) \subset \mathrm{D}$, resp. $\mathcal{R}_{\mathrm{K},\mathrm{E}}(\phi_n z^{\mathbf{h}_1}) \subset \mathrm{D}'$). As $\mathrm{Ker}(t_{\mathrm{D}_\sigma}) = \mathrm{Ker}(t_{\mathrm{D}'_\sigma})$, we have $\mathrm{Ker}(t_{\mathrm{D}_\sigma, \mathscr{F}}) = \mathrm{Ker}(t_{\mathrm{D}'_\sigma, \mathscr{F}})$ and $\mathrm{Ker}(t_{\mathrm{D}_{\sigma,\mathscr{G}}}) = \mathrm{Ker}(t_{\mathrm{D}'_{\sigma,\mathscr{G}}})$ by Corollary 3.32 (2). By (3.51), we have $\mathrm{Ker}(t_{\mathrm{D}_{1,\sigma}}) = \mathrm{Ker}(t_{\mathrm{D}'_{1,\sigma}})$ and $\mathrm{Ker}(t_{\mathrm{C}_{1,\sigma}}) = \mathrm{Ker}(t_{\mathrm{C}'_{1,\sigma}})$, hence $\mathrm{D}_{1,\sigma} \cong \mathrm{D}'_{1,\sigma}$ and $\mathrm{C}_{1,\sigma} \cong \mathrm{C}'_{1,\sigma}$ by induction hypothesis.

Let $\pi := \pi_{\min}(\mathrm{D})_\sigma \cong \pi_{\min}(\mathrm{D}')_\sigma$. Let π^- (resp. π^+) be the extension of $\mathrm{Ker}(t_{\mathrm{D}_\sigma, \mathscr{F}}) \otimes_{\mathrm{E}} \pi_{\mathrm{alg}}(\phi, \mathbf{h})$ (resp. $\mathrm{Ker}(t_{\mathrm{D}_\sigma, \mathscr{G}}) \otimes_{\mathrm{E}} \pi_{\mathrm{alg}}(\phi, \mathbf{h})$) by $\pi_{1,\sigma}(\phi, \mathbf{h})$ (which stays unchanged if D_σ is replaced by D'_σ). So $\pi^\pm \hookrightarrow \pi$. Let $\mathcal{L}$ be the kernel of the following natural (push-forward) map

$$\overline{\mathrm{Ext}}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi^-\big) \oplus \overline{\mathrm{Ext}}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi^+\big) \twoheadrightarrow \overline{\mathrm{Ext}}^1_\sigma\big(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi\big). \tag{3.53}$$

We have a commutative diagram of exact sequences (see (2.42) for the bottom one)

$$\begin{array}{ccccc} \mathcal{L} & \hookrightarrow & \overline{\mathrm{Ext}}^1_\sigma(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi^-) \oplus \overline{\mathrm{Ext}}^1_\sigma(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi^+) & \twoheadrightarrow & \overline{\mathrm{Ext}}^1_\sigma(\pi_{\mathrm{alg}}(\phi, \mathbf{h}), \pi) \\ \downarrow & & \downarrow \sim & & {\scriptstyle t_{\mathrm{D}_\sigma}} \downarrow \sim \\ \mathcal{L}(\mathrm{D}_\sigma, \mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma}) & \hookrightarrow & \mathrm{V}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})_\sigma & \twoheadrightarrow & \overline{\mathrm{Ext}}^1_\sigma(\mathrm{D}_\sigma, \mathrm{D}_\sigma) \end{array} \tag{3.54}$$

where the middle (bijective) map is induced by (3.52). We then deduce $\mathcal{L} \xrightarrow{\sim} \mathcal{L}(\mathrm{D}_\sigma, \mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})$. Similarly, replacing D_σ by D'_σ, we obtain $\mathcal{L} \xrightarrow{\sim} \mathcal{L}(\mathrm{D}_\sigma, \mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})$. Note the middle map in (3.54) does not change when D_σ is replaced by D'_σ by the discussion in the first paragraph. Hence $\mathcal{L}(\mathrm{D}_\sigma, \mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma}) \cong \mathcal{L}(\mathrm{D}'_\sigma, \mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})$ as subspace of $\mathrm{V}(\mathrm{D}_{1,\sigma}, \mathrm{C}_{1,\sigma})_\sigma$. But this implies $\mathrm{D}_\sigma \cong \mathrm{D}'_\sigma$ by Corollary 2.38 (1). □

3.2.2. *Universal extensions.* — We give a reformulation of Theorem 3.21 using deformation rings of (φ, Γ)-modules, which will be useful in our proof of the local-global compatibility.

Let $\mathrm{D} \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$. Note $\mathrm{End}_{(\varphi,\Gamma)}(\mathrm{D}) \cong \mathrm{E}$. Let R_{D} be the universal deformation ring of deformations of D over local Artinian E-algebras. Let $\mathrm{R}_{\mathrm{D},w}$ be the universal deformation ring of $\mathscr{T}_w$-deformations of D (i.e. the trianguline deformations of D with respect to the refinement $w(\phi)$), and $\mathrm{R}_{\mathrm{D},g}$ be the universal deformation ring of de Rham deformations or equivalently crystabelline deformations. All of these rings are formally smooth complete local Noetherian E-algebras (using the fact ϕ is generic). See [3, Section 2.3.5, Section 2.5.3] [51, Section 2] for detailed discussions. For a continuous character δ of $\mathrm{T}(\mathrm{K})$, denote by R_δ the universal deformation ring of deformations of δ over local Artinian E-algebras, which is also formally smooth complete local Noetherian. If δ is locally algebraic, denote by $\mathrm{R}_{\delta,g}$ the universal deformation ring of locally algebraic deformations of δ. For a complete local Noetherian E-algebra R, we use $\mathfrak{m}_\mathrm{R}$ to denote its maximal ideal and we will use $\mathfrak{m}$ for simplicity when it does not cause confusion.

We have natural surjections $\mathrm{R}_\mathrm{D} \twoheadrightarrow \mathrm{R}_{\mathrm{D},w} \twoheadrightarrow \mathrm{R}_{\mathrm{D},g}$, $\mathrm{R}_\delta \twoheadrightarrow \mathrm{R}_{\delta,g}$. Moreover, we have natural isomorphisms of E-vector spaces for the tangent spaces $(\mathfrak{m}_{\mathrm{R}_{\mathrm{D},*}}/\mathfrak{m}^2_{\mathrm{R}_{\mathrm{D},*}})^\vee \cong \mathrm{Ext}^1_*(\mathrm{D},\mathrm{D})$ for $* \in \{\emptyset, g, w\}$, $(\mathfrak{m}_{\mathrm{R}_\delta}/\mathfrak{m}^2_{\mathrm{R}_\delta})^\vee \cong \mathrm{Hom}(\mathrm{T}(\mathrm{K}),\mathrm{E})$ and $(\mathfrak{m}_{\mathrm{R}_{\delta,g}}/\mathfrak{m}^2_{\mathrm{R}_{\delta,g}})^\vee \cong \mathrm{Hom}_{\mathrm{sm}}(\mathrm{T}(\mathrm{K}),\mathrm{E})$. For $w \in \mathrm{S}_n$, by Proposition 2.10 (2)(3), the map κ_w in (2.12) induces a commutative Cartesian diagram (of local Artinian E-algebras):

$$\begin{array}{ccc} \mathrm{R}_{w(\phi)z^{\mathbf{h}}}/\mathfrak{m}^2 & \twoheadrightarrow & \mathrm{R}_{w(\phi)z^{\mathbf{h}},g}/\mathfrak{m}^2 \\ \downarrow & & \downarrow \\ \mathrm{R}_{\mathrm{D},w}/\mathfrak{m}^2 & \twoheadrightarrow & \mathrm{R}_{\mathrm{D},g}/\mathfrak{m}^2. \end{array} \tag{3.55}$$

As in the proof of Proposition 3.3 (2), let $\mathcal{H}$ (resp. $\mathcal{H}_i \cong \mathbf{G}_m$) be the Bernstein centre over E associated to the smooth representation $\pi_{\mathrm{sm}}(\phi)$ of $\mathrm{GL}_n(\mathrm{K})$ (resp. ϕ_i of $\mathrm{K}^\times$) (cf. [23, Section 3.13]). For $w \in \mathrm{S}_n$, there is a natural morphism $\mathcal{J}_w : \prod_{i=1}^n \mathrm{Spec}\,\mathcal{H}_{w^{-1}(i)} \to \mathrm{Spec}\,\mathcal{H}$ (see the proof of Proposition 3.3 (2)), which, by [23, Lem. 3.22], induces an isomorphism between completions at closed points. The completion of $\prod_{i=1}^n \mathrm{Spec}\,\mathcal{H}_{w^{-1}(i)}$ at the point $w(\phi)$ is naturally isomorphic to $\mathrm{R}_{w(\phi),g}$. Let $\widehat{\mathcal{H}}_\phi$ be the completion of $\mathcal{H}$ at $\pi_{\mathrm{sm}}(\phi)$. So $\mathcal{J}_w$ induces $\mathcal{J}_w : \widehat{\mathcal{H}}_\phi \xrightarrow{\sim} \mathrm{R}_{w(\phi),g} \xrightarrow{\sim} \mathrm{R}_{w(\phi)z^{\mathbf{h}},g}$ where the second map is given by twisting $z^{-\mathbf{h}}$. By Lemma 2.11 and Proposition 3.3 (2), the composition

$$\mathrm{A}_0 := \widehat{\mathcal{H}}_\phi/\mathfrak{m}^2 \xrightarrow[\sim]{\mathcal{J}_w} \mathrm{R}_{w(\phi)z^{\mathbf{h}},g}/\mathfrak{m}^2 \hookrightarrow \mathrm{R}_{\mathrm{D},g}/\mathfrak{m}^2$$

is independent of the choice of w. We let $\mathrm{A}_\mathrm{D} := \mathrm{R}_\mathrm{D}/\mathfrak{m}^2 \times_{\mathrm{R}_{\mathrm{D},g}/\mathfrak{m}^2} \mathrm{A}_0$ and $\mathrm{A}_{\mathrm{D},w} := \mathrm{R}_{\mathrm{D},w}/\mathfrak{m}^2 \times_{\mathrm{R}_{\mathrm{D},g}/\mathfrak{m}^2} \mathrm{A}_0$ ($\cong \mathrm{R}_{w(\phi)z^{\mathbf{h}}}/\mathfrak{m}^2$ by (3.55)). The tangent space of A_D (resp. $\mathrm{A}_{\mathrm{D},w}$) is naturally isomorphic to $\overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D})$ (resp. $\overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D}) \cong \mathrm{Hom}(\mathrm{T}(\mathrm{K}),\mathrm{E})$). We let $\mathcal{I}_w$ be the kernel of $\mathrm{A}_\mathrm{D} \twoheadrightarrow \mathrm{A}_{\mathrm{D},w}$. By Proposition 2.12, the natural morphism $\mathrm{A}_\mathrm{D} \to \prod_w \mathrm{A}_{\mathrm{D},w}$ is injective.

Let $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}$ (resp. $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}_w$) be the tautological extension of $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \otimes_\mathrm{E} \pi_{\mathrm{alg}}(\phi,\mathbf{h})$ (resp. of $\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \otimes_\mathrm{E} \pi_{\mathrm{alg}}(\phi,\mathbf{h})$) by $\pi_1(\phi,\mathbf{h})$ (cf. Section 3.1.1). For $w \in \mathrm{S}_n$, denote by $\delta_w := w(\phi)z^{\mathbf{h}}\varepsilon^{-1} \circ \theta$, and $\widetilde{\delta}^{\mathrm{univ}}_w$ the tautological extension of $\mathrm{Ext}^1_{\mathrm{T}(\mathrm{K})}(\delta_w,\delta_w) \otimes_\mathrm{E} \delta_w$ ($\cong \mathrm{Hom}(\mathrm{T}(\mathrm{K}),\mathrm{E}) \otimes_\mathrm{E} \delta_w$) by δ_w.

Lemma **3.35.** — *The induced representation* $I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_w^{univ}$ *is the universal extension of* $\pi_{alg}(\phi, \boldsymbol{h}) \otimes_E \mathrm{Ext}^1_{GL_n(K)}(\pi_{alg}(\phi, \boldsymbol{h}), PS_1(w(\phi), \boldsymbol{h}))$ *by* $PS_1(w(\phi), \boldsymbol{h})$.

Proof. — By Remark 3.7, $I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_w^{univ}$ is an extension of $\pi_{alg}(\phi, \mathbf{h})^{\oplus(n(d_K+1))}$ by a certain subrepresentation V of $PS_1(w(\phi), \mathbf{h})$. However, again by Remark 3.7, as $\widetilde{\delta}_w^{univ}$ is universal, any extension in the image of (3.10) comes from an extension of $\pi_{alg}(\phi, \mathbf{h})$ by V by push-forward via $V \hookrightarrow PS_1(w(\phi), \mathbf{h})$. As (3.10) is bijective (by Proposition 3.6 (1)), using the surjectivity of the last map in (3.14), we see V has to be the entire $PS_1(w(\phi), \mathbf{h})$. Using again Proposition 3.6 (1), we see $I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_w^{univ}$ is in fact the universal extension. □

We have hence an isomorphism of $GL_n(K)$-representations

$$(\mathbf{3.56}) \qquad I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_w^{univ} \oplus_{PS_1(w(\phi),\mathbf{h})} \pi_1(\phi, \mathbf{h}) \xrightarrow{\sim} \pi_1(\phi, \mathbf{h})_w^{univ}.$$

There is a natural action of $A_{D,w} \cong R_{w(\phi)z^{\mathbf{h}}}/\mathfrak{m}^2$ on $\widetilde{\delta}_w^{univ}$ where an element $x \in \mathfrak{m}_{R_{w(\phi)z^{\mathbf{h}}}}/\mathfrak{m}^2_{R_{w(\phi)z^{\mathbf{h}}}} \cong \mathrm{Hom}(T(K), E)^\vee$ acts via $x : \widetilde{\delta}_w^{univ} \twoheadrightarrow \mathrm{Hom}(T(K), E) \otimes_E \delta_w \xrightarrow{x} \delta_w \hookrightarrow \widetilde{\delta}_w^{univ}$. Hence $I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_w^{univ}$ is equipped with an induced $R_{w(\phi)z^{\mathbf{h}}}/\mathfrak{m}^2$-action. Similarly $\pi_1(\phi, \mathbf{h})_w^{univ}$ is equipped with an action of $A_{D,w}$ given by

$$(\mathbf{3.57}) \qquad \begin{aligned} x : \pi_1(\phi, \mathbf{h})_w^{univ} &\twoheadrightarrow \mathrm{Ext}^1_w\big(\pi_{alg}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big) \otimes_E \pi_{alg}(\phi, \mathbf{h}) \\ &\xrightarrow{x} \pi_{alg}(\phi, \mathbf{h}) \hookrightarrow \pi_1(\phi, \mathbf{h})_w^{univ}, \end{aligned}$$

for $x \in \mathfrak{m}_{A_{D,w}}/\mathfrak{m}^2_{A_{D,w}} \cong \mathrm{Hom}(T(K), E)^\vee \xrightarrow[\sim]{\zeta_w} \mathrm{Ext}^1_w(\pi_{alg}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))^\vee$. The injection $I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_w^{univ} \hookrightarrow \pi_1(\phi, \mathbf{h})_w^{univ}$ (induced by (3.56)) is $A_{D,w}$-equivariant.

For a commutative E-algebra A acting on an E-vector space V, and an ideal I of A, we denote by V[I] the subspace of V annihilated by elements in I. The following theorem is a reformulation of Theorem 3.21.

Theorem **3.36.** — *There is a unique* A_D*-action on* $\pi_1(\phi, \boldsymbol{h})^{univ}$ *such that for all* $w \in S_n$*, we have an* $A_{D,w} \times GL_n(K)$*-equivariant injection* $\pi_1(\phi, \boldsymbol{h})_w^{univ} \hookrightarrow \pi_1(\phi, \boldsymbol{h})^{univ}[\mathcal{I}_w]$.

Proof. — By Theorem 3.21, we define an A_D-action by letting $x \in \mathfrak{m}_{A_D}/\mathfrak{m}^2_{A_D} \cong \overline{\mathrm{Ext}}^1(D, D)^\vee \hookrightarrow \mathrm{Ext}^1_{GL_n(K)}(\pi_{alg}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))^\vee$ act via

$$(\mathbf{3.58}) \qquad \begin{aligned} x : \pi_1(\phi, \mathbf{h})^{univ} &\twoheadrightarrow \mathrm{Ext}^1\big(\pi_{alg}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})\big) \otimes_E \pi_{alg}(\phi, \mathbf{h}) \\ &\xrightarrow{x} \pi_{alg}(\phi, \mathbf{h}) \hookrightarrow \pi_1(\phi, \mathbf{h})_w^{univ}. \end{aligned}$$

By definition, the restriction of t_D to $\mathrm{Ext}^1_w(\pi_{alg}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h}))$ coincides with the composition $\mathrm{Ext}^1_w(\pi_{alg}(\phi, \mathbf{h}), \pi_1(\phi, \mathbf{h})) \xrightarrow[\sim]{\zeta_w^{-1}} \mathrm{Hom}(T(K), E) \xrightarrow[\sim]{\kappa_w^{-1}} \overline{\mathrm{Ext}}^1_w(D, D)$. We deduce the action in (3.58) is compatible with (3.57) when $x \in \mathrm{A}_{\mathrm{D},w}$ hence satisfies the properties in the theorem. The uniqueness follows from the fact $\pi_1(\phi, \mathbf{h})^{\mathrm{univ}}$ is generated by $\pi_1(\phi, \mathbf{h})^{\mathrm{univ}}_w$ for $w \in \mathrm{S}_n$. □

By the construction of $\pi_{\min}(\mathrm{D})$, we have:

Corollary **3.37.** — *We have* $\pi_{\min}(\mathrm{D}) \cong \pi_1(\phi, \boldsymbol{h})^{\mathrm{univ}}[\mathfrak{m}_{\mathrm{A}_\mathrm{D}}]$.

4. Local-global compatibility

4.1. *The patched setting.* — Let M_∞ be the patched module of [23, Section 2]. Then $\Pi_\infty := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_\mathrm{E}}(\mathrm{M}_\infty, \mathrm{E})$, equipped with the usual maximum norm, is a unitary Banach representation of $\mathrm{GL}_n(\mathrm{K})$ (where K is the field F of *loc. cit.*), which is equipped with an action of the patched Galois deformation ring $\mathrm{R}_\infty \cong \mathrm{R}^{\square}_{\overline{\rho}} \widehat{\otimes}_{\mathcal{O}_\mathrm{E}} \mathrm{R}^{\wp}_\infty$ (where $\wp$ is "$\widetilde{\mathfrak{p}}$" and $\overline{\rho}$ is the local Galois representation $\overline{r}$ of *loc. cit.*). We refer to [23, Section 2.8] for details. Let $\widehat{\mathrm{T}}$ be the rigid space over E parametrizing continuous characters of $\mathrm{T}(\mathrm{K})$. Let

$$\mathcal{E} \longleftrightarrow \left(\mathrm{Spf}\,\mathrm{R}^{\square}_{\overline{\rho}}\right)^{\mathrm{rig}} \times \widehat{\mathrm{T}} \times \left(\mathrm{Spf}\,\mathrm{R}^{\wp}_\infty\right)^{\mathrm{rig}} \tag{4.1}$$

be the associated patched eigenvariety (see [32, Section 4.1.2], that is an easy variation of the patched eigenvariety introduced in [21]), $\mathcal{M}$ be the natural coherent sheaf on $\mathcal{E}$ such that there is a $\mathrm{T}(\mathrm{K}) \times \mathrm{R}_\infty$-equivariant isomorphism $\Gamma(\mathcal{E}, \mathcal{M})^\vee \cong \mathrm{J}_\mathrm{B}(\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}})$ (see [21, Section 3.1] for "$\mathrm{R}_\infty - \mathrm{an}$"). Recall a point $x = (\rho_{x,\wp}, \delta_x, \mathfrak{m}^{\wp}_x) \in (\mathrm{Spf}\,\mathrm{R}^{\square}_{\overline{\rho}})^{\mathrm{rig}} \times \widehat{\mathrm{T}} \times (\mathrm{Spf}\,\mathrm{R}^{\wp}_\infty)^{\mathrm{rig}}$ lies in $\mathcal{E}$ if and only if $\mathrm{Hom}_{\mathrm{T}(\mathrm{K})}(\delta_x, \mathrm{J}_\mathrm{B}(\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}})[\mathfrak{m}_x]) \neq 0$ where $\mathfrak{m}_x = (\rho_{x,\wp}, \mathfrak{m}^{\wp}_x)$ is the associated maximal ideal of $\mathrm{R}_\infty[1/p]$.

Let $\mathrm{X}^{\square}_{\mathrm{tri}}(\overline{\rho}) \hookrightarrow (\mathrm{Spf}\,\mathrm{R}^{\square}_{\overline{\rho}})^{\mathrm{rig}} \times \widehat{\mathrm{T}}$ be the trianguline variety [21, Section 2.2], and ι_p be the twisting map (see Section 3.1.1 for δ_B and θ)

$$\iota_p : \left(\mathrm{Spf}\,\mathrm{R}^{\square}_{\overline{\rho}}\right)^{\mathrm{rig}} \times \widehat{\mathrm{T}} \xrightarrow{\sim} \left(\mathrm{Spf}\,\mathrm{R}^{\square}_{\overline{\rho}}\right)^{\mathrm{rig}} \times \widehat{\mathrm{T}},\ (\rho_p, \chi) \mapsto \left(\rho_p, \chi\delta_\mathrm{B}\left(\varepsilon^{-1} \circ \theta\right)\right).$$

Recall (4.1) factors through an embedding (cf. [21, Thm. 1.1]).

$$\mathcal{E} \longleftrightarrow \iota_p\left(\mathrm{X}^{\square}_{\mathrm{tri}}(\overline{\rho})\right) \times \left(\mathrm{Spf}\,\mathrm{R}^{\wp}_\infty\right)^{\mathrm{rig}}, \tag{4.2}$$

which identifies $\mathcal{E}$ with a union of irreducible components of the latter. Recall $\dim \mathrm{X}^{\square}_{\mathrm{tri}}(\overline{\rho}) = n^2 + d_\mathrm{K}\frac{n(n+1)}{2}$ (cf. [21, Thm. 2.6], where "n^2" comes from the framing).

Let $\rho : \mathrm{Gal}_\mathrm{K} \to \mathrm{GL}_n(\mathrm{E})$ be a continuous representation such that ρ admits a Gal_K-invariant $\mathcal{O}_\mathrm{E}$-lattice whose modulo p reduction is equal to $\overline{\rho}$ and that $\mathrm{D} := \mathrm{D}_{\mathrm{rig}}(\rho) \in \Phi\Gamma_{\mathrm{nc}}(\phi, \mathbf{h})$ (ϕ generic, $\mathbf{h}$ strictly dominant given as in Section 2.2). Let $\mathfrak{m}_\rho \subset \mathrm{R}^{\square}_{\overline{\rho}}[1/p]$ be the maximal ideal associated to ρ.

Lemma **4.1.** — *For a continuous character* $\delta : \mathrm{T}(\mathrm{K}) \to \mathrm{E}^\times$, $(\rho, \delta) \in \mathrm{X}^{\square}_{\mathrm{tri}}(\overline{\rho})$ *if and only if* $\delta = w(\phi)z^{\boldsymbol{h}}$ *for some* $w \in \mathrm{S}_n$.

Proof. — The "if" part follows the very construction (cf. [21, Section 2.2]). Indeed all these points lie in the space $U_{\mathrm{tri}}^{\square}(\overline{\rho})^{\mathrm{reg}}$ of *loc. cit.* The "only if" part follows from the fact that D is non-critical hence does not admit companion points of non-dominant weight (e.g. see [22, Thm. 4.2.3], [16, Cor. 6.4.12]). □

Suppose there exists a maximal ideal $\mathfrak{m}^{\wp}$ of $R_{\infty}^{\wp}[1/p]$ such that $\Pi_{\infty}[\mathfrak{m}_x]^{\mathrm{lalg}} \neq 0$ for $\mathfrak{m}_x = (\mathfrak{m}_{\rho}, \mathfrak{m}^{\wp})$, the corresponding maximal ideal of $R_{\infty}[1/p]$. By [23, Thm. 4.35], we have $\Pi_{\infty}[\mathfrak{m}_x]^{\mathrm{lalg}} \cong \pi_{\mathrm{alg}}(\phi, \mathbf{h})$. By taking Jacquet-Emerton modules, this implies $x_w := (x_{w,\wp}, \mathfrak{m}^{\wp}) = (\rho, \delta_w \delta_B = w(\phi) z^{\mathbf{h}} \delta_B(\varepsilon^{-1} \circ \theta), \mathfrak{m}^{\wp}) \in \mathcal{E}$ for all $w \in S_n$. By Lemma 4.1, these give all the points on $\mathcal{E}$ associated to $\mathfrak{m}_x$. By [21, Thm. 2.6 (iii)], $X_{\mathrm{tri}}^{\square}(\overline{\rho})$ is smooth at the points $\iota_p^{-1}(x_{w,\wp})$ and (4.2) is a local isomorphism at x_w. As $(\mathrm{Spf} R_{\infty}^{\wp})^{\mathrm{rig}}$ is also smooth at $\mathfrak{m}^{\wp}$ (e.g. see the proof of [31, Cor. 4.4]), $\mathcal{E}$ is smooth at all x_w. By [20, Lem. 3.8] and [23, Thm. 4.35], we see $\mathcal{M}$ is locally free of rank one at all x_w.

Let $R_D^{\square} := R_D \otimes_{R_{\rho}} R_{\rho}^{\square} \cong R_{\rho}^{\square}$ (where $R_{\rho}^{\square}$ is the framed universal deformation ring of ρ of deformations over local Artinian E-algebras). Let $R_{D,w}^{\square} := R_D^{\square} \otimes_{R_D} R_{D,w}$ for $w \in S_n$ and $R_{D,g}^{\square} := R_D^{\square} \otimes_{R_D} R_{D,g}$. We have commutative Cartesian diagrams (see Section 3.2.2, in particular the discussion below (3.55) for A_D, $A_{D,w}$, A_0, where $\mathfrak{m}$ denotes the corresponding maximal ideals):

$$\begin{array}{ccccc} A_D & \hookrightarrow & R_D/\mathfrak{m}^2 & \hookrightarrow & R_D^{\square}/\mathfrak{m}^2 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ A_{D,w} & \hookrightarrow & R_{D,w}/\mathfrak{m}^2 & \hookrightarrow & R_{D,w}^{\square}/\mathfrak{m}^2 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ A_0 & \hookrightarrow & R_{D,g}/\mathfrak{m}^2 & \hookrightarrow & R_{D,g}^{\square}/\mathfrak{m}^2. \end{array} \tag{4.3}$$

Let $\mathfrak{a} \supset \mathfrak{m}_{R_D^{\square}}^2$ be an ideal of $R_D^{\square}$ such that $\mathfrak{a}/\mathfrak{m}_{R_D^{\square}}^2 \oplus \mathfrak{m}_{A_D}/\mathfrak{m}_{A_D}^2 \xrightarrow{\sim} \mathfrak{m}_{R_D^{\square}}/\mathfrak{m}_{R_D^{\square}}^2$ (noting $\mathfrak{m}_{A_D}/\mathfrak{m}_{A_D}^2 \hookrightarrow \mathfrak{m}_{R_D^{\square}}/\mathfrak{m}_{R_D^{\square}}^2$). The composition $A_D \hookrightarrow R_D^{\square}/\mathfrak{m}_{R_D^{\square}}^2 \twoheadrightarrow R_D^{\square}/\mathfrak{a}$ is hence an isomorphism. We use $\mathfrak{a}$ to denote its image in $R_{D,w}^{\square}$ and $R_{D,g}^{\square}$. By (4.3), $\mathfrak{a}/\mathfrak{m}_{R_{D,w}^{\square}}^2 \oplus \mathfrak{m}_{A_{D,w}}/\mathfrak{m}_{A_{D,w}}^2 \xrightarrow{\sim} \mathfrak{m}_{R_{D,w}^{\square}}/\mathfrak{m}_{R_{D,w}^{\square}}^2$ and $\mathfrak{a}/\mathfrak{m}_{R_{D,g}^{\square}}^2 \oplus \mathfrak{m}_{A_0}/\mathfrak{m}_{A_0}^2 \xrightarrow{\sim} \mathfrak{m}_{R_{D,g}^{\square}}/\mathfrak{m}_{R_{D,g}^{\square}}^2$. Moreover, the compositions $A_{D,w} \hookrightarrow R_{D,w}^{\square}/\mathfrak{m}^2 \twoheadrightarrow R_{D,w}^{\square}/\mathfrak{a}$ for $w \in S_n$, and $A_0 \hookrightarrow R_{D,g}^{\square}/\mathfrak{m}^2 \twoheadrightarrow R_{D,g}^{\square}/\mathfrak{a}$ are all isomorphisms.

Recall the completion of $R_{\overline{\rho}}^{\square}[1/p]$ at ρ is naturally isomorphic to $R_{\rho}^{\square} \cong R_D^{\square}$ (cf. [45]). Let $\mathfrak{a} \subset R_{\overline{\rho}}^{\square}[1/p]$ denote the preimage of $\mathfrak{a} \subset R_D^{\square}$. Let $\mathfrak{a}_x := (\mathfrak{a}, \mathfrak{m}^{\wp}) \subset R_{\infty}[1/p]$. So $\Pi_{\infty}^{R_{\infty}-\mathrm{an}}[\mathfrak{a}_x]$ is equipped with a natural $R_D^{\square}/\mathfrak{a} \cong A_D$-action.

Lemma **4.2.** — *(1) We have* $\Pi_{\infty}^{R_{\infty}-\mathrm{an}}[\mathfrak{m}_x] \xrightarrow{\sim} \Pi_{\infty}^{R_{\infty}-\mathrm{an}}[\mathfrak{a}_x][\mathfrak{m}_{A_D}]$.
(2) $\mathrm{Hom}_{\mathrm{GL}_n}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \Pi_{\infty}^{R_{\infty}-\mathrm{an}}[\mathfrak{a}_x]) \cong \mathrm{Hom}_{\mathrm{GL}_n}(\pi_{\mathrm{alg}}(\phi, \boldsymbol{h}), \Pi_{\infty}^{R_{\infty}-\mathrm{an}}[\mathfrak{m}_x]) \cong E$.

Proof. — By definition, $\mathfrak{a}+\mathfrak{m}_{\mathrm{A_D}}=\mathfrak{m}_{\mathrm{R_D^\square}}$ hence $\mathfrak{a}_x+\mathfrak{m}_{\mathrm{A_D}}=\mathfrak{m}_x$. (1) follows. By [23, Lem. 4.17], the $\mathrm{R_D^\square}$-action on $\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]^{\mathrm{lalg}}$ factors through $\mathrm{R}_{\mathrm{D},g}^\square/\mathfrak{a}$. Let σ_{sm} be a smooth irreducible representation of $\mathrm{GL}_n(\mathcal{O}_\mathrm{K})$ over E associated to the Bernstein component of $\pi_{\mathrm{sm}}(\phi)$. We have $\mathcal{H}\cong\mathrm{End}_{\mathrm{GL}_n(\mathrm{K})}(\text{c-ind}_{\mathrm{GL}_n(\mathcal{O}_\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\sigma_{\mathrm{sm}})$ (recalling $\mathcal{H}$ is the Bernstein centre associated to $\pi_{\mathrm{sm}}(\phi)$). Consider

$$\begin{aligned}\mathrm{M}&:=\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{K})}\Big(\big(\text{c-ind}_{\mathrm{GL}_n(\mathcal{O}_\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\sigma_{\mathrm{sm}}\big)\otimes_\mathrm{E}\mathrm{L}(\lambda),\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]^{\mathrm{lalg}}\Big)\\&\cong\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{K})}\Big(\text{c-ind}_{\mathrm{GL}_n(\mathcal{O}_\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\sigma_{\mathrm{sm}},\big(\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]^{\mathrm{lalg}}\otimes_\mathrm{E}\mathrm{L}(\lambda)^\vee\big)_{\mathrm{sm}}\Big)\end{aligned}$$

which is naturally an $\mathcal{H}$-module (with $\mathcal{H}$ acting on $\text{c-ind}_{\mathrm{GL}_n(\mathcal{O}_\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\sigma_{\mathrm{sm}}$). Hence $\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]^{\mathrm{lalg}})\cong\mathrm{M}[\mathfrak{m}_{\mathrm{A}_0}]$. By [23, Thm. 4.19], this $\mathcal{H}$-action on M coincides with the one induced by $\mathcal{H}\to\widehat{\mathcal{H}}_\phi\twoheadrightarrow\mathrm{A}_0\xrightarrow{\sim}\mathrm{R}_{\mathrm{D},g}^\square/\mathfrak{a}$ acting on $\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]^{\mathrm{lalg}}$. This implies $\mathrm{M}[\mathfrak{m}_{\mathrm{A}_0}]\cong\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{K})}((\text{c-ind}_{\mathrm{GL}_n(\mathcal{O}_\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\sigma_{\mathrm{sm}})\otimes_\mathrm{E}\mathrm{L}(\lambda),\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]^{\mathrm{lalg}}[\mathfrak{m}_{\mathrm{A}_0}])$. However, as ideals of $\mathrm{R}_{\mathrm{D},g}^\square/\mathfrak{m}^2$, we have $\mathfrak{m}_{\mathrm{A}_0}+\mathfrak{a}=\mathfrak{m}$ hence $\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]^{\mathrm{lalg}}[\mathfrak{m}_{\mathrm{A}_0}]\cong\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]^{\mathrm{lalg}}\cong\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ (by [23, Thm. 4.3.5]). So $\mathrm{M}[\mathfrak{m}_{\mathrm{A}_0}]\cong\mathrm{E}$, proving (2). □

For $w\in\mathrm{S}_n$, let $\mathrm{U}=\mathrm{U}_\wp\times\mathrm{U}^\wp\subset\iota_p(\mathrm{X}_{\mathrm{tri}}^\square(\overline{\rho}))\times(\mathrm{Spf}\,\mathrm{R}_\infty^\wp)^{\mathrm{rig}}$ be a smooth affinoid neighbourhood of x_w such that $x_{w'}\notin\mathrm{U}$ for $w'\neq w$. Let $\mathfrak{m}_{x_{w,\wp}}$ be the maximal ideal of $\mathcal{O}(\mathrm{U}_\wp)$ at $x_{w,\wp}$. As $x_{w,\wp}$ is the only point in $\mathrm{U}_\wp$ associated to ρ (by assumption and Lemma 4.1), $\mathfrak{m}_{x_{w,\wp}}\subset\mathcal{O}(\mathrm{U}_\wp)$ is the closed ideal generated by $\mathfrak{m}_\rho\subset\mathrm{R}_{\overline{\rho}}^\square[1/p]$. Let $\mathfrak{a}\supset\mathfrak{m}_{x_{w,\wp}}^2$ be the closed ideal generated by $\mathfrak{a}\subset\mathrm{R}_{\overline{\rho}}^\square[1/p]$. Consider $\mathcal{M}_{\tilde{x}_w}:=\mathcal{M}(\mathrm{U})/(\mathfrak{a}+\mathfrak{m}^\wp)$. By definition, we have a $\mathrm{T}(\mathrm{K})\times\mathrm{R}_\infty$-equivariant map

$$\mathcal{M}_{\tilde{x}_w}^\vee\hookrightarrow\mathrm{J_B}\big(\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}\big)[\mathfrak{a}_x]\cong\mathrm{J_B}\big(\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]\big).\tag{4.4}$$

Recall as D is non-critical, the completion of $\mathrm{X}_{\mathrm{tri}}^\square(\overline{\rho})$ at $\iota_p^{-1}(x_{w,\wp})$ is naturally isomorphic to $\mathrm{R}_{\mathrm{D},w}^\square$. As $\mathcal{M}$ is locally free of rank one at x_w, we see $\mathcal{M}_{\tilde{x}_w}\cong\mathrm{R}_{\mathrm{D},w}^\square/\mathfrak{a}$. In particular, $\dim_\mathrm{E}\mathcal{M}_{\tilde{x}_w}=1+(n+nd_\mathrm{K})$. The $\mathrm{T}(\mathrm{K})$-action on $\mathcal{M}_{\tilde{x}_w}$ is encoded in the $\mathrm{A}_{\mathrm{D},w}$-action. Indeed, $\mathrm{T}(\mathrm{K})$ acts on the $\mathrm{A}_{\mathrm{D},w}$-module $\mathcal{M}_{\tilde{x}_w}$ via the composition:

$$\mathrm{T}(\mathrm{K})\longrightarrow\mathrm{R}_{\delta_w\delta_\mathrm{B}}/\mathfrak{m}^2\xrightarrow{\sim}\mathrm{R}_{w(\phi)z^{\mathbf{h}}}/\mathfrak{m}^2\cong\mathrm{A}_{\mathrm{D},w}\tag{4.5}$$

where the first map is induced by the universal deformation, and the second is induced by twisting $\delta_\mathrm{B}(\varepsilon^{-1}\circ\theta)$ (which corresponds to the twist in ι_p). We equip $\mathrm{A}_{\mathrm{D},w}$ with the $\mathrm{T}(\mathrm{K})$-action as in (4.5). Then the $\mathrm{T}(\mathrm{K})$-representation $\mathrm{A}_{\mathrm{D},w}^\vee$ is just isomorphic to the universal extension $\widetilde{\delta}_w^{\mathrm{univ}}\delta_\mathrm{B}$ (see the discussion below (3.56)). In summary, we have a $\mathrm{T}(\mathrm{K})\times\mathrm{A}_{\mathrm{D},w}$-equivariant isomorphism $\mathcal{M}_{\tilde{x}_w}^\vee\cong\widetilde{\delta}_w^{\mathrm{univ}}\delta_\mathrm{B}$.

Lemma **4.3.** — *The map (4.4) is balanced in the sense of [37, Def. 0.8], hence (by [37, Thm. 0.13]) induces a* $\mathrm{GL}_n(\mathrm{K})\times\mathrm{R}_\infty$*-equivariant injection*

$$\iota_w:\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\widetilde{\delta}_w^{\mathrm{univ}}\hookrightarrow\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x],$$

where the R_∞*-action on* $I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_w^{\rm univ}=I_{B^-(K)}^{GL_n(K)}((\widetilde{\delta}_w^{\rm univ}\delta_B)\delta_B^{-1})$ *is induced by* $R_\infty\to(R_{D,w}^{\square}/\mathfrak{a})\otimes_E(R_\infty^{\wp}[1/p]/\mathfrak{m}^{\wp})\xrightarrow{\sim}A_{D,w}$ *acting on* $\widetilde{\delta}_w^{\rm univ}\delta_B$.

Proof. — The lemma follows by the same argument as in [32, Lem. 4.11], using Lemma 4.1. ☐

Lemma **4.4.** — *We have* $(\operatorname{Im}\iota_w)[\mathfrak{m}_x]\cong PS_1(w(\phi),\boldsymbol{h})$.

Proof. — By Lemma 3.35, we have

$$PS_1\big(w(\phi),\mathbf{h}\big)\longrightarrow I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_w^{\rm univ}\xhookrightarrow{\iota_w}\Pi_\infty^{R_\infty-\rm an}[\mathfrak{a}_x]. \tag{4.6}$$

By Lemma 4.2, it sends $\pi_{\rm alg}(\phi,\mathbf{h})$ to $\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x]$. By Lemma 4.1 and Lemma 3.2 (1), $\operatorname{Hom}_{GL_n(K)}(PS_1(w(\phi),\mathbf{h}),\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x])=0$. For any $\alpha\in\mathfrak{m}_x$, the induced ($GL_n(K)$-equivariant) map $\Pi_\infty^{R_\infty-\rm an}[\mathfrak{a}_x]\xrightarrow{\alpha}\Pi_\infty^{R_\infty-\rm an}[\mathfrak{a}_x]$ composed with (4.6) factors through $PS_1(w(\phi),\mathbf{h})\to\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x]$ (noting $\mathfrak{a}_x\supset\mathfrak{m}_x^2$), hence has to be zero. So (4.6) has image in $\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x]$.

By [20, Lem. 4.16] (which directly generalizes to the crystabelline case), we have (where $\{-\}$ denotes the generalized eigenspace):

$$J_B\big(\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x]\big)\big[T(K)=\delta_w\delta_B\big]\xrightarrow{\sim}J_B\big(\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x]\big)\big\{T(K)=\delta_w\delta_B\big\} \tag{4.7}$$

which is hence one dimensional. Indeed, consider the tangent map of $U_\wp\to(\operatorname{Spf}R_{\overline{\rho}}^{\square})^{\rm rig}\times\widehat{T}$ at the point $x_{w,\wp}$:

$$\begin{aligned}(\kappa_1,\kappa_2):T_{x_{w,\wp}}U_\wp&\longrightarrow T_\rho\big(\operatorname{Spf}R_{\overline{\rho}}^{\square}\big)^{\rm rig}\times T_{\delta_w\delta_B}\widehat{T}\\&\longrightarrow \operatorname{Ext}^1(D,D)\times\operatorname{Hom}\big(T(K),E\big)\end{aligned}$$

(where T_xX denotes the tangent space of a rigid analytic space X at a closed point x). For $v\in T_{x_{w,\wp}}U_\wp$, let $\widetilde{D}_v$ be the deformation of D over $\mathcal{R}_{K,E[\epsilon]/\epsilon^2}$ associated to $\kappa_1(v)$. As D is generic non-critical, the global triangulation theory (e.g. using [5, Thm. 5.3] and an induction argument) implies $\widetilde{D}_v$ is trianguline of parameter $\delta_w(1+\kappa_2(v)\epsilon)$. In particular, if $\kappa_1(v)=0$, then $\kappa_2(v)$ must be zero as well. As $\mathcal{M}$ is locally free at x_w, we deduce $J_B(\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x])[\mathfrak{m}_{\delta_w\delta_B}]\xrightarrow{\sim}J_B(\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x])[\mathfrak{m}_{\delta_w\delta_B}^2]$, where $\mathfrak{m}_{\delta_w\delta_B}$ denotes the maximal ideal of $\mathcal{O}(\widehat{T})$ at the point $\delta_w\delta_B$. But this implies $J_B(\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x])[\mathfrak{m}_{\delta_w\delta_B}]\xrightarrow{\sim}J_B(\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x])[\mathfrak{m}_{\delta_w\delta_B}^k]$ for all $k\geq 1$. (4.7) follows.

Now if the injection $PS_1(w(\phi),\mathbf{h})\hookrightarrow(\operatorname{Im}\iota_w)[\mathfrak{m}_x]$ is not surjective, by Proposition 3.6 (1) and Remark 3.7, there exists $\psi\in\operatorname{Hom}(T(K),E)$ such that $I_{B^-(K)}^{GL_n(K)}(w(\phi)\eta z^\lambda(1+\psi\epsilon))\hookrightarrow(\operatorname{Im}\iota_w)[\mathfrak{m}_x]$. Hence

$$\delta_w\delta_B(1+\psi\epsilon)\longrightarrow J_B\big(\Pi_\infty^{R_\infty-\rm an}[\mathfrak{m}_x]\big)\big\{T(K)=\delta_w\delta_B\big\},$$

a contradiction. The lemma follows. ☐

Let $\widetilde{\pi}$ be the closed subrepresentation of $\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]$ generated by $\mathrm{Im}\,\iota_w$ for all $w\in \mathrm{S}_n$. It is clear that $\widetilde{\pi}$ is stabilized by R_∞. In particular, $\widetilde{\pi}$ has an induced A_{D}-action via $\mathrm{A}_{\mathrm{D}} \xrightarrow{\sim} \mathrm{R}_{\mathrm{D}}^{\square}/\mathfrak{a} \xrightarrow{\sim} \mathrm{R}_\infty[1/p]/\mathfrak{a}_x$.

Theorem **4.5.** — *We have a* $\mathrm{GL}_n(\mathrm{K})\times \mathrm{A}_{\mathrm{D}}$*-equivariant isomorphism* $\pi_1(\phi,\boldsymbol{h})^{\mathrm{univ}}\cong\widetilde{\pi}$ *(see Theorem 3.36 for the* A_{D}*-action on* $\pi_1(\phi,\boldsymbol{h})^{\mathrm{univ}}$*).*

Proof. — We first show $\widetilde{\pi}\cong\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}$ as $\mathrm{GL}_n(\mathrm{K})$-representation. It is clear that the irreducible constituents of $\widetilde{\pi}$ are given by $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$, and all $\mathscr{C}(\mathrm{I},s_{i,\sigma})$ (with multiplicities no less than one, cf. (3.1)). By Lemma 4.1 and Lemma 3.2 (1), $\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{K})}(\mathscr{C}(\mathrm{I},s_{i,\sigma}),\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x])=0$. Together with the fact $\widetilde{\pi}^{\mathrm{lalg}}\cong\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ by Lemma 4.2 (2), we see $\mathrm{soc}_{\mathrm{GL}_n(\mathrm{K})}\,\widetilde{\pi}\cong\pi_{\mathrm{alg}}(\phi,\mathbf{h})$. It is also clear from the definition that all $\mathscr{C}(\mathrm{I},s_{i,\sigma})$ lie in the socle of $\widetilde{\pi}/\pi_{\mathrm{alg}}(\phi,\mathbf{h})$, hence all have multiplicity one by Lemma 3.5 (1). These together with $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\widetilde{\delta}_w^{\mathrm{univ}}\subset\widetilde{\pi}$ for all w, imply $\widetilde{\pi}\cong\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}$.

We equip $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}$ with an A_{D}-action induced by the A_{D}-action on $\widetilde{\pi}$ (induced from R_∞). By Lemma 4.3, the composition $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\widetilde{\delta}_w^{\mathrm{univ}}\hookrightarrow\widetilde{\pi}\cong\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}$ is then A_{D}-equivariant hence factors through an $\mathrm{A}_{\mathrm{D},w}$-equivariant injection $\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\widetilde{\delta}_w^{\mathrm{univ}}\hookrightarrow\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}[\mathcal{I}_w]$ (recalling $\mathcal{I}_w$ is the kernel of $\mathrm{A}_{\mathrm{D}}\twoheadrightarrow\mathrm{A}_{\mathrm{D},w}$). By Theorem 3.36, we see this (global) A_{D}-action coincides with the one given there. This concludes the proof. □

Together with Corollary 3.37, Lemma 4.2 (1), we get

Corollary **4.6.** — *We have a* $\mathrm{GL}_n(\mathrm{K})$*-equivariant injection*

$$\pi_{\min}(\mathrm{D})\cong\pi_1(\phi,\boldsymbol{h})^{\mathrm{univ}}[\mathfrak{m}_{\mathrm{A}_{\mathrm{D}}}]\cong\widetilde{\pi}[\mathfrak{m}_{\mathrm{A}_{\mathrm{D}}}]\hookrightarrow\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x][\mathfrak{m}_{\mathrm{D}}]\cong\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]. \tag{4.8}$$

Remark **4.7.** — By [17, Thm. 5.12], $\pi_{\mathrm{alg}}(\phi,\mathbf{h})\hookrightarrow\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]$ uniquely extends to $\pi(\phi,\mathbf{h})\hookrightarrow\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]$. Using (3.44), we deduce from (4.8) an injection $\pi_{\mathrm{fs}}(\mathrm{D})\hookrightarrow\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]$. Remark that $\pi_{\mathrm{fs}}(\mathrm{D})$ should still be far from the entire $\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]$.

Corollary **4.8.** — *The representation* $\pi_{\min}(\mathrm{D})$ *is the maximal subrepresentation of* $\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]$ *given by extensions of* $\pi_{\mathrm{alg}}(\phi,\boldsymbol{h})$ *by* $\pi_1(\phi,\boldsymbol{h})$*.*

Proof. — Let V be an extension of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ by $\pi_1(\phi,\mathbf{h})$ contained in $\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]\subset\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]$. If V is not a subrepresentation of $\widetilde{\pi}$, then $\mathrm{V}\oplus_{\pi_1(\phi,\mathbf{h})}\widetilde{\pi}\hookrightarrow\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x]$. As $\widetilde{\pi}$ is isomorphic to the universal extension of $\pi_1(\phi,\mathbf{h})$ by $\pi_{\mathrm{alg}}(\phi,\mathbf{h})$, this implies $\dim_{\mathrm{E}}\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{a}_x])\geq 2$ contradicting Lemma 4.2 (2). So $\mathrm{V}\subset\widetilde{\pi}\cap\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]=\widetilde{\pi}[\mathfrak{m}_x]=\widetilde{\pi}[\mathfrak{m}_{\mathrm{A}_{\mathrm{D}}}]\cong\pi_{\min}(\mathrm{D})$. □

By [22, Thm. 5.3.3] (for the crystalline case) and [16, Thm. 1.3] (for the crystabelline non-crystalline case), the information that D is non-critical can be determined by $\Pi_\infty^{\mathrm{R}_\infty-\mathrm{an}}[\mathfrak{m}_x]$. By Corollary 4.8, Theorem 3.34, we then obtain

Corollary **4.9.** — *Keep the situation, then* $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}_x]$ *determines* $\{D_\sigma\}_{\sigma\in\Sigma_K}$ *for* $D = D_{\mathrm{rig}}(\rho)$. *In particular, when* $K=\mathbf{Q}_p$, *it determines* ρ.

4.2. *Some other cases.* — We discuss the local-global compatibility in the space of p-adic automorphic representations for certain definite unitary groups (with fewer global hypotheses than Section4.1).

4.2.1. *Some formal results.* — We first discuss some corollaries of the results in Section 3.2. Let $D\in \Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$ and $\mathrm{Ext}^1_U(D,D)$ be a certain subspace of $\mathrm{Ext}^1(D,D)$. For $w\in S_n$, set $\mathrm{Ext}^1_{U,w}(D,D):=\mathrm{Ext}^1_U(D,D)\cap \mathrm{Ext}^1_w(D,D)$. We assume the following hypotheses.

Hypothesis **4.10.** — *(1)* $\mathrm{Ext}^1_U(D,D)\cap\mathrm{Ext}^1_g(D,D)=0$.
(2) For $w\in S_n$, $\dim_E \mathrm{Ext}^1_{U,w}(D,D)=nd_K$.

Corollary **4.11.** — *The natural map* $\oplus_{w\in S_n}\mathrm{Ext}^1_{U,w}(D,D)\to\mathrm{Ext}^1_U(D,D)$ *is surjective, and* $\dim_E\mathrm{Ext}^1_U(D,D)=\frac{n(n+1)}{2}d_K$.

Proof. — We have a commutative diagram

$$\begin{array}{ccc}
\oplus_{w\in S_n}\mathrm{Ext}^1_{U,w}(D,D) & \longrightarrow & \mathrm{Ext}^1_U(D,D)\\
\downarrow\wr & & \big\downarrow\\
\oplus_{w\in S_n}\mathrm{Ext}^1_w(D,D)/\mathrm{Ext}^1_g(D,D) & \twoheadrightarrow & \mathrm{Ext}^1(D,D)/\mathrm{Ext}^1_g(D,D)
\end{array}$$

where the vertical maps are injective by Hypothesis 4.10 hence the left one is bijective by comparing dimensions (cf. Proposition 2.10 (1)), the surjectivity of the bottom map follows from Proposition 2.12. We deduce the top and right maps are also surjective. The second part follows then by Proposition 2.10 (1). ☐

Denote by $\mathrm{Ext}^1_{U,w}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ the image of the composition (see (2.15) and (3.17)): $\mathrm{Ext}^1_{U,w}(D,D)\hookrightarrow\overline{\mathrm{Ext}}^1_w(D,D)\xrightarrow[\sim]{\zeta_w\circ\kappa_w}\mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$, where the injectivity of the first map follows from Hypothesis 4.10 (1) (recalling $\mathrm{Ext}^1_0(D,D)\subset\mathrm{Ext}^1_g(D,D)$). Denote by $\mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ the subspace of $\mathrm{Ext}^1_{\mathrm{GL}_n(K)}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ generated by $\mathrm{Ext}^1_{U,w}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ for all $w\in S_n$.

Corollary **4.12.** — *(1) The map* $t_{\phi,\boldsymbol{h}}:\oplus_{w\in S_n}\mathrm{Ext}^1_{U,w}(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h}))\cong\oplus_{w\in S_n}\mathrm{Ext}^1_{U,w}(D,D)\twoheadrightarrow\mathrm{Ext}^1_U(D,D)$ *(uniquely) factors through a surjection*

$$t_{D,U}:\mathrm{Ext}^1_U\big(\pi_{\mathrm{alg}}(\phi,\boldsymbol{h}),\pi_1(\phi,\boldsymbol{h})\big)\longrightarrow\!\!\!\!\!\rightarrow\mathrm{Ext}^1_U(D,D).$$

(2) We have $\mathrm{Ker}\, t_{D,U}\xrightarrow{\sim}\mathrm{Ker}\, t_D$ *(cf. (3.39)).*

Proof. — We have a commutative diagram

$$(\mathbf{4.9})\qquad \begin{array}{ccccc} \oplus_w \mathrm{Ext}^1_{\mathrm{U},w}(\mathrm{D},\mathrm{D}) & \hookrightarrow & \oplus_w \overline{\mathrm{Ext}}^1_w(\mathrm{D},\mathrm{D}) & \xrightarrow[\sim]{(\zeta_w\circ\kappa_w)} & \oplus_w \mathrm{Ext}^1_w(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathrm{U}}(\mathrm{D},\mathrm{D}) & \hookrightarrow & \overline{\mathrm{Ext}}^1(\mathrm{D},\mathrm{D}) & \xleftarrow[(3.39)]{t_{\mathrm{D}}} & \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})). \end{array}$$

By (4.9), $t_{\mathrm{D},\mathrm{U}} := t_{\mathrm{D}}|_{\mathrm{Ext}^1_{\mathrm{U}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))}$ satisfies the property in (1). We have

$$(\mathbf{4.10})\qquad \mathrm{Ker}\, t_{\mathrm{D},\mathrm{U}} = \mathrm{Ker}\, t_{\mathrm{D}} \cap \mathrm{Ext}^1_{\mathrm{U}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big) \subset \mathrm{Ker}\, t_{\mathrm{D}}.$$

Denote by $\mathrm{Ext}^1_{\mathrm{U}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_1(w(\phi),\mathbf{h}))$ the image of the composition $\mathrm{Ext}^1_{\mathrm{U},w}(\mathrm{D},\mathrm{D})\overset{\kappa_w}{\hookrightarrow} \mathrm{Hom}(\mathrm{T}(\mathrm{K}),\mathrm{E}) \xrightarrow[\sim]{(3.10)} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_1(w(\phi),\mathbf{h}))$, which has dimension nd_{K} by Hypothesis 4.10 (2). By (3.14), we have an exact sequence

$$(\mathbf{4.11})\qquad \begin{aligned} 0 &\longrightarrow \mathrm{W} \longrightarrow \mathrm{Ext}^1_{\mathrm{U}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathrm{PS}_1\big(w(\phi),\mathbf{h}\big)\big) \\ &\longrightarrow \oplus_{\substack{i=1,\dots,n-1\\ \sigma\in\Sigma_{\mathrm{K}}}} \mathrm{Ext}^1\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}\big(w,s_{i,\sigma}\big)\big), \end{aligned}$$

where W is a subspace of $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$. By Hypothesis 4.10 (1) ((2.15) and Proposition 3.3 (1)), $\mathrm{W}\cap \mathrm{Ext}^1_{\mathrm{lalg}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))=0$. So $\dim_{\mathrm{E}}\mathrm{W}\leq (n+d_{\mathrm{K}})-n=d_{\mathrm{K}}$. By comparing dimensions, the last map in (4.11) must be surjective and $\dim_{\mathrm{E}}\mathrm{W}=d_{\mathrm{K}}$. Similarly by (3.12), $\mathrm{Ext}^1_{\mathrm{U}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))$ lies in an exact sequence

$$(\mathbf{4.12})\qquad \begin{aligned} 0 &\longrightarrow \mathrm{W}' \longrightarrow \mathrm{Ext}^1_{\mathrm{U}}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h})\big) \\ &\longrightarrow \oplus_{\substack{i=1,\dots,n-1,\sigma\in\Sigma_{\mathrm{K}}\\ \mathrm{I}\subset\{1,\dots,n-1\},\#\mathrm{I}=i}} \mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}\big(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\mathscr{C}(\mathrm{I},s_{i,\sigma})\big), \end{aligned}$$

where $\mathrm{W}'\supset \mathrm{W}$ is a subspace of $\mathrm{Ext}^1_{\mathrm{GL}_n(\mathrm{K})}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_{\mathrm{alg}}(\phi,\mathbf{h}))$. By the surjectivity of the last map in (4.11) and varying w (see also the proof of Proposition 3.8), the last map of (4.12) is surjective as well. Hence $\dim_{\mathrm{E}}\mathrm{Ext}^1_{\mathrm{U}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\geq d_{\mathrm{K}}+(2^n-2)d_{\mathrm{K}}$. As $\dim_{\mathrm{E}}\mathrm{Ext}^1_{\mathrm{U}}(\mathrm{D},\mathrm{D})=\frac{n(n+1)}{2}d_{\mathrm{K}}$ by Corollary 4.11, we see $\dim_{\mathrm{E}}\mathrm{Ker}\, t_{\mathrm{D},\mathrm{U}}\geq (2^n-\frac{n(n+1)}{2}-1)d_{\mathrm{K}}=\dim_{\mathrm{E}}\mathrm{Ker}\, t_{\mathrm{D}}$. By (4.10), (2) follows. □

We set $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}_{\mathrm{U}}$ (resp. $\pi_1(\phi,\mathbf{h})^{\mathrm{univ}}_{\mathrm{U},w}$) to be the tautological extension of $\mathrm{Ext}^1_{\mathrm{U}}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\otimes_{\mathrm{E}}\pi_{\mathrm{alg}}(\phi,\mathbf{h})$ (resp. $\mathrm{Ext}^1_{\mathrm{U},w}(\pi_{\mathrm{alg}}(\phi,\mathbf{h}),\pi_1(\phi,\mathbf{h}))\otimes_{\mathrm{E}}\pi_{\mathrm{alg}}(\phi,\mathbf{h})$) by $\pi_1(\phi,\mathbf{h})$ (cf. Section 3.1.1). Let $\mathrm{A}_{\mathrm{D},\mathrm{U}}$ (resp. $\mathrm{A}_{\mathrm{D},\mathrm{U},w}$) be the quotient of $\mathrm{R}_{\mathrm{D}}/\mathfrak{m}^2$ (resp. $\mathrm{R}_{\mathrm{D},w}/\mathfrak{m}^2$) associated to $\mathrm{Ext}^1_{\mathrm{U}}(\mathrm{D},\mathrm{D})$ (resp. $\mathrm{Ext}^1_{\mathrm{U},w}(\mathrm{D},\mathrm{D})$). Let $\mathrm{A}_{\mathrm{U},w}$ be the quotient of $\mathrm{R}_{w(\phi)z^{\mathbf{h}}}/\mathfrak{m}^2$ associated to the image $\mathrm{Ext}^1_{\mathrm{U}}(w(\phi)z^{\mathbf{h}},w(\phi)z^{\mathbf{h}})$ of $\kappa_w:\mathrm{Ext}^1_{\mathrm{U},w}(\mathrm{D},\mathrm{D})\hookrightarrow \mathrm{Ext}^1_{\mathrm{T}(\mathrm{K})}(w(\phi)z^{\mathbf{h}},w(\phi)z^{\mathbf{h}})(\cong \mathrm{Hom}(\mathrm{T}(\mathrm{K}),\mathrm{E}))$. The map κ_w then induces a natural isomorphism of Artinian E-algebras:

$$(\mathbf{4.13})\qquad \mathrm{A}_{\mathrm{U},w}\xrightarrow{\sim}\mathrm{A}_{\mathrm{D},\mathrm{U},w}.$$

Recall $\delta_w = w(\phi)z^{\mathbf{h}}(\varepsilon^{-1}\circ\theta)$. We equip $A_{U,w}$ with the T(K)-action via

$$T(K) \longrightarrow R_{\delta_w}/\mathfrak{m}^2 \xrightarrow{\sim} R_{w(\phi)z^{\mathbf{h}}}/\mathfrak{m}^2 \twoheadrightarrow A_{U,w}$$

where the middle isomorphism is induced by twisting by $\varepsilon^{-1}\circ\theta$. The T(K)-representation $A_{U,w}^\vee$ is isomorphic to the tautological extension $\widetilde{\delta}_{U,w}^{\mathrm{univ}}$ of $\mathrm{Ext}^1_U(\delta_w,\delta_w)\otimes_E \delta_w$ by δ_w, where $\mathrm{Ext}^1_U(\delta_w,\delta_w)$ consists of characters $\widetilde{\delta}_w$ over $E[\epsilon]/\epsilon^2$ such that $\widetilde{\delta}_w(\varepsilon\circ\theta)\in \mathrm{Ext}^1_U(w(\phi)z^{\mathbf{h}}, w(\phi)z^{\mathbf{h}})$. Reciprocally, the T(K)-representation $\widetilde{\delta}_{U,w}^{\mathrm{univ}}$ is equipped with a natural $A_{U,w}$-action (hence an $A_{D,U,w}$-action via (4.13)) as in the discussion below (3.56) (identifying the tangent space of $A_{U,w}$ with a subspace of that of R_{δ_w}). Note the natural map $E[T(K)]\to R_{\delta_w}/\mathfrak{m}^2$ is surjective. Thus the action of T(K) and $A_{U,w}$ actually determine each other.

Consider $I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_{U,w}^{\mathrm{univ}}$. By similar arguments as in the proof of Lemma 3.35 and (the surjectivity of the last map in) (4.11), $I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_{U,w}^{\mathrm{univ}}$ is isomorphic to the universal extension of $\pi_{\mathrm{alg}}(\phi,\mathbf{h})\otimes_E \mathrm{Ext}^1_U(\pi_{\mathrm{alg}}(\phi,\mathbf{h}), \mathrm{PS}_1(w(\phi),\mathbf{h}))$ by $\mathrm{PS}_1(w(\phi),\mathbf{h})$. Moreover, similarly as in the discussion below (3.56), we have a $GL_n(K)\times A_{D,U,w}$-equivariant injection $I_{B^-(K)}^{GL_n(K)}\widetilde{\delta}_{U,w}^{\mathrm{univ}}\hookrightarrow \pi_1(\phi,\mathbf{h})_{U,w}^{\mathrm{univ}}$, where the $A_{D,U,w}$-action on the left hand side is induced by its action on $\widetilde{\delta}_{U,w}^{\mathrm{univ}}$ as discussed in the precedent paragraph and the $A_{D,U,w}$-action on the right hand side is given in a similar way as in (3.57) (using also (4.13)). The following corollary follows by similar arguments as in Theorem 3.36 and Corollary 3.37.

Corollary **4.13.** — *(1) There is a unique $A_{D,U}$-action on $\pi_1(\phi,\boldsymbol{h})_U^{\mathrm{univ}}$ such that for all $w\in S_n$, there is a $GL_n(K)\times A_{D,U,w}$-equivariant injection $\pi_1(\phi,\boldsymbol{h})_{U,w}^{\mathrm{univ}}\hookrightarrow \pi_1(\phi,\boldsymbol{h})_U^{\mathrm{univ}}[\mathcal{I}_w]$.*

(2) We have $\pi_1(\phi,\boldsymbol{h})_U^{\mathrm{univ}}[\mathfrak{m}_{A_{D,U}}]\cong \pi_{\min}(D)$.

4.2.2. *Local-global compatibility.* — We prove a local-global compatibility result in a non-patched setting. We briefly introduce the setup and some notation.

Let F/F^+ be a CM extension and G/F^+ be a unitary group attached to the quadratic extension F/F^+ (e.g. see [3, Section 6.2.1]) such that $G\times_{F^+}F\cong GL_n$ ($n\geq 2$) and $G(F^+\otimes_{\mathbf{Q}}\mathbf{R})$ is compact. For a finite place v of F^+ which is split in F and $\widetilde{v}$ a place of F dividing v, we have isomorphisms $\iota_{\widetilde{v}}: G(F_v^+)\xrightarrow{\sim} G(F_{\widetilde{v}})\xrightarrow{\sim} GL_n(F_{\widetilde{v}})$. We let S_p denote the set of places of F^+ dividing p and we assume that each place in S_p is split in F. For each $v\in S_p$, we fix a place $\widetilde{v}$ of F dividing v.

We fix a place $\wp$ of F^+ above p, and set $K:=F^+_\wp=F_{\widetilde{\wp}}$. We have thus an isomorphism $G(F^+_\wp)\xrightarrow{\sim} GL_n(K)$. For each $v\in S_p$, $v\neq\wp$, let ξ_v be a dominant weight of $\mathrm{Res}^{F_v^+}_{\mathbf{Q}_p} GL_n$, and $\tau_v: I_{F_v^+}\to GL_n(E)$ be an inertial type. To τ_v, one can associate a smooth irreducible representation $\sigma(\tau_v)$ of $GL_n(\mathcal{O}_{F_v^+})$ over E (see for example [23, Thm 3.7]). Let $W_{\xi,\tau}$ be a $GL_n(\mathcal{O}_{F_v^+})$-invariant $\mathcal{O}_E$-lattice of the locally algebraic representation $\sigma(\tau_v)\otimes_E L(\xi_v)$ (see also [23, Section 2.3]).

Let $U^\wp = U^pU_p^\wp=\prod_{v\nmid p}U_v\times\prod_{v\in S_p\setminus\{\wp\}}U_v$ be a sufficiently small (cf. [27]) compact open subgroup of $G(\mathbf{A}_{F^+}^{\infty,\wp})$ with $\iota_{\widetilde{v}}(U_v)=GL_n(\mathcal{O}_{F_v^+})$ for $v\in S_p\setminus\{\wp\}$. We also assume that U_v is hyperspecial if v is inert in F. Let S be the union of S_p and of the places $v \notin \mathrm{S}_p$ such that U_v is not hyperspecial.

For $k \in \mathbf{Z}_{\geq 1}$ and a compact open subgroup $\mathrm{U}_\wp$ of $\mathrm{G}(\mathcal{O}_{\mathrm{F}^+_\wp})$, consider the $\mathcal{O}_\mathrm{E}/\varpi_\mathrm{E}^k$-module $\mathrm{S}_{\xi,\tau}(\mathrm{U}^\wp\mathrm{U}_\wp, \mathcal{O}_\mathrm{E}/\varpi_\mathrm{E}^k) = \{f : \mathrm{G}(\mathrm{F}^+)\backslash\mathrm{G}(\mathbf{A}^\infty_{\mathrm{F}^+}) \to \mathrm{W}_{\xi,\tau}/\varpi_\mathrm{E}^k \mid f(gu) = u^{-1}f(g),\ \forall g \in \mathrm{G}(\mathbf{A}^\infty_{\mathrm{F}^+}), u \in \mathrm{U}^\wp\mathrm{U}_\wp\}$ where $\mathrm{U}^\wp\mathrm{U}_\wp$ acts on $\mathrm{W}_{\xi,\tau}/\varpi_\mathrm{E}^k$ via $\mathrm{U}^\wp\mathrm{U}_\wp \twoheadrightarrow \prod_{v\in \mathrm{S}_p\setminus\{\wp\}} \mathrm{U}_v$. Put

$$\widehat{\mathrm{S}}_{\xi,\tau}\big(\mathrm{U}^\wp, \mathcal{O}_\mathrm{E}\big) := \varprojlim_k \mathrm{S}_{\xi,\tau}\big(\mathrm{U}^\wp, \mathcal{O}_\mathrm{E}/\varpi_\mathrm{E}^k\big) := \varprojlim_k \varinjlim_{\mathrm{U}_\wp} \mathrm{S}_{\xi,\tau}\big(\mathrm{U}^\wp\mathrm{U}_\wp, \mathcal{O}_\mathrm{E}/\varpi_\mathrm{E}^k\big),$$

and $\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathrm{E}) := \widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathcal{O}_\mathrm{E}) \otimes_{\mathcal{O}_\mathrm{E}} \mathrm{E}$. Then $\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathrm{E})$ is an admissible unitary Banach representation of $\mathrm{GL}_n(\mathrm{K})$. Recall that $\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathrm{E})$ is equipped with a natural action of $\mathbf{T}(\mathrm{U}^\wp)$ commuting with $\mathrm{GL}_n(\mathrm{K})$, where $\mathbf{T}(\mathrm{U}^\wp)$ is the polynomial $\mathcal{O}_\mathrm{E}$-algebra generated by Hecke operators: $\mathrm{T}^{(j)}_{\widetilde{v}} = [\mathrm{U}_v \iota_{\widetilde{v}}^{-1} \begin{pmatrix} 1_{n-j} & 0 \\ 0 & \varpi_{\widetilde{v}} 1_j \end{pmatrix} \mathrm{U}_v]$, for $v \notin \mathrm{S}$ which splits to $\widetilde{v}\widetilde{v}^c$ in F and $j = 1, \ldots, n$, where $\varpi_{\widetilde{v}}$ is a uniformizer of $\mathrm{F}_{\widetilde{v}}$.

Using Emerton's method [35, (2.3)], one can construct an eigenvariety $\mathcal{E}(\mathrm{U}^\wp)$ from $\mathrm{J}_\mathrm{B}(\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathrm{E})^{\mathbf{Q}_p-\mathrm{an}})$. There is a natural morphism of rigid spaces $\kappa : \mathcal{E}(\mathrm{U}^\wp) \to \widehat{\mathrm{T}}$. The strong dual $\mathrm{J}_\mathrm{B}(\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathrm{E})^{\mathbf{Q}_p-\mathrm{an}})^\vee$ gives rise to a coherent sheaf $\mathcal{M}(\mathrm{U}^\wp)$ over $\mathcal{E}(\mathrm{U}^\wp)$. An E-point of $\mathcal{E}(\mathrm{U}^\wp)$ can be parametrized by (δ, ω) where δ is a continuous character of $\mathrm{T}(\mathrm{K})$, and ω is a morphism of E-algebras $\mathbf{T}(\mathrm{U}^\wp) \twoheadrightarrow \mathrm{E}$ which corresponds to a maximal ideal $\mathfrak{m}_\omega$ of $\mathbf{T}(\mathrm{U}^\wp)$. Moreover, $(\delta, \omega) \in \mathcal{E}(\mathrm{U}^\wp)$ if and only if $\mathrm{Hom}_{\mathrm{T}(\mathrm{K})}(\delta, \mathrm{J}_\mathrm{B}(\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathrm{E})^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\omega])) \neq 0$. Recall a point $(\delta, \omega) \in \mathcal{E}(\mathrm{U}^\wp)$ is called *classical* if $\mathrm{Hom}_{\mathrm{T}(\mathrm{K})}(\delta, \mathrm{J}_\mathrm{B}(\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathrm{E})^{\mathrm{lalg}}[\mathfrak{m}_\omega])) \neq 0$. In fact, by [14, (6.3)], such points are associated to classical automorphic representations. Note also if (δ, ω) is classical, then δ is locally algebraic hence has the form $\delta_{\mathrm{sm}}\delta_{\mathrm{alg}}$, where δ_{alg} is an algebraic character of $\mathrm{T}(\mathrm{K})$. We call a classical point (δ, ω) generic if $\delta_{\mathrm{sm}}\delta_\mathrm{B}^{-1}|\cdot|_\mathrm{K} \circ \theta =: (\phi'_i)$ is generic, i.e. $\phi'_i(\phi'_j)^{-1} \neq 1, |\cdot|_\mathrm{K}$ for $i \neq j$.

The following proposition is well-known.

Proposition **4.14.** — *(1) $\mathcal{E}(\mathrm{U}^\wp)$ is equidimensional of dimension nd_K.*
(2) The coherent sheaf $\mathcal{M}(\mathrm{U}^\wp)$ is Cohen-Macaulay over $\mathcal{E}(\mathrm{U}^\wp)$.
(3) $\mathcal{E}(\mathrm{U}^\wp)$ is reduced.

Proof. — By [14, Lem. 6.1], for a compact open subgroup H of $\mathrm{GL}_n(\mathcal{O}_\mathrm{K})$, we have $\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathrm{E})|_\mathrm{H} \cong \mathcal{C}(\mathrm{H}, \mathrm{E})^{\oplus s}$ for some $s \geq 1$ (where $\mathcal{C}(\mathrm{H}, \mathrm{E})$ denotes the space of continuous functions on H). (1) (2) then follows verbatim from [21, Lem. 3.10, Prop. 3.11, Cor. 3.12][20, Lem. 3.8], applying [21, Section 5.2] to $\Pi := \widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp, \mathrm{E})^{\mathbf{Q}_p-\mathrm{an}}$. (3) follows by the same argument as in [25, Prop. 3.9] (see also [21, Cor. 3.20]). □

Let F^S be the maximal algebraic extension of F unramified outside the places dividing those in S, and $\mathrm{Gal}_{\mathrm{F},\mathrm{S}} := \mathrm{Gal}(\mathrm{F}^\mathrm{S}/\mathrm{F})$. Let $\rho : \mathrm{Gal}_{\mathrm{F},\mathrm{S}} \to \mathrm{GL}_n(\mathrm{E})$ be a continuous representation satisfying $\rho^c \cong \rho^\vee \otimes_\mathrm{E} \varepsilon^{1-n}$ where $\rho^c(g) := \rho(cgc)$ for $g \in \mathrm{Gal}_{\mathrm{F},\mathrm{S}}$ with c being the complex conjugation. To ρ, one naturally associates a maximal ideal $\mathfrak{m}_\rho$ of $\mathbf{T}(\mathrm{U}^\wp)$ generated by $((-1)^j(\#k_{\widetilde{v}})^{j(j-1)/2}\mathrm{T}_{\widetilde{v}}^{(j)}-a_{\widetilde{v}}^{(j)})$, where $k_{\widetilde{v}}$ is the residue field of $\mathrm{F}_{\widetilde{v}}$, and the characteristic polynomial of $\rho(\mathrm{Frob}_{\widetilde{v}})$ (for a geometric Frobenius $\mathrm{Frob}_{\widetilde{v}}$ at $\widetilde{v}$) is given by $\mathrm{X}^n+a_{\widetilde{v}}^{(1)}\mathrm{X}^{n-1}+\cdots+a_{\widetilde{v}}^{(n-1)}\mathrm{X}+a_{\widetilde{v}}^{(n)}$. Let ω_ρ denote the morphism $\mathbf{T}(\mathrm{U}^\wp)\twoheadrightarrow \mathbf{T}(\mathrm{U}^\wp)/\mathfrak{m}_\rho\cong \mathrm{E}$. Assume $\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^\wp,\mathrm{E})^{\mathrm{lalg}}[\mathfrak{m}_\rho]\neq 0$ and $\mathrm{D}:=\mathrm{D}_{\mathrm{rig}}(\rho_{\widetilde{\wp}})\in\Phi\Gamma_{\mathrm{nc}}(\phi,\mathbf{h})$ (with ϕ generic, and $\mathbf{h}$ strictly dominant, cf. Section 2.1), where $\rho_w:=\rho|_{\mathrm{Gal}_{\mathrm{F}_w}}$ for a place w of F. There exists hence $r\in\mathbf{Z}_{\geq 1}$ such that (e.g. by [2, Thm. 1.1] and [14, (6.3)])

$$\pi_{\mathrm{alg}}(\phi,\mathbf{h})^{\oplus r}\xrightarrow{\sim}\widehat{\mathrm{S}}_{\xi,\tau}\big(\mathrm{U}^\wp,\mathrm{E})[\mathfrak{m}_\rho]^{\mathrm{lalg}}. \tag{4.14}$$

Taking Jacquet-Emerton modules, we see $z_w:=(\delta_w\delta_{\mathrm{B}},\omega_\rho)\in\mathcal{E}(\mathrm{U}^\wp)$ for $w\in\mathrm{S}_n$ (with $\delta_w=w(\phi)z^{\mathbf{h}}(\varepsilon^{-1}\circ\theta)$). Moreover, similarly as in Lemma 4.1, using the global triangulation theory (cf. [44, 49]), $(\delta,\omega_\rho)\in\mathcal{E}(\mathrm{U}^\wp)$ if and only if $\delta=\delta_w\delta_{\mathrm{B}}$ for some $w\in\mathrm{S}_n$ (cf. [12, Prop. 9.2]).

Let $\mathrm{Ext}^1_{\mathrm{U}}(\rho,\rho)$ be the subspace of $\mathrm{Ext}^1_{\mathrm{Gal}_{\mathrm{F},\mathrm{S}}}(\rho,\rho)$ consisting of $\widetilde{\rho}$ such that $\widetilde{\rho}^c\cong\widetilde{\rho}^\vee\otimes_{\mathrm{E}}\varepsilon^{1-n}$. For $v\in\mathrm{S}_p$, we have a natural map $\mathrm{Ext}^1_{\mathrm{U}}(\rho,\rho)\to\mathrm{Ext}^1_{\mathrm{Gal}_{\mathrm{F}_{\widetilde{v}}}}(\rho_{\widetilde{v}},\rho_{\widetilde{v}})$. Set

$$\begin{aligned}&\mathrm{Ext}^1_{g,\mathrm{S}_p\setminus\{\wp\}}(\rho,\rho)\\ &:=\mathrm{Ker}\Big[\mathrm{Ext}^1_{\mathrm{U}}(\rho,\rho)\to\prod_{v\in\mathrm{S}_p\setminus\{\wp\}}\mathrm{Ext}^1_{\mathrm{Gal}_{\mathrm{F}_{\widetilde{v}}}}(\rho_{\widetilde{v}},\rho_{\widetilde{v}})/\mathrm{Ext}^1_g(\rho_{\widetilde{v}},\rho_{\widetilde{v}})\Big].\end{aligned}$$

Let $\mathrm{Ext}^1_{\mathrm{U}}(\mathrm{D},\mathrm{D})$ be the image of the subspace $\mathrm{Ext}^1_{g,\mathrm{S}_p\setminus\{\wp\}}(\rho,\rho)$ via $\mathrm{Ext}^1_{\mathrm{U}}(\rho,\rho)\to\mathrm{Ext}^1_{\mathrm{Gal}_{\mathrm{F}_{\widetilde{\wp}}}}(\rho_{\widetilde{\wp}},\rho_{\widetilde{\wp}})\xrightarrow{\sim}\mathrm{Ext}^1(\mathrm{D},\mathrm{D})$.

We make the following vanishing hypothesis on the adjoint Selmer group:

Hypothesis **4.15.** — *Suppose the composition* $\mathrm{Ext}^1_{g,\mathrm{S}_p\setminus\{\wp\}}(\rho,\rho)\to\mathrm{Ext}^1(\mathrm{D},\mathrm{D})\to\mathrm{Ext}^1(\mathrm{D},\mathrm{D})/\mathrm{Ext}^1_g(\mathrm{D},\mathrm{D})$ *is injective. In particular,* $\mathrm{Ext}^1_{\mathrm{U}}(\mathrm{D},\mathrm{D})\cap\mathrm{Ext}^1_g(\mathrm{D},\mathrm{D})=0$.

Remark **4.16.** — Hypothesis 4.15 is known to hold in many cases, see [1, Thm. A] [21, Cor. 4.12] [52, Thm. A].

Let $\mathrm{R}_{\rho,\mathrm{U}}$ be the universal deformation ring of deformations ρ_{A} of the $\mathrm{Gal}_{\mathrm{F},\mathrm{S}}$-representation ρ over local Artinian E-algebras A satisfying $\rho_{\mathrm{A}}^c\cong\rho_{\mathrm{A}}^\vee\otimes_{\mathrm{E}}\varepsilon^{1-n}$. Note $\mathrm{R}_{\rho,\mathrm{U}}$ exists as $\mathrm{End}_{\mathrm{Gal}_{\mathrm{F},\mathrm{S}}}(\rho)\hookrightarrow\mathrm{End}(\mathrm{D})\cong\mathrm{E}$. Let $\mathfrak{a}_\rho\supset\mathfrak{m}^2_{\mathrm{R}_{\rho,\mathrm{U}}}$ (resp. $\mathfrak{a}_{\mathrm{D}}\supset\mathfrak{m}^2_{\mathrm{R}_{\mathrm{D}}}$) be the ideal associated to $\mathrm{Ext}^1_{g,\mathrm{S}_p\setminus\{\wp\}}(\rho,\rho)$ (resp. $\mathrm{Ext}^1_{\mathrm{U}}(\mathrm{D},\mathrm{D})$). By Hypothesis 4.15, $\mathrm{Ext}^1_{g,\mathrm{S}_p\setminus\{\wp\}}(\rho,\rho)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{U}}(\mathrm{D},\mathrm{D})$. The natural morphism $\mathrm{R}_{\mathrm{D}}\to\mathrm{R}_{\rho,\mathrm{U}}$ induces hence an isomorphism (of local Artinian E-algebras) $\mathrm{A}_{\mathrm{D},\mathrm{U}}=\mathrm{R}_{\mathrm{D}}/\mathfrak{a}_{\mathrm{D}}\xrightarrow{\sim}\mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho$. Let $\widetilde{\rho}_{\mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho}$ be the universal deformation of ρ over $\mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho$. We have a natural morphism $\mathbf{T}(\mathrm{U}^\wp)\to\mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho$ sending $\mathrm{T}^{(j)}_{\widetilde{v}}$ to $(-1)^j(\#k_{\widetilde{v}})^{-j(j-1)/2}\widetilde{a}^{(j)}_{\widetilde{v}}$ where $\widetilde{a}^{(j)}_{\widetilde{v}}\in\mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho$, $j=1,\dots,n$, satisfy that the characteristic polynomial of $\widetilde{\rho}_{\mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho}(\mathrm{Frob}_{\widetilde{v}})$ is given by $\mathrm{X}^n + \widetilde{a}_{\widetilde{v}}^{(1)}\mathrm{X}^{n-1} + \cdots + \widetilde{a}_{\widetilde{v}}^{(n-1)}\mathrm{X} + \widetilde{a}_{\widetilde{v}}^{(n)}$. Let $\mathfrak{a}_{\mathrm{T}}$ be its kernel. The induced morphism $\mathbf{T}(\mathrm{U}^{\wp})/\mathfrak{a}_{\mathrm{T}} \to \mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho$ is an isomorphism when ρ is absolutely irreducible. Indeed, it suffices to show the morphism sends $\mathfrak{m}_\rho$ onto $\mathfrak{m}_{\mathrm{R}_{\rho,\mathrm{U}}}/\mathfrak{a}_\rho$. Consider the universal representation $\widetilde{\rho}$ of $\mathrm{Gal}_{\mathrm{F},\mathrm{S}}$ over $\mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{m}_\rho$. By the definition of the morphism, we deduce the associated pseudo-character $\mathrm{tr}(\widetilde{\rho})$ takes values in E. This implies $\mathrm{tr}(\widetilde{\rho})$ is a trivial deformation of $\mathrm{tr}(\rho)$. As ρ is absolutely irreducible, deforming ρ is equivalent to deforming $\mathrm{tr}(\rho)$ (cf. [53, Thm. 1]). We deduce $\widetilde{\rho}$ is a trivial deformation of ρ hence $\mathfrak{m}_\rho(\mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho) = \mathfrak{m}_{\mathrm{R}_{\rho,\mathrm{U}}}/\mathfrak{a}_\rho$.

Proposition **4.17.** — *Assume Hypothesis 4.15 and ρ is absolutely irreducible. Then $\mathcal{E}(\mathrm{U}^{\wp})$ is smooth at z_w for all $w \in \mathrm{S}_n$. Moreover, $\mathrm{Ext}^1_{\mathrm{U}}(\mathrm{D},\mathrm{D})$ satisfies Hypothesis 4.10 (1) (2).*

Proof. — There is a natural family of $\mathrm{Gal}_{\mathrm{F},\mathrm{S}}$-representations on $\mathcal{E}(\mathrm{U}^{\wp})$. We quickly recall some of its properties that we need. Let $\mathrm{X} \subset \mathcal{E}(\mathrm{U}^{\wp})$ be a sufficiently small affinoid neighbourhood of x_w such that $x_{w'} \notin \mathrm{X}$ for $w' \neq w$, and that the generic classical points are Zariski dense in X. Recall to each classical point $z \in \mathrm{X}$, we can associate an n-dimensional continuous representation ρ_z of $\mathrm{Gal}_{\mathrm{F},\mathrm{S}}$, hence an n-dimension pseudo-character of $\mathrm{Gal}_{\mathrm{F},\mathrm{S}}$. By [24, Prop. 7.1.1], there is a pseudo-character $\mathcal{T}_{\mathrm{X}} : \mathrm{Gal}_{\mathrm{F},\mathrm{S}} \to \mathcal{O}(\mathrm{X})$ interpolating those associated to the classical points. By [4, Lem. 5.5] and the assumption ρ is absolutely irreducible, shrinking X if needed, $\mathcal{T}_{\mathrm{X}}$ gives rise to a continuous representation $\rho_{\mathrm{X}} : \mathrm{Gal}_{\mathrm{F},\mathrm{S}} \to \mathrm{GL}_n(\mathcal{O}(\mathrm{X}))$ satisfying that for all points $z \in \mathrm{X}$, $\rho_z = z^*\rho_{\mathrm{X}}$ is absolutely irreducible. Let $\delta_{\mathrm{X}} = (\delta_{\mathrm{X},1}, \ldots, \delta_{\mathrm{X},n}) : \mathrm{T}(\mathrm{K}) \to \mathcal{O}(\mathrm{X})^\times$ be the natural character associated to κ. Let $\mathcal{R}_{\mathrm{K},\mathrm{X}}$ be the relative Robba ring over $\mathcal{O}(\mathrm{X})$ for K (cf. [44, Def. 2.2.2]). By shrinking X if needed, ρ_{X} has the following properties:

(i) $\rho_{\mathrm{X}}^c \cong \rho_{\mathrm{X}} \otimes_{\mathrm{E}} \varepsilon^{1-n}$.
(ii) For $v \in \mathrm{S}_p \setminus \{\wp\}$, $\rho_{\mathrm{X},\widetilde{v}}$ is de Rham of Hodge-Tate weights $\xi_v - \theta^{\mathrm{F}_v^+}$.
(iii) The (φ,Γ)-module $\mathrm{D}_{\mathrm{rig}}(\rho_{\mathrm{X},\widetilde{\wp}})$ over $\mathcal{R}_{\mathrm{K},\mathrm{X}}$ is isomorphic to a successive extension of the rank one (φ,Γ)-modules $\mathcal{R}_{\mathrm{K},\mathrm{X}}(\delta_{\mathrm{X},i}|\cdot|_{\mathrm{K}}^{2i-(n+1)}\varepsilon^{1-i})$ (cf. [44, Def. 6.2.1]) for $i = 1, \ldots, n$.

(i) follows easily from the fact that for all the classical points, $\rho_z^c \cong \rho_z \otimes_{\mathrm{E}} \varepsilon^{1-n}$. For a place $v \in \mathrm{S}_p \setminus \{\wp\}$, $\rho_{z,\widetilde{v}}$ is de Rham (of inertial type τ_v) of Hodge-Tate weights $\xi_v - \theta^{\mathrm{F}_v^+}$ for all classical points z. (ii) follows then by [8, Thm. B]. Finally, as $\mathrm{D} := \mathrm{D}_{\mathrm{rig}}(\rho_{x_w})$ is non-critical and (φ-)generic, by [5, Thm. 5.3] (and an easy induction argument), (iii) follows (by shrinking X if needed).

Let T_{z_w} be the tangent space of $\mathcal{E}(\mathrm{U}^{\wp})$ at the point z_w. By (i), we have a map $\mathrm{T}_{z_w} \to \mathrm{Ext}^1_{\mathrm{U}}(\rho,\rho)$, sending $t : \mathrm{Spec}\,\mathrm{E}[\epsilon]/\epsilon^2 \to \mathrm{X}$ to $t^*\rho_{\mathrm{X}}$. Denote by $\kappa_{z_w} : \mathrm{T}_{z_w} \to \mathrm{Ext}^1_{\mathrm{T}(\mathrm{K})}(\delta_w\delta_{\mathrm{B}}, \delta_w\delta_{\mathrm{B}})$ the tangent map of $\mathcal{E}(\mathrm{U}^{\wp}) \to \widehat{\mathrm{T}}$ at z_w.

Claim. The induced map $f_{z_w} : \mathrm{T}_{z_w} \to \mathrm{Ext}^1_{\mathrm{U}}(\rho,\rho)$ is injective and has image in $\mathrm{Ext}^1_{g,\mathrm{S}_p\setminus\{\wp\}}(\rho,\rho)$.

Let ν be in the kernel. So $\nu^*\rho_{\mathrm{X}} \cong \rho \oplus \rho$. This implies the composition $\mathbf{T}(\mathrm{U}^{\wp}) \to \mathcal{O}(\mathrm{X}) \xrightarrow{\nu} \mathrm{E}[\epsilon]/\epsilon^2$ factors through $\mathfrak{m}_\rho$. However, by (iii), $\kappa_{z_w}(\nu)$ (as a character of $\mathrm{T}(\mathrm{K})$ over $\mathrm{E}[\epsilon]/\epsilon^2$) is a trianguline parameter of the (φ,Γ)-module $\mathrm{D}_{\mathrm{rig}}(\nu^*\rho_{\mathrm{X}})$ over $\mathcal{R}_{\mathrm{K},\mathrm{E}[\epsilon]/\epsilon^2}$. Hence $\kappa_{z_w}(\nu)=0$. But by the construction of $\mathcal{E}(\mathrm{U}^{\wp})$, $\mathbf{T}(\mathrm{U}^{\wp}) \otimes_{\mathrm{E}} \mathrm{E}[\mathrm{T}(\mathrm{K})]$ is dense in $\mathcal{O}(\mathrm{X})$, hence the map $\mathrm{T}_{z_w} \xrightarrow{(f_{z_w},\kappa_{z_w})} \mathrm{Ext}^1_{\mathrm{U}}(\rho,\rho) \times \mathrm{Ext}^1_{\mathrm{T}(\mathrm{K})}(\delta_w\delta_{\mathrm{B}},\delta_w\delta_{\mathrm{B}})$ is injective. We deduce ν is zero. The second part of the claim follows from (ii).

By (iii), the composition $f_{\mathrm{D}}: \mathrm{T}_{z_w} \to \mathrm{Ext}^1_{\mathrm{U}}(\rho,\rho) \to \mathrm{Ext}^1(\mathrm{D},\mathrm{D})$ has image in $\mathrm{Ext}^1_w(\mathrm{D},\mathrm{D})$ hence (by the claim) in $\mathrm{Ext}^1_{\mathrm{U},w}(\mathrm{D},\mathrm{D})$. Together with Hypothesis 4.15, we deduce $\dim_{\mathrm{E}} \mathrm{T}_{z_w} \leq \dim \mathrm{Ext}^1_{\mathrm{U},w}(\mathrm{D},\mathrm{D}) \leq \dim_{\mathrm{E}} \mathrm{Ext}^1_w(\mathrm{D},\mathrm{D})/\mathrm{Ext}^1_g(\mathrm{D},\mathrm{D}) = nd_{\mathrm{K}}$. As $\dim \mathcal{E}(\mathrm{U}^{\wp}) = nd_{\mathrm{K}}$, we see z_w is a smooth point and $\dim_{\mathrm{E}} \mathrm{Ext}^1_{\mathrm{U},w}(\mathrm{D},\mathrm{D}) = nd_{\mathrm{K}}$. This finishes the proof. □

By Proposition 4.14 (2), Proposition 4.17 and (4.14), $\mathcal{M}$ is locally free of rank r at each z_w for $w \in \mathrm{S}_n$. Let X be a sufficiently small smooth affinoid neighbourhood of z_w, and $\mathfrak{m}_{z_w} \subset \mathcal{O}(\mathrm{X})$ be the maximal ideal associated to z_w. We use the notation of Section 4.2: $\mathrm{A}_{\mathrm{D},\mathrm{U}}$, $\mathrm{A}_{\mathrm{D},\mathrm{U},w}$, $\mathrm{A}_{\mathrm{U},w}$ etc. By the proof of Proposition 4.17, the composition $\mathrm{A}_{\mathrm{D},\mathrm{U},w} \hookrightarrow \mathrm{A}_{\mathrm{D},\mathrm{U}} = \mathrm{R}_{\mathrm{D}}/\mathfrak{a}_{\mathrm{D}} \xrightarrow{\sim} \mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho \twoheadrightarrow \mathcal{O}(\mathrm{X})/\mathfrak{m}_{z_w}^2$ is an isomorphism (where the last map is obtained by sending an element in the tangent space of X at z_w to the associated $\mathrm{Gal}_{\mathrm{F},\mathrm{S}}$-representation). The map $\mathrm{A}_{\mathrm{U},w} \xrightarrow{\sim} \mathrm{A}_{\mathrm{D},\mathrm{U},w} \xrightarrow{\sim} \mathcal{O}(\mathrm{X})/\mathfrak{m}_{z_w}^2$ coincides with the natural map induced by κ, and the map $\mathbf{T}(\mathrm{U}^{\wp})/\mathfrak{a}_{\mathrm{T}} \xrightarrow{\sim} \mathrm{R}_{\rho,\mathrm{U}}/\mathfrak{a}_\rho \twoheadrightarrow \mathcal{O}(\mathrm{X})/\mathfrak{m}_{z_w}^2$ coincides with the one induced by $\mathbf{T}(\mathrm{U}^{\wp}) \to \mathcal{O}(\mathrm{X})$. We deduce a $\mathrm{T}(\mathrm{K}) \times \mathrm{A}_{\mathrm{D},w,\mathrm{U}}$-equivariant injection

$$\widetilde{\delta}_{\mathrm{U},w}^{\mathrm{univ},\oplus r} \cong \big(\mathcal{M}/\mathfrak{m}_{z_w}^2\big)^\vee \longrightarrow \mathrm{J}_{\mathrm{B}}\big(\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^{\wp},\mathrm{E})_{\overline{\rho}}^{\mathbf{Q}_p-\mathrm{an}}\big)[\mathfrak{a}_{\mathrm{T}}][\mathcal{I}_w]\big\{\mathrm{T}(\mathrm{K}) = \delta_w\delta_{\mathrm{B}}\big\}, \tag{4.15}$$

where the $\mathrm{A}_{\mathrm{D},w,\mathrm{U}}$-action on the left hand side is given as in the discussion below (4.13) and it acts on the right hand side via $\mathrm{A}_{\mathrm{D},w,\mathrm{U}} \cong \mathrm{A}_{\mathrm{D},\mathrm{U}}/\mathcal{I}_w \twoheadleftarrow \mathrm{A}_{\mathrm{D},\mathrm{U}} \cong \mathrm{R}_{\mathrm{D},\mathrm{U}}/\mathfrak{a}_\rho \cong \mathbf{T}(\mathrm{U}^{\wp})/\mathfrak{a}_{\mathrm{T}}$. Note as in the discussion below (4.13), the action of $\mathrm{A}_{\mathrm{D},w,\mathrm{U}}$ and $\mathrm{T}(\mathrm{K})$ determine each other. Similarly in Lemma 4.3, the map (4.15) is balanced and induces (by [37, Thm. 0.13]) a $\mathrm{GL}_n(\mathrm{K}) \times \mathrm{R}_{\mathrm{D},w,\mathrm{U}}$-equivariant injection

$$\iota_w : \big(\mathrm{I}_{\mathrm{B}^-(\mathrm{K})}^{\mathrm{GL}_n(\mathrm{K})}\widetilde{\delta}_{\mathrm{U},w}^{\mathrm{univ}}\big)^{\oplus r} \longrightarrow \widehat{\mathrm{S}}_{\xi,\tau}\big(\mathrm{U}^{\wp},\mathrm{E}\big)_{\overline{\rho}}^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{a}_{\mathrm{T}}]. \tag{4.16}$$

Let $\widetilde{\pi}$ be the closed subrepresentation of $\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^{\wp},\mathrm{E})_{\overline{\rho}}^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{a}_{\mathrm{T}}]$ generated by $\mathrm{Im}\,\iota_w$ for all w. Note $\widetilde{\pi}$ inherits from $\widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^{\wp},\mathrm{E})_{\overline{\rho}}^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{a}_{\mathrm{T}}]$ a (global) $\mathrm{A}_{\mathrm{D},\mathrm{U}}$ ($\cong \mathbf{T}(\mathrm{U}^{\wp})/\mathfrak{a}_{\mathrm{T}}$)-action.

Theorem **4.18.** — *Suppose Hypothesis 4.15 and ρ is absolutely irreducible. We have a* $\mathrm{GL}_n(\mathrm{K}) \times \mathrm{A}_{\mathrm{D},\mathrm{U}}$*-equivariant isomorphism* $\widetilde{\pi} \cong \pi_1(\phi,\boldsymbol{h})_{\mathrm{U}}^{\mathrm{univ},\oplus r}$ *(cf. Corollary 4.13). Consequently, we have* $\pi_{\min}(\mathrm{D})^{\oplus r} \hookrightarrow \widehat{\mathrm{S}}_{\xi,\tau}(\mathrm{U}^{\wp},\mathrm{E})^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$.

Proof. — We first show $\widetilde{\pi} \cong \pi_1(\phi,\mathbf{h})_{\mathrm{U}}^{\mathrm{univ},\oplus r}$ as $\mathrm{GL}_n(\mathrm{K})$-representation. By (4.16) and similar arguments as in the proof of Corollary 4.4 (or using the same argument as in the proof of [17, Thm. 5.12]), the injection $\pi_{\mathrm{alg}}(\phi, \mathbf{h})^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$ extends uniquely to an injection $\pi_1(\phi, \mathbf{h})^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$. Note that $\mathrm{Im}\,\iota_w \cap \pi_1(\phi, \mathbf{h})^{\oplus r} \cong \mathrm{PS}_1(w(\phi), \mathbf{h})^{\oplus r}$ (by the same argument as in the proof of Corollary 4.4). As in the proof of Theorem 4.5, $\widetilde{\pi}$ is isomorphic to an extension of certain copies of $\pi_{\mathrm{alg}}(\phi, \mathbf{h})$ by $\pi_1(\phi, \mathbf{h})^{\oplus r}$. Using (4.14) (4.16) (and the structure of $\pi_1(\phi, \mathbf{h})_{U}^{\mathrm{univ}}$), it is not difficult to see $\widetilde{\pi}$ has to be isomorphic to $\pi_1(\phi, \mathbf{h})_{U}^{\mathrm{univ}, \oplus r}$.

For the compatibility of the $A_{D,U}$-action, it suffices to show any injection $\iota : \pi_1(\phi, \mathbf{h})_{U}^{\mathrm{univ}} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{a}_T]$ (extending $\pi_1(\phi, \mathbf{h}) \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$) is $A_{D,U}$-equivariant. As $\pi_1(\phi, \mathbf{h})_{U}^{\mathrm{univ}}$ is generated by $I_{B^-(K)}^{\mathrm{GL}_n(K)} \widetilde{\delta}_{U,w}^{\mathrm{univ}}$, it suffices to show the restriction of ι to $I_{B^-(K)}^{\mathrm{GL}_n(K)} \widetilde{\delta}_{U,w}^{\mathrm{univ}}$ is $A_{D,U}$-equivariant. The restriction of ι to $I_{B^-(K)}^{\mathrm{GL}_n(K)} \widetilde{\delta}_{U,w}^{\mathrm{univ}}$ corresponds to a unique $T(K)$-equivariant injection

$$\widetilde{\delta}_{U,w}^{\mathrm{univ}} \delta_B \longrightarrow J_B\Big(\widehat{S}_{\xi,\tau}\big(U^{\wp}, E\big)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{a}_\rho]\Big). \tag{4.17}$$

Its image is clearly contained in the image of (4.15) (by the definition of $\mathcal{M}$). However, any $T(K)$-equivariant injection $\widetilde{\delta}_{U,w}^{\mathrm{univ}} \delta_B \hookrightarrow (\widetilde{\delta}_{U,w}^{\mathrm{univ}} \delta_B)^{\oplus r}$ has to be $A_{D,U,w}$-equivariant (by the discussion below (4.13)), so is (4.17). Thus $\iota|_{I_{B^-(K)}^{\mathrm{GL}_n(K)} \widetilde{\delta}_{U,w}^{\mathrm{univ}}}$ is $A_{D,U,w}$-equivariant for all w, so ι is $A_{D,U}$-equivariant. The second part follows from Corollary 4.13 (2). □

Remark **4.19.** — By the same argument as in the proof of [17, Thm. 5.12], the injection $\pi_{\mathrm{alg}}(\phi, \mathbf{h})^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$ uniquely extends to $\pi(\phi, \mathbf{h})^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$. Similarly as in Remark 4.7, we see the injection $\pi_{\min}(D)^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$ (in Theorem 4.18) extends uniquely to $\pi_{\mathrm{fs}}(D)^{\oplus r} \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$.

By similar arguments as in Corollary 4.8 (replacing Lemma 4.2 (2) by (4.14)), we have:

Corollary **4.20.** — *The representation $\pi_{\min}(D)^{\oplus r}$ is the maximal subrepresentation of $\widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$, which is generated by extensions of $\pi_{\mathrm{alg}}(\phi, \boldsymbol{h})$ by $\pi_1(\phi, \boldsymbol{h})$.*

The information that D is non-critical can be read out from $\widehat{S}_{\xi,\tau}(U^{\wp}, E)[\mathfrak{m}_\rho]$ by [12, Thm. 9.3]. Together with Corollary 4.20 and Theorem 3.34, we get:

Corollary **4.21.** — *For $D' \in \Phi\Gamma_{\mathrm{nc}}(\phi, \boldsymbol{h})$, $\pi_{\min}(D') \hookrightarrow \widehat{S}_{\xi,\tau}(U^{\wp}, E)[\mathfrak{m}_\rho]$ if and only if $D'_\sigma \cong D_\sigma$ for all $\sigma \in \Sigma_K$. In particular, when $K = \mathbf{Q}_p$, the $\mathrm{GL}_n(\mathbf{Q}_p)$-representation $\widehat{S}_{\xi,\tau}(U^{\wp}, E)^{\mathbf{Q}_p-\mathrm{an}}[\mathfrak{m}_\rho]$ determines $\rho_{\widetilde{\wp}}$.*

Acknowledgements

I thank Xiaozheng Han, Zicheng Qian, Zhixiang Wu for helpful discussions. I especially thank Christophe Breuil for the discussions and his interest, which substantially accelerates the work, and for his comments on a preliminary version of the paper.

I heartily thank the anonymous referee for the careful reading, and valuable suggestions, which greatly improve the clarity of the paper.

Funding information

This work is supported by the NSFC Grant No. 8200800065, No. 8200907289 and No. 8200908310, and partially supported by the New Cornerstone Science Foundation.

Declarations:

Competing interests

The author declares no competing interests.

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Publisher's note

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Manuscrit reçu le 20 août 2024
Manuscrit accepté le 11 juin 2025
publié en ligne le 10 juillet 2025.