GENERALISED ANDRÉ-PINK-ZANNIER CONJECTURE FOR SHIMURA VARIETIES OF ABELIAN TYPE

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ABSTRACT

In this paper we prove the generalised André-Pink-Zannier conjecture (an important case of the Zilber-Pink conjecture) for all Shimura varieties of abelian type. Questions of this type were first asked by Y. André in 1989. We actually prove a general statement for all Shimura varieties, subject to certain assumptions that are satisfied for Shimura varieties of abelian type and are expected to hold in general. We also prove another result, a p-adic Kempf-Ness theorem, on the relation between good reduction of homogeneous spaces over p-adic integers with Mumford stability property in p-adic geometric invariant theory.

CONTENTS

1. Introduction

The central object of study in this article is the following conjecture.

Conjecture **1.1** *(André-Pink-Zannier). — Let* S *be a Shimura variety and* Σ *a subset of a generalised Hecke orbit in* S *(as in [40, Def. 2.1]). Then the irreducible components of the Zariski closure of* Σ *are weakly special subvarieties.*

This conjecture is an important special case of the Zilber-Pink conjectures for Shimura varieties, which has recently been and continues to be a subject of active research.

A special case of Conjecture 1.1 was first formulated in 1989 by Y. André in [1, §X 4.5, p. 216 (Problem 3)]. Conjecture 1.1 was then stated in the introduction to the second author's 2000 PhD thesis [57],[1] following discussions with Bas Edixhoven. Both statements refer to classical Hecke orbits, rather than *generalised* Hecke orbits (cf. [40, §2.5.1]).

[1] The statement there uses the terminology 'totally geodesic subvarieties' instead of 'weakly special', but Moonen had proved in [30] that the two notions are equivalent.

https://doi.org/10.1007/s10240-025-00154-4

Zannier has considered questions of this type in the context of abelian schemes and tori. Richard Pink, in his 2005 paper [35], has formulated and studied this question.

Pink proves it for "Galois generic" points of Shimura varieties:[2] this implies in particular that such points are Hodge generic in their connected component. Pink uses equidistribution of Hecke points proved in [8] (or in [18]).

We refer to the introduction of [40] for further background on Conjecture 1.1.

In the Pila-Zannier approach and most other approaches to Zilber-Pink conjectures, one of the major difficulties is to obtain suitable lower bounds for Galois orbits of points in the "unlikely locus" (see [9]).

In [40], we develop a general approach to Conjecture 1.1 based on the Pila-Zannier strategy (o-minimality and functional transcendence). In [40], we define generalised Hecke orbits, we define a natural height function on these orbits, and we prove precise lower Galois bounds [40, Th. 6.4] under the "weakly adélic Mumford-Tate conjecture" [40, §6.1].

Let (G, X) be a Shimura datum and let $\mathrm{K} \leq \mathrm{G}(\mathbf{A}_f)$ be a compact open subgroup and let $\mathrm{S} = \mathrm{S}h_{\mathrm{K}}(\mathrm{G}, \mathrm{X})$ be the associated Shimura variety. The main result of [40] is as follows.

Theorem **1.2** *(Theorem 1.2 of [40]). — Let $x_0 \in \mathrm{X}$. Assume that x_0 satisfies the weakly adélic Mumford-Tate conjecture.*

Then the conclusion of Conjecture 1.1 holds for any subset of the generalised Hecke orbit of $[x_0, 1]$.

In the present article we prove conclusions of this theorem *unconditionally* for all Shimura varieties *of abelian type*. This completely generalises the main result of [34] by M. Orr.

Our main result is as follows.

Theorem **1.3.** — *Let s_0 be a point in a Shimura variety* $\mathrm{S}h_{\mathrm{K}}(\mathrm{G}, \mathrm{X})$ *of abelian type. Let* Z *be a subvariety whose intersection with the generalised Hecke orbit of s_0 is Zariski dense in* Z. *Then* Z *is a finite union of weakly special subvarieties of* S.

We actually prove the more general statement below, which we believe to be of independent interest. Its assumption is weaker than 'weakly adélic Mumford-Tate conjecture' in Th. 1.2. It is the 'uniform integral Tate conjecture' assumption explained in §2. We refer to [40, Def. 2.1] for the notion of geometric Hecke orbit. By [40, Th. 2.4], a generalised Hecke orbit is a finite union of geometric Hecke orbits.

Theorem **1.4.** — *Let* $s_0 = [x_0, 1]$ *be a point in a Shimura variety* $\mathrm{S}h_{\mathrm{K}}(\mathrm{G}, \mathrm{X})$, *and assume the uniform integral Tate conjecture for x_0 in* X *in the sense of Definition 2.3. Let* Z *be a subvariety whose*

[2] Roughly, the image of the corresponding Galois representation intersects the derived subgroup of the ambient group in an adélically open subgroup. This may be too strong to hold in general. See the first author's 2009 PhD thesis [37, III.§7, p. 59] for a weaker assumption.

intersection with the geometric Hecke orbit of s_0 is Zariski dense in Z. *Then* Z *is a finite union of weakly special subvarieties of* S.

Using Faltings' theorems, we prove in §4.5 that points on Shimura varieties of adjoint abelian type and satisfy this 'uniform integral Tate assumption'. Thus Theorem 1.3, in the adjoint type case, is a special case of Theorem 1.4. Because Conjecture 1.1 can be reduced to the adjoint case, we deduce Theorem 1.3 for any Shimura variety of abelian type.

At the heart of this article is obtaining polynomial lower bounds [40, Th. 6.4] which are unconditional for Shimura varieties of abelian type, or in general under the assumption of the Tate hypothesis. We emphasize that Shimura varieties of abelian type constitute the most important class of Shimura varieties.

The Tate hypothesis is used to compare the sizes of Galois orbits with that of the adélic orbits of [40, App. B]. In our setting, we can easily recover former results of [33] which were only concerned with S-Hecke orbits (involving a finite set S of primes). In order to work with whole Hecke orbits, and even geometric Hecke orbits, we use an "integral and uniform" refined version of the Tate conjecture. Using generalised Hecke orbits is important for our strategy to work, in particular for the reduction steps in [40, §7].

Some of the new ideas in this article relate the notion of "Stability" in the Mumford sense to the Tate hypothesis. The fine estimates we need use stability not only over complex numbers, but in a broader context, over $\mathbf{Z}_p$ and $\mathbf{Z}$. This is where the "uniformity and integrality" in our Tate hypothesis is essential. These ideas originate from [38], part of the first author's 2009 PhD thesis.

This article also develops several results of independent interest. Theorem 7.1 is a p-adic version of a Theorem of Kempf-Ness [26]. We expect it to be useful in other contexts, and it is proved in more generality than needed here. Theorem 6.1 gives precise and uniform comparison on norms along two closed orbits of reductive groups. In Appendix B, we give some consequences of Faltings theorems, in the axiomatic form given in §2, for factorisations of Galois images, and in particular ℓ-independence. Our arguments rely only on group theory, they do not involve ramification properties, and therefore apply to more general groups than images of Galois representations.

Outline of the paper. — We define the uniform integral hypothesis in Section 2.

In Section 3, we reduce Th. 1.4 to the bounds on Galois orbits established in the rest of the paper, and the functorial invariance properties of the Tate hypothesis of Section 4. Since the formal strategy is almost identical to that of [40, §7] we only give a sketch indicating necessary adjustments and provide precise references to [40].

In Section 4, we also derive the refined version of Faltings' theorems that we use, using arguments of Serre and Noot. We deduce that the uniform integral Tate hypothesis holds in Shimura varieties which are of abelian type and also of adjoint type.

The central and technically hardest parts of the paper are §§5–7. There we establish the lower bounds for the Galois orbits of points in geometric Hecke orbits as in [40] under assumptions of Th. 1.4.

The main result Th. 6.1 of Section 6 is essential to the proofs in Section 5. We derive it in Section 6 from the results of Sections 7 and 8.

Section 7 gives a p-adic analogue Th. 7.1 of a Theorem of Kempf-Ness. We prove in greater generality than required for Th. 6.1, as we believe it will be useful in other contexts. It involves good reduction properties of homogeneous spaces of reductive groups over $\mathbf{Z}_p$, and of closed orbits in linear representations over $\mathbf{Z}_p$.

The ideas behind the convexity and slope estimates in §8 can be better understood in the context of Bruhat-Tits buildings as in [38]. The height functions which are central in our implementation of the Pila-Zannier strategy give examples of the type of functions studied in §8.

The Appendix A describes results about closed orbits of tuple in the adjoint representation of reductive groups in arbitrary characteristic. These are used in the proof of Prop. 5.7.

The Appendix B is used in §5.1.1 in the proof of Theorem 5.1.

2. Uniform integral Tate conjecture

In this section, we define in Def. 2.3 our main assumption in this paper, the 'uniform integral Tate conjecture' property. This is an extension of the conclusions of Faltings' theorem in the form given in Th. 4.7, to all Shimura varieties.

2.1. *Uniform integral Tate conjecture.* — In §2.1.1 and §2.1.2 we consider an abstract setting. In §2.1.4 we specialise it to the context of Shimura varieties.

2.1.1. Let $\mathrm{M} \leq \mathrm{G}$ be (connected) reductive algebraic groups over $\mathbf{Q}$. We identify G with its image by a faithful representation

$$\rho_{\mathrm{G}} : \mathrm{G} \to \mathrm{GL}(d).$$

Def. 2.1 and Theorem 1.4 will not depend on this choice. The Zariski closure in $\mathrm{GL}(d)_{\mathbf{Z}}$ of the algebraic groups M and G and $\mathrm{Z}_{\mathrm{G}}(\mathrm{M})$ define models over $\mathbf{Z}$. We write $\mathrm{G}_{\mathbf{F}_p}$ for the special fibre[3] and

$$\mathrm{G}(\mathbf{Z}_p) = \mathrm{G}(\mathbf{Q}_p) \cap \mathrm{GL}(d, \mathbf{Z}_p) \text{ and}$$

$$\mathrm{G}(\widehat{\mathbf{Z}}) = \prod_p \mathrm{G}(\mathbf{Z}_p) = \mathrm{G}(\mathbf{A}_f) \cap \mathrm{GL}(d, \widehat{\mathbf{Z}}).$$

[3] For almost all primes p the group $\mathrm{G}(\mathbf{Z}_p)$ is hyperspecial and $\mathrm{G}_{\mathbf{F}_p}$ is a connected reductive algebraic group over $\mathbf{F}_p$.

We also have a reduction map $G(\mathbf{Z}_p) \to G_{\mathbf{F}_p}(\mathbf{F}_p)$. These constructions apply to M and $Z_G(M)$ as well.

2.1.2. Let $U \leq M(\mathbf{A}_f)$ be a compact subgroup.

For every prime p, we define $U_p = M(\mathbf{Z}_p) \cap U$. We denote by $U(p)$ the image of U_p in $G(\mathbf{F}_p)$. We define $U_p{}^0 = U_p \cap H_p{}^0(\mathbf{Q}_p)$ where $H_p = \overline{U_p}^{Zar} \leq G_{\mathbf{Q}_p}$ is the Zariski closure as a $\mathbf{Q}_p$-algebraic subgroup, and $H_p{}^0$ is its neutral Zariski connected component.

Definition **2.1** *(Uniform integral Tate property). — We say that a compact subgroup* $U \leq M(\mathbf{A}_f)$ "satisfies the uniform integral Tate" property with respect to M, G and ρ_G *if:*

(1) For every p,

$$(\mathbf{1a}) \qquad Z_{G_{\mathbf{Q}_p}}(U_p) = Z_{G_{\mathbf{Q}_p}}\left(U_p{}^0\right) = Z_G(M)_{\mathbf{Q}_p}.$$

and

$$(\mathbf{1b}) \qquad \textit{the action of } U_p \textit{ on } \mathbf{Q}_p{}^d \textit{ is semisimple.}$$[4]

(This (1b) is equivalent to: H_p *is reductive.)*

(2) For every D*, there exists an integer* M(D) *such that for every* $p \geq M(D)$ *and every* $U' \leq U_p$ *of index* $[U_p : U'] \leq D$*, we have*

$$(\mathbf{2a}) \qquad Z_{G_{\mathbf{F}_p}}\left(U'(p)\right) = Z_{G_{\mathbf{F}_p}}(M_{\mathbf{F}_p})$$

and

$$(\mathbf{2b}) \qquad \textit{the action of } U'(p) \textit{ on } \overline{\mathbf{F}_p}^d \textit{ is semisimple.}$$

(When $p > d$, (2b) is equivalent to: the Nori group, defined below, of $U'(p)$ *is semisimple.)*

In our terminology, *integrality* refers to the second property over $\mathbf{F}_p$ on $U(p)$ and *uniformity* to the fact that the integer M(D) depends on D only.

2.1.3. *Remarks.* — We collect here some facts that will be used throughout this article.

(1) For p large enough, in terms of d, we can use Nori theory [32]. For a subgroup $U'(p) \leq G(\mathbf{F}_p)$, the group $U'(p)^\dagger$ generated by unipotent elements of $U'(p)$ is of the form $H(\mathbf{F}_p)^\dagger$ for an algebraic group $H \leq G_{\mathbf{F}_p}$ over $\mathbf{F}_p$. We call this H the **Nori group** of $U'(p)$. The property (2b) is then equivalent to the fact that H is a **reductive** group $H \leq G_{\mathbf{F}_p}$ over $\mathbf{F}_p$ (see [46, Th. 5.3]). We also note that $[H(\mathbf{F}_p) : H(\mathbf{F}_p)^\dagger]$ can be bounded in terms of dim(G) (see. [32, 3.6(v)]).

(2) If $U' \leq U$ has index $[U : U'] \leq p$, then $U'(p)^\dagger = U(p)^\dagger$.

[4] Some authors refer to this as completely reducible.

(3) This "uniform integral Tate" property does not depend [5] on the choice of a faithful representation ρ_G.
(4) The semisimplicity of the action over $\overline{\mathbf{F}_p}$ is equivalent to the semisimplicity over $\mathbf{F}_p$.(cf. [5, Alg. VIII, §12. N.8 Prop. 8 Cor 1 a)] with $K = \mathbf{F}_p$, $L = \overline{\mathbf{F}_p}$, $A = K[U'(p)]$, $M = \overline{\mathbf{F}_p}^d$.)
(5) The group $U \leq M(\mathbf{A}_f)$ "satisfies the uniform integral Tate" property with respect to M, G and ρ if and only if the subgroup $\prod_p U_p \leq U$ does so.
(6) Part (1) of Def. 2.1 is satisfied for U if and only it is satisfied for some subgroup of finite index in U. It follows from Lem. A.2 and Cor. A.3 that, for any subgroup $U' \leq U$, if Part (1) of Def. 2.1 is satisfied for U', then it is satisfied for U.
(7) Let $U' \leq U_p \leq M(\mathbf{Q}_p)$ be subgroups.
If Property (1a) of part (1) of Def. 2.1 is satisfied for U', then it is satisfied for U_p: we will have $U'^0 \leq U_p{}^0 \leq U_p \leq M_{\mathbf{Q}_p}$, and $Z_G(M) = Z_G(U'^0) \geq Z_G(U_p{}^0) \geq Z_G(U_p) \geq Z_G(M)$.
Assume that U' is of finite index in U_p. Then Property (1b) of part (1) of Def. 2.1 is satisfied for U' if and only if it is satisfied for U_p: if H'_p is the Zariski closure of U', the finite index property implies $H'^0_p = H^0_p$.
(8) If a compact subgroup $U \leq M(\mathbf{A}_f)$ *"satisfies the uniform integral Tate" property with respect to* M, G *and* ρ_G, then, for any $M \leq G' \leq G$, the group U satisfies the uniform integral Tate property *with respect to* M, G′ *and* ρ_G.
(9) For every p, let $V_p \leq U_p$ be a finite index subgroup, and assume that there exists $D' \in \mathbf{Z}_{\geq 1}$ such that

$$\forall p, \quad [U_p : V_p] \leq D'. \tag{3}$$

If (2) is satisfied for U, then (2) is satisfied for $\prod_p V_p$ with the function $D \mapsto M(D \cdot D')$.
In view of Remark 2.1.3 (3) we may, from now on, just say "satisfies the uniform integral Tate property" without referring to a particular faithful representation ρ_G.

We deduce from the above facts the following.

Lemma **2.2.** — *Let* $U'' \leq U \leq M(\widehat{\mathbf{Z}})$ *be such that* U'' *satisfies the uniform integral Tate property with respect to* M, G *and* ρ_G *and such that* U'' *is of finte index in* U. *Then* U *satisfies the uniform integral Tate" property with respect to* M, G *and* ρ_G.

2.1.4. We denote by (G, X) a Shimura datum, by $K \leq G(\mathbf{A}_f)$ a compact open subgroup, and by $S = Sh_K(G, X)$ the associated Shimura variety. Fix $x_0 \in X$ and let $M \leq G$ be the Mumford-Tate group of x_0. Let E be a field of finite type over $\mathbf{Q}$ such that $s_0 = [x_0, 1] \in S(E)$ (such an E always exists). We denote by $\rho_{x_0} : \mathrm{Gal}(\overline{E}/E) \to M(\mathbf{A}_f)$

[5] Indeed, Def. 2.1 does not involve ρ_G itself, but only the induced models of G and M. The algebraic groups $G_{\mathbf{Q}_p}$ and $M_{\mathbf{Q}_p}$ do not depend on the integral models, and two models, for almost all p, induce the same local models $G_{\mathbf{Z}_p}$ and $M_{\mathbf{Z}_p}$.

the representation associated to x_0 (see [40, §3.1, Def. 3.1]), and by $U \leq M(\mathbf{A}_f) \cap K$ its image.

The main hypothesis in Theorem 1.4 is the following.

Definition **2.3.** — *We say that* x_0 *"satisfies the uniform integral Tate conjecture" if* $U = \rho_{x_0}(\mathrm{Gal}(\overline{E}/E))$ *"satisfies the uniform integral Tate" property with respect to* M, G *in the sense of Def. 2.1.*

2.1.5. We will make use of the following terminology.

Definition **2.4.** — *We say that a subgroup* $U \leq M(\mathbf{A}_f)$ *satisfies the* ℓ*-independence property if it is of the form*

$$U = \prod_p U_p$$

with $U_p \leq M(\mathbf{Q}_p)$ *for every prime* p.

3. Proof of the main result

The structure of the proof of Th. 1.4 is essentially the same as in [40]. The main difference is that our hypothesis is the integral uniform Tate property instead of the "weakly adélic Mumford-Tate conjecture". Using the results of §§4, 5, we may follow the same proof as [40, §7] making the following changes.

3.1. In the step "reduction to the Hodge generic case" [40, 7.1.1] we make the following changes.

Since we work with geometric Hecke orbits $\mathcal{H}^g(x_0)$ instead of generalised Hecke orbits $\mathcal{H}(x_0)$, we use [40, Cor. 2.7] to remark that, with $\Sigma^g = \mathcal{H}^g([x_0, 1]) \cap Z$ the following set is a finite union

$$\Sigma'^g := \overset{-1}{\Psi}(\Sigma^g) = \mathcal{H}^g([x'_1, 1]) \cup \cdots \cup \mathcal{H}^g([x'_k, 1])$$

of geometric Hecke orbits in $Sh_{K \cap G'(\mathbf{A}_f)}(G', X')$. According to Prop. 4.1 and Prop. 4.2, each of the $[x'_1, 1], \ldots, [x'_k, 1]$ satisfy the uniform integral Tate conjecture (relative to M and G′).

We replace "On the other hand, the Mumford-Tate hypothesis [...]" by the observation that if a point of the geometric Hecke orbit $\mathcal{H}^g([x_0, 1])$ in $Sh(G, X)$ satisfies the uniform integral Tate conjecture (relative to M and G), then, by Prop. 4.2, each point of each of the geometric Hecke orbits $\mathcal{H}^g([x'_1, 1]), \ldots, \mathcal{H}^g([x'_k, 1])$ satisfy the uniform integral Tate conjecture (relative to M and G′).

3.2. In the step "reduction to the adjoint datum" [40, 7.1.2] we make the following changes.

Instead of "Using §3, the Mumford-Tate hypothesis will still be valid even [...]", we use Prop. 4.1.

Instead of "In view of § 7, the Mumford-Tate hypothesis [...]" we use Prop. 4.3.

3.3. In [40, 7.1.3], "Induction argument for factorable subvarieties", we make the following changes.

Instead of "As explained in § 7, the Mumford-Tate hypothesis [...]" we use Prop. 4.5.

3.4. The last change from [40, §§3.1–3.3, 7] is in [40, 7.2.3] where we use our Th. 5.1, instead of the lower bound on the size of Galois orbits [40, Th. 6.4].

We may apply Th. 5.1 to the Galois image U, because: the hypothesis on M^{ab} is satisfied for Galois images (cf. [40, Lem. §6.11]); the other hypotheses are satisfied by assumption. (In the case of Shimura data of abelian type, see §4.5.)

4. Functoriality of the Tate condition and independence condition

In this section, we verify that the conditions in Definition 2.1 and 2.4 are preserved by various natural operations. This is necessary to make simplifying assumptions in the proof of the main theorems (cf. [40, §7.1]). We also show that the conditions of 2.1 and 2.4 hold for all Shimura varieties of abelian type.

According to Remark 2.1.3, the Definition 2.1 does not depend on ρ_G. It follows from Definition 2.4, that the property that the Galois image satisfies the ℓ-independence property does not depend on G, nor on ρ_G.

4.1. *Invariance on the geometric Hecke orbit.* — We refer to [40, Def. 2.1] for the notion of geometric Hecke orbit $\mathcal{H}^g(x_0)$, the variety $W = G \cdot \phi_0$ and the notation $x_\phi = x_0 \circ \phi$ for $\phi \in W(\mathbf{Q})$.

Proposition **4.1.** — *Let $x_\phi \in \mathcal{H}^g(x_0)$ for some $\phi \in W(\mathbf{Q})$.*

If x_0 "satisfies the uniform integral Tate conjecture", then x_ϕ "satisfies the uniform integral Tate conjecture".

If the image U of ρ_{x_0} satisfies the ℓ-independence property in the sense of Def. 2.4, then the image U′ of ρ_{x_ϕ} satisfies the ℓ-independence property in the sense of Def. 2.4.

Proof. — Let $g \in G(\overline{\mathbf{Q}})$ be such that $\phi = g\phi_0 g^{-1}$, and let L be a number field such that $g \in G(L)$, and let $\mathbf{A}_{f,L} = \mathbf{A}_f \otimes_{\mathbf{Q}} L$ be the ring of adèles of the number field L. Denote by U′ the image of ρ_{x_ϕ}. According to [40, Prop. 3.4] we have

$$U' = \phi(U).$$

We prove the assertion about ℓ-independence. Assume $U = \prod_p U_p$. Since ϕ is defined over $\mathbf{Q}$, we have

$$U' = \prod_p \phi(U_p).$$

This proves the assertion about ℓ-independence.

We treat the semisimplicity over $\mathbf{Q}_p$ in Def. 2.1. Assume that the action of U_p is semisimple. Equivalently, the Zariski closure $\overline{U_p}^{Zar}$ is reductive. As ϕ is defined over $\mathbf{Q}$, the algebraic group $\overline{\phi(U_p)}^{Zar} = \phi(\overline{U_p}^{Zar})$ is reductive, or equivalently, the action of $U'_p = \phi(U_p)$ is semisimple.

We now treat the centraliser property of part (1) of Def. 2.1. For every prime p, let us choose an embedding $L \to \overline{\mathbf{Q}_p}$. We have $gmg^{-1} = \phi(m)$ for $m \in M(\overline{\mathbf{Q}_p})$, and thus $U'_p = gU_pg^{-1} \leq \phi(M)(\overline{\mathbf{Q}_p})$. We have

$$\begin{aligned}
(\mathbf{4}) \qquad Z_{G_{\overline{\mathbf{Q}_p}}}(U'_p) &= gZ_{G_{\overline{\mathbf{Q}_p}}}(U_p)g^{-1} \\
&= gZ_{G_{\overline{\mathbf{Q}_p}}}({U_p}^0)g^{-1} = gZ_G(M)_{\overline{\mathbf{Q}_p}}g^{-1} \\
&= Z_G\left(gMg^{-1}\right)_{\overline{\mathbf{Q}_p}} = Z_G\left(\phi(M)\right)_{\overline{\mathbf{Q}_p}}
\end{aligned}$$

As $\phi(M)$ is the Mumford-Tate group of x_ϕ we have proved (1) of Def. 2.1 for x_ϕ.

We now deal with part (2) of Def. 2.1.

Note that the component g_p of g as an adélic element is in $G(O_{L\otimes\mathbf{Q}_p})$ for p large enough. For p large enough, the group $\phi(M)(\mathbf{Z}_p) = gM(\mathbf{Z}_p)g^{-1}$ is hyperspecial and the reduction map

$$g \mapsto \overline{g} : \phi(M)(\overline{\mathbf{Z}_p}) \to \phi(M)(\overline{\mathbf{F}}_p)$$

is well defined.

Let $m_0 \in \mathbf{Z}_{\geq 1}$ be such that the above apply for $p \geq m_0$. Let D and M(D) be as in (2) of Def. 2.1, and let $V \leq U_p$ be a subgroup of index $[U_p : V] \leq D$, and denote by $V(p) \leq U(p)$ the corresponding subgroup as in (2) of Def. 2.1, and define $V' := gVg^{-1} \leq \phi(U)$ and $V'(p)$ accordingly. Then, for $p \geq M'(D) := \max\{m_0; M(D)\}$ we have

$$V'(p) = \overline{g_p}V(p)\overline{g_p}^{-1}$$

with $\overline{g}$ the reduction of g in $G(\kappa_L) \leq G(\overline{\mathbf{F}}_p)$, where κ_L is the residue field of L at a prime above p. The semisimplicity follows.

For $p \geq M'(D)$, we also have

$$\begin{aligned}
Z_{G_{\overline{\mathbf{F}_p}}}\left(V'(p)\right) &= \overline{g_p}Z_{G_{\overline{\mathbf{F}_p}}}\left(V(p)\right)\overline{g_p}^{-1} = \overline{g_p}Z_{G_{\overline{\mathbf{F}_p}}}(M)\overline{g_p}^{-1} \\
&= Z_{G_{\overline{\mathbf{F}_p}}}\left(\overline{g_p}M\overline{g_p}^{-1}\right) = Z_{G_{\overline{\mathbf{F}_p}}}\left(\phi(M)\right).
\end{aligned}$$

□

4.2. *Passing to a Shimura subdatum.*

Proposition **4.2.** — *Let* $\Psi : (G', X') \to (G, X)$ *be an injective morphism of Shimura data. Let* $x_\phi \in \mathcal{H}^g(x_0)$ *be such that there exists* x'_ϕ *such that* $x_\phi = \Psi \circ x'_\phi$.

If x_0 *"satisfies the uniform integral Tate conjecture", then* x'_ϕ *(and* x_ϕ*)"satisfy the uniform integral Tate conjecture".*

If the image U *of* ρ_{x_0} *satisfies the* ℓ*-independence property in the sense of Def. 2.4, then the image* U′ *of* $\rho_{x'_\phi}$ *satisfies the* ℓ*-independence property in the sense of Def. 2.4.*

Proof. — By Prop. 4.1 we may assume $x_\phi = x_0$. We identify G′ with its image in G. By [40, Prop. 3.4] we have

$$U = \Psi\big(U'\big) = U'$$

where we denote by U′ the image of $\rho'_{x'_0}$ in $G'(\mathbf{A}_f)$.

The semisimplicity of the action of U′ is automatic. It follows readily from the definitions and the remark that

$$Z_{G'_{\mathbf{Q}_p}}\big(U'_p\big) = Z_{G_{\mathbf{Q}_p}}\big(U'_p\big) \cap G'_{\mathbf{Q}_p} = Z_{G_{\mathbf{Q}_p}}(M) \cap G'_{\mathbf{Q}_p} = Z_{G'_{\mathbf{Q}_p}}(M).$$

and similarly for $\mathbf{F}_p$ for p big enough.

The last statement follows from

$$U' = \Psi\big(U'\big) = U = \prod_p U_p.$$

□

4.3. *Passing to quotients by central subgroups.*

Proposition **4.3.** — *Let* $F \subset Z(G)$ *(*$Z(G)$ *is the centre of* G*) be a subgroup and let* G′ *be the quotient* G/F. *Let* $\Psi : (G, X) \to (G', X')$ *be the morphism of Shimura data induced by the quotient* $G \longrightarrow G'$.

If x_0 *"satisfies the uniform integral Tate conjecture", then* $x'_0 = \Psi \circ x_0$ *"satisfies the uniform integral Tate conjecture".*

If the image U *of* ρ_{x_0} *satisfies the* ℓ*-independence property in the sense of Def. 2.4, then the image* U′ *of* $\rho_{x'_0}$ *satisfies the* ℓ*-independence property in the sense of Def. 2.4.*

Proof. — Arguments are similar. Firstly, by [40, Prop. 3.4] we have

$$U' = \Psi(U).$$

We use, remarking $\Psi(M)$ is the Mumford-Tate group of x'_0,

$$Z_{G^{ad}}\big(U'_p\big) = Z_G(U_p)/F = Z_G(M)/F = Z_{G^{ad}}\big(M/Z(G)\big) = Z_{G^{ad}}\big(\Psi(M)\big).$$

(For p big enough Ψ will be compatible with the integral models.) Let H'_p be the Zariski closure of U'_p. Observe that $\Psi(U_p) \leq U'_p \leq \Psi(H_p)$, and $\Psi(H_p) \leq G'_{\mathbf{Q}_p}$ is closed, and $\Psi(U_p) \leq \Psi(H_p)$ is Zariski dense. Thus $H'_p = \Psi(H_p)$. As H_p is reductive, so is H'_p. This proves the semisimplicity (1b).

Finally, if $U = \prod_p U_p$, then $U' = \prod_p \Psi(U_p)$. This proves the assertion about ℓ-independence.

For the semisimplicity property over $\mathbf{F}_p$, let $\rho : G \to GL(n)$ and $\rho' : G' \to GL(m)$ be faithful representations, and $G_{\mathbf{F}_p}$ and $G'_{\mathbf{F}_p}$ be the corresponding models over $\mathbf{F}_p$, which are reductive for $p \geq c_1(G, G')$. Let $U' \leq U(p)$ be of index $[U' : U(p)] \leq D$. By Def. 2.1, the representation $U(p) \to GL(n)_{\mathbf{F}_p}$ is a semisimple representation for $p \geq M(D)$. If furthermore $p \geq c_2(\rho)$, by [46, Th. 5.4 (ii)], the subgroup $U' \leq G$ is G-cr, for $p \geq c_2(\rho)$. If furthermore $p \geq c_3(\rho' \circ \Psi)$, by [46, Th. 5.4 (i)], the representation $\rho' \circ \Psi : U' \to GL(m)$ is semisimple. This proves that, for $p \geq M'(D) := \max\{M(D); c_1(G, G'); c_2(\rho); c_3(\rho' \circ \Psi)\}$, the representation $U' \to GL(m)$ is semisimple. This proves (2b) of Def. 2.1 with $p > M'(D)$. □

Remark **4.4.** — This proposition in particular shows that we can restrict ourselves to the case of Shimura varieties where G is semisimple of adjoint type (by taking $F = Z(G)$ in this proposition).

4.4. *Compatibility to products.*

Proposition **4.5.** — *Assume* G *to be of adjoint type and not simple. Let*

$$(G, X) = (G_1, X_1) \times (G_2, X_2)$$

be a decomposition of (G, X) *as a product. We denote* $\pi_1 : G \to G_1$ *and* $\pi_2 : G \to G_2$ *the projection maps.*

If x_0 *"satisfies the uniform integral Tate conjecture", then* $x_i = \pi_i \circ x_0$ *"satisfies the uniform integral Tate conjecture".*

If the image U *of* ρ_{x_0} *satisfies the* ℓ*-independence property in the sense of Def. 2.4, then the image* U_i *of* ρ_{x_i} *satisfies the* ℓ*-independence property in the sense of Def. 2.4.*

The proof is the same above. We recall that $\rho_{x_i} = \pi_i \circ \rho_{x_0}$ by [40, Prop. 3.4]. We also recall that the Mumford-Tate group of x_i is $M_i = \pi_i(M)$, and that

$$G_i \cap Z_G(M) = Z_{G_i}(M_i).$$

4.5. *Shimura varieties of Abelian type.*

Proposition **4.6.** — *All Shimura varieties of abelian type and adjoint type satisfy conditions of Def. 2.3 and Def. 2.4.*

More precisely, let (G, X) *be a Shimura datum of abelian type with* G *of adjoint type, and let* S *be an associated Shimura variety. Then for every* $s_0 = [x_0, 1] \in S$,

- *the point* x_0 *satisfies the uniform integral Tate conjecture,*
- *and there exists a field of finite type* E *over* $\mathbf{Q}$ *such that* $U = \rho_{x_0}(\mathrm{G}al(\overline{E}/E))$ *satisfies the* ℓ*-independence condition.*

Proof. — By definition of abelian type Shimura data ([53, §3.2], [10, Prop. 2.3.10]), there exists an isomorphism of Shimura data

$$(G, X) \simeq \left(G'^{ad}, X'^{ad}\right)$$

with (G', X') of Hodge type. Using Prop. 4.1 we may replace $s_0 = [x_0, 1]$ by any point of its geometric Hecke orbit, and assume x_0 belongs to the image of X' in $X \simeq X'^{ad}$: there exists $x'_0 \in X'$ such that $x_0 = {x'_0}^{ad}$.

According to §4.3, it is enough to prove the conclusion for x'_0 instead of x_0.

By definition of Hodge type data, there exists an injective morphism of Shimura data $(G', X') \to (\mathrm{GS}p(2g), \mathfrak{H}_g)$, the latter being the Shimura datum of the moduli space $\mathcal{A}_g$. Let τ_0 be the image of x'_0 in $\mathfrak{H}_g$.

According to §4.2, we may assume $(G, X) = (\mathrm{GS}p(2g), \mathfrak{H}_g)$ and $x_0 = \tau_0$. It then follows from Th. 4.9 and Cor. 4.11. □

4.6. *Uniform integral Faltings' theorem over fields of finite type.*

Theorem **4.7** *(Faltings). — Let* K *be a field of finite type over* $\mathbf{Q}$*, and let* A/K *be an abelian variety.*

Fix an algebraic closure $\overline{K}$ *of* K*. Denote*

$$T_A \approx \widehat{\mathbf{Z}}^{2\dim(A)}$$

the $\widehat{\mathbf{Z}}$*-linear Tate module, on which we have a continuous* $\widehat{\mathbf{Z}}$*-linear representation*

$$(\mathbf{5}) \qquad \rho = \rho_A : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{\widehat{\mathbf{Z}}}(T_A).$$

We assume that $\mathrm{End}(A/K) = \mathrm{End}(A/\overline{K})$ *and we let* $\mathrm{End}(A/K)$ *act on* T_A *and denote*

$$Z := \left\{ b \in \mathrm{End}_{\widehat{\mathbf{Z}}}(T_A) | \forall a \in \mathrm{End}(A/K), [b, a] = 0 \right\}$$

the $\widehat{\mathbf{Z}}$*-algebra which is the centraliser of* $\mathrm{End}(A/K)$ *in* $\mathrm{End}_{\widehat{\mathbf{Z}}}(T_A) \approx \mathrm{M}at(2\dim(A), \widehat{\mathbf{Z}})$.

We denote the image of ρ *by*

$$(\mathbf{6}) \qquad U := \rho\left(\mathrm{G}al(\overline{K}/K)\right).$$

Then, for every $d \in \mathbf{Z}_{\geq 1}$*, there exists some* $M(A, K, d) \in \mathbf{Z}_{\geq 1}$ *such that: for every open subgroup* $U' \leq U$ *of index at most* d*, we have*

$$\widehat{\mathbf{Z}}\left[U'\right] \leq Z$$

is an open subalgebra of index at most $M(A, K, d)$.

4.6.1. *The number field case.* — This statement follows from Faltings' theorems. In the case where K is a number field, we deduce it from [28, Th. 1, Cor. 1, Th. 2] as follows.

Proof of Th. 4.7 if K *is a number field.* — Consider $d \in \mathbf{Z}_{\geq 1}$, and let $\mathrm{M}'(\mathrm{A}, \mathrm{K}, d)$ be the M of [28, Th. 1], with their d our $d \cdot [\mathrm{K} : \mathbf{Q}]$. Consider U' as in the statement of Theorem 4.7, denote by $\Gamma := \overset{-1}{\rho_{\mathrm{A}}}(\mathrm{U}') \leq \mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$ its inverse image, and denote by $k = \overline{\mathrm{K}}^{\Gamma}/\mathrm{K}$ the corresponding finite field extension. For every prime ℓ, we denote by $\mathrm{U}'(\ell)$ be the image of U' in $\mathrm{E}_\ell := \mathrm{End}_{\mathbf{F}_\ell}(\mathrm{A}[\ell])$.

We note that (cf. [11, Th. 2.7])

$$\widehat{\mathbf{Z}}[\mathrm{U}'] = \prod_\ell \mathbf{Z}_\ell[\mathrm{U}'] \quad \text{in } \mathrm{End}_{\widehat{\mathbf{Z}}}(\mathrm{T}_{\mathrm{A}}).$$

We will prove the following.

(1) For every prime ℓ, we have

$$\mathrm{M}_\ell := \max_{[\mathrm{U}:\mathrm{U}'] \leq d} \left[\mathrm{Z} \otimes \mathbf{Z}_\ell : \mathbf{Z}_\ell[\mathrm{U}']\right] < +\infty.$$

(2) There exists $\mathrm{M}_{\mathrm{A}} \in \mathbf{Z}_{\geq 1}$ such that for every prime $\ell > \max\{\mathrm{M}_{\mathrm{A}}; \mathrm{M}'(\mathrm{A}, \mathrm{K}, d)\}$,

$$\left[\mathrm{Z} \otimes \mathbf{Z}_\ell : \mathbf{Z}_\ell[\mathrm{U}']\right] = 1.$$

The conclusion of Theorem 4.7 will then follow with

$$\mathrm{M}(\mathrm{A}, \mathrm{K}, d) := \prod_{\ell \leq \max\{\mathrm{M}_{\mathrm{A}}; \mathrm{M}'(\mathrm{A},\mathrm{K},d)\}} \mathrm{M}_\ell.$$

We prove (1). Fix a prime ℓ. We recall that $\mathrm{End}(\mathrm{A}/\overline{\mathrm{K}}) = \mathrm{End}(\mathrm{A}/\mathrm{K}) \leq \mathrm{End}(\mathrm{A}/k) \leq \mathrm{End}(\mathrm{A}/\overline{\mathrm{K}})$. By Faltings theorems, for A/K and A/k, we have

$$\mathbf{Q}_\ell[\mathrm{U}] = \mathbf{Q}_\ell\left[\mathrm{U}'\right] = \mathrm{Z} \otimes \mathbf{Q}_\ell$$

in $\mathrm{End}_{\mathbf{Q}_\ell}(\mathrm{V}_\ell)$, where $\mathrm{V}_\ell = \mathrm{T}_{\mathrm{A}} \otimes_{\widehat{\mathbf{Z}}} \mathbf{Q}_\ell$ is the $\mathbf{Q}_\ell$ Tate module of A.

It follows that

$$\mathbf{Z}_\ell\left[\mathrm{U}'\right] \leq \mathbf{Z}_\ell[\mathrm{U}] \leq \mathrm{Z} \otimes \mathbf{Z}_\ell$$

is of finite index.

We denote by U_ℓ and U'_ℓ the image of U and U' in $\mathrm{End}_{\mathbf{Q}_\ell}(\mathrm{V}_\ell)$. We have $[\mathrm{U}_\ell : \mathrm{U}'_\ell] \leq [\mathrm{U} : \mathrm{U}'] \leq d$ and

$$\mathbf{Z}_\ell[\mathrm{U}] = \mathbf{Z}_\ell[\mathrm{U}_\ell] \text{ and } \mathbf{Z}_\ell\left[\mathrm{U}'\right] = \mathbf{Z}_\ell\left[\mathrm{U}'_\ell\right].$$

By [45, Ch. III, §4.1, Prop. 8 and 9] and [44, Prop. 2], the group U_ℓ has at most finitely many open subgroup U'_ℓ of index at most d. We also have $[U_\ell : U'_\ell] \leq [U : U']$. Thus, as U' ranges through subgroups of U of index at most d, we have the finiteness

$$M_\ell := \max_{[U:U'] \leq d} \left[Z \otimes \mathbf{Z}_\ell : \mathbf{Z}_\ell[U']\right] < +\infty.$$

This implies (1).

We prove (2). We define

$$B := \widehat{\mathbf{Z}}\left[U'\right], \quad D := \mathrm{End}(A/K).$$

and denote the images of B, resp. D, in $E_\ell := \mathrm{End}_{\mathbf{F}_\ell}(A[\ell])$ by

$$B_\ell = \mathbf{F}_\ell\left[U'(\ell)\right] \text{ and } D_\ell.$$

When $\ell > M'(A, K, d)$:

- the subalgebra $B_\ell := \mathbf{F}_\ell[U'(\ell)] \leq E_\ell$ is semisimple, by [28, Th. 2],
- and its centraliser

$$C_\ell := Z_{E_\ell}(B_\ell) = \mathrm{End}_{U'(\ell)}\left(A[\ell]\right) = \mathrm{End}_{U'}\left(A[\ell]\right)$$

satisfies, by [28, Cor. 1],

$$C_\ell = \mathrm{End}(A/K) \otimes \mathbf{F}_\ell.$$

By the double centraliser theorem (cf. [5, VIII§5, Th. 3]), we have

$$B_\ell = Z_{E_\ell}(C_\ell).$$

Recall that D_ℓ is the image of End(A/K) in E_ℓ. Let $Z_\ell := Z_{E_\ell}(D_\ell)$ be its centraliser.

We claim that there exists M_A such that for $\ell > M_A$, the map

$$Z \otimes \mathbf{F}_\ell \to Z_{E_\ell}(D_\ell) \tag{7}$$

is an isomorphism.

Proof of the claim. — Let us recall a fact. For any subgroup $\Lambda \leq \mathbf{Z}^n \leq \mathbf{Q}^n$, the map $\Lambda \otimes \mathbf{F}_\ell \to \mathbf{Z}^n \otimes \mathbf{F}_\ell$ is injective if and only if $\ell \nmid [\Lambda \cdot \mathbf{Q} \cap \mathbf{Z}^n : \Lambda] < \infty$. This holds for $\ell \gg 0$, and, if Λ is primitive, for every ℓ.

We choose an embedding $\overline{K} \to \mathbf{C}$ and consider the Betti cohomology $H_1(A/\mathbf{C}; \mathbf{Z}) \approx \mathbf{Z}^{2g}$. We define $E_0 = \mathrm{End}_{\mathbf{Z}}(H_1(A/\mathbf{C}; \mathbf{Z})) \approx \mathbf{Z}^{(2g)^2}$, and $Z_0 = Z_{E_0}(\mathrm{End}(A/K))$.

Let us prove the injectivity of (7). We apply the recalled fact for $n = (2g)^2$ and $\Lambda = Z_0$: the map

$$Z_0 \otimes \mathbf{F}_\ell \to E_\ell.$$

is injective for $\ell \gg 0$ (actually, for every prime ℓ). We note that $Z \simeq Z_0 \otimes \widehat{\mathbf{Z}}$. Thus $Z \otimes \mathbf{F}_\ell \simeq Z_0 \otimes \mathbf{F}_\ell$, and the map (7) is injective for $\ell \gg 0$.

Let us prove the surjectivity of (7), for $\ell \gg 0$. As we proved that (7) is injective, it is enough to prove that

$$\forall \ell \gg 0, \quad \dim_{\mathbf{F}_\ell} Z_0 \otimes \mathbf{F}_\ell = \dim_{\mathbf{F}_\ell} Z_\ell. \tag{8}$$

We may write $D = \mathbf{Z} \cdot d_1 + \cdots + \mathbf{Z} \cdot d_i$. Denote by $\mathrm{ad}(d) : E_0 \to E_0$ the map $e \mapsto [d, e] = d \circ e - e \circ d$. We choose an isomorphism $E \simeq \mathbf{Z}^{(2g)^2}$, and, for $1 \leq j \leq i$, denote the coordinates of $ad(d_j)$ by

$$ad(d_j) = (\delta_{j,1}, \ldots, \delta_{j,(2g)^2}).$$

We denote by $W \leq E_0^\vee := \mathrm{Hom}(E_0, \mathbf{Z})$ the subgroup generated by $\{\delta_{j,h} | 1 \leq j \leq i, 1 \leq h \leq (2g)^2\}$. Then $Z_0 \leq E_0$ is the dual $\{d \in E_0 | \forall w \in W, w(d) = 0\}$ of W, and $Z_{E_\ell}(D_\ell) \leq E_\ell$ is the dual of the image W_ℓ of W in $E_\ell^\vee$. We have thus $\dim_{\mathbf{F}_\ell}(Z_0 \otimes \mathbf{F}_\ell) = (2g)^2 - \dim_{\mathbf{F}_\ell} W \otimes \mathbf{F}_\ell$. By the recalled fact, we have $\dim_{\mathbf{F}_\ell}(W_\ell) = \dim_{\mathbf{F}_\ell} W \otimes \mathbf{F}_\ell = \mathrm{rank}(W)$ for $\ell > M_A := [W \cdot \mathbf{Q} \cap E_0 : W]$. This implies (8). We have proved the claim. □

It follows that, when $\ell > \max\{M_A; M(A, K, d)\}$ the map

$$\widehat{\mathbf{Z}}[U'] \to Z \to Z_\ell = \mathbf{F}_\ell[U'(\ell)]$$

is surjective. By Nakayama's lemma [5, VIII§9.3 Cor. 2], one deduces that

$$\mathbf{Z}_\ell[U'] = Z \otimes \mathbf{Z}_\ell.$$

We have proven (2). This concludes our proof of Th. 4.7 if K is a number field. □

4.6.2. *General case.* — Because we lack a reference, we give a specialisation argument which reduces the theorem for general fields of finite over $\mathbf{Q}$ to the case of number fields.

In view of Def. 2.1 we will prove the following refinement of Th. 4.7.

Proposition **4.8.** — *The same conclusions holds if we replace* (6) *by*

$$U := \prod_p {U_p}^0 \tag{9}$$

with $U_p := \rho(\mathrm{Gal}(\overline{K}/K)) \cap GL_{\mathbf{Z}_p}(T_A \otimes \mathbf{Z}_p)$.

The proof of Prop. 4.8 will use the following results.

Theorem **4.9** *(Serre). — In the same situation, assume moreover that* K *is a number field. Then there exists a finite extension* L/K *such that Galois image* $U := \rho(\mathrm{Gal}(\overline{L}/L))$

(1) is such that U *satisfies the* ℓ*-independence condition in the sense of Def. 2.4, ([48, 136. Th. 1, p.34], [47, §3.1])*
(2) and such that the U_p *are Zariski connected, ([48, 133. p. 15;135. 2.2.3 p. 31], [27, 6.14, p 623])*
(3) and such that $\mathrm{U} \leq \mathrm{M}(\widehat{\mathbf{Z}})$ *where* M *is the Mumford-Tate group of* A*. (cf. [40, §3] and §4.5.)*

The assertions in Theorem 4.9 are found in the indicated references. We deduce Prop. 4.8.

Proof of Prop. 4.8. — Let $\eta = \mathrm{Spec}(\mathrm{K})$ and $\overline{\eta} = Spec(\overline{\mathrm{K}})$. Following [31, §1.2 and Cor. 1.5],[6] there exists a number field $\mathrm{F} \leq \overline{\mathrm{K}}$, and an abelian variety A_{F} over F such that

- We have an identification of Tate modules $\mathrm{T} := \mathrm{T}_{\mathrm{A}} \simeq \mathrm{T}_{\mathrm{A}_{\mathrm{F}}}$
- We have an identity (cf. [31, Cor. 1.5]

$$\mathrm{End}(\mathrm{A}/\mathrm{K}) \simeq \mathrm{End}(\mathrm{A}/\mathrm{F}) \simeq \mathrm{End}(\mathrm{A}/\overline{\mathrm{F}}) \tag{10}$$

  as subalgebras of $\mathrm{B} := \mathrm{End}_{\widehat{\mathbf{Z}}}(\mathrm{T})$,
- we have a diagram

$$\begin{array}{ccccc} \mathrm{G}al(\overline{\mathrm{K}}/\mathrm{K}) & \hookleftarrow & \mathrm{D}_{\mathrm{F}} & \twoheadrightarrow & \mathrm{G}al(\overline{\mathrm{F}}/\mathrm{F}) \\ \downarrow{\scriptstyle \rho} & & & & \downarrow{\scriptstyle \rho'} \\ \mathrm{End}(\mathrm{T}_{\mathrm{A}}) & = & \mathrm{End}(\mathrm{T}) & = & \mathrm{End}(\mathrm{T}_{\mathrm{A}_{\mathrm{F}}}). \end{array}$$

The commutativity implies that $\mathrm{U}_{\mathrm{F}} := \rho(\mathrm{D}_{\mathrm{F}})$ satisfies

$$\rho(\mathrm{D}_{\mathrm{F}}) = \mathrm{U}_{\mathrm{F}} := \rho'\big(\mathrm{G}al(\overline{\mathrm{F}}/\mathrm{F})\big)$$

and

$$\rho(\mathrm{D}_{\mathrm{F}}) \leq \mathrm{U} := \rho\big(\mathrm{G}al(\overline{\mathrm{K}}/\mathrm{K})\big).$$

By Th. 4.9, after possibly passing to a finite extension of F and the corresponding finite extension of K, we may assume

$$\mathrm{U}_{\mathrm{F}} = \prod_p (\mathrm{U}_{\mathrm{F}})_p{}^0.$$

We note that $(\mathrm{U}_{\mathrm{F}})_p \leq \mathrm{U}_p$ and thus $(\mathrm{U}_{\mathrm{F}})_p{}^0 \leq \mathrm{U}_p{}^0$. We deduce

$$\mathrm{U}_{\mathrm{F}} \leq \widetilde{\mathrm{U}} = \prod_p \mathrm{U}_p{}^0.$$

[6] We apply [31] with $\mathrm{F} := \mathbf{Q}$, and $\mathrm{F}(\mathrm{S}) := \mathrm{K}$. Our F is a finite extension of their $\mathbf{Q}(\sigma)$ such that $\mathrm{End}(\mathrm{A}_\sigma/\overline{\mathbf{Q}(\sigma)}) \simeq \mathrm{End}(\mathrm{A}_\sigma/\mathrm{F})$, and our A_{F} is the base change of their A_σ.

We will prove the refinement Prop. 4.8 of Th. 4.7 with

$$M(A, K, d) = M(A_F, F, d).$$

Fix d and an open subgroup $U' \leq \widetilde{U}$ of index at most d. We denote

$$U'_F = U' \cap U_F.$$

We first note that $U'_F \leq U_F$ is a subgroup of index at most d. We have, as $\widehat{\mathbf{Z}}$-subalgebras of $B := Mat(2\dim(A), \widehat{\mathbf{Z}})$,

$$\widehat{\mathbf{Z}}[U'_F] \leq \widehat{\mathbf{Z}}[U'].$$

From (10) we have

$$Z = Z_F := \left\{b \in \mathrm{End}_{\widehat{\mathbf{Z}}}(T_A) | \forall a \in \mathrm{End}(A_F/F), [b, a] = 0\right\}.$$

We use the number field case 4.6.1 of the theorem for A_F and d and U'_F and get

$$\left[Z_F : \widehat{\mathbf{Z}}[U'_F]\right] \leq M(A_F, F, d).$$

We note that $\widehat{\mathbf{Z}}[U'_F] \leq Z$ because $U \geq U'$ commutes with the action of End(A) (all the endomorphisms are rational over K).

Finally

$$\widehat{\mathbf{Z}}[U'_F] \leq \widehat{\mathbf{Z}}[U'] \leq Z$$

hence

$$\left[Z : \widehat{\mathbf{Z}}[U']\right] \leq \left[Z : \widehat{\mathbf{Z}}[U'_F]\right] = \left[Z_F : \widehat{\mathbf{Z}}[U'_F]\right] \leq M(A_F, F, d). \qquad \square$$

Corollary **4.10.** — *We consider the setting of Th. 4.7 and denote by g the dimension of* A. *Choose an isomorphism* $H_1(A; \mathbf{Z}) \simeq \mathbf{Z}^{2g}$ *and denote* $\sigma : GL(H_1(A; \mathbf{Z})) \to GL(2g)$ *the corresponding isomorphism. There exists* $c(A)$ *such the following holds.*

The subgroup $U \leq M(\widehat{\mathbf{Z}})$ *satisfies the uniform integral Tate property with respect to* M, $G = GL(H_1(A; \mathbf{Q}))$ *and* $\rho_G = \sigma : G \to GL(2g)$ *in the sense of Def. 2.1, with* $M(D) := \max\{c(A); M(A, K, D)\}$.

In the following, we denote $GSp(H_1(A; \mathbf{Q})) \approx GSp(2g)$ *the* $\mathbf{Q}$*-algebraic group of symplectic similitudes of the Riemann symplectic form on* $H_1(A; \mathbf{Q})$.

The subgroup $U \leq M(\widehat{\mathbf{Z}})$ *satisfies the uniform integral Tate property with respect to* M, $G' = GSp(H_1(A; \mathbf{Q}))$ *and* $\rho_{G'} = \rho_G \restriction_{G'}: G' \to GL(2g)$ *in the sense of Def. 2.1 with* $M(D) := \max\{c(A); M(A, K, D)\}$.

We only treat the case of $GL(2g)$, as the case of $GSp(2g)$ follows directly using Remark 2.1.3 (8).

Proof. — Thanks to Lemma 2.2, we may assume $U = \prod_p {U_p}^0$. We have then

$$\widehat{\mathbf{Z}}[U] = \prod_p \mathbf{Z}_p[{U_p}^0].$$

We use Prop. 4.8. Then

- for every p, the algebra $\mathbf{Q}_p[U_p] = \mathbf{Q}_p[{U_p}^0]$ is the commutant of $\mathrm{End}(A/K) \otimes \mathbf{Q}_p = Z_{\mathrm{End}_{\mathbf{Q}_p}(H^1(A;\mathbf{Q}_p))}(M)$. This implies (1a). Because $\mathrm{End}(A/K) \otimes \mathbf{Q}_p$ is a semisimple algebra, so is it commutant in $\mathrm{End}_{\mathbf{Q}_p}(H^1(A; \mathbf{Q}_p))$ and thus the action of U_p is semisimple.
- for every D, and every $p \geq M(A, K, D)$, and every $U' \leq U(p)$ of index at most D, we have $\mathbf{Z}_p[U'] = Z \otimes \mathbf{Z}_p$, with Z as in (7). Then the algebra $\mathbf{F}_p[U']$ is $Z \otimes \mathbf{F}_p$. For $p \gg 0$, depending only on $\mathrm{End}(A/K)$, the image $Z \otimes \mathbf{F}_p$ of Z is the commutant of $\mathrm{End}(A/K) \otimes \mathbf{F}_p$, by (7). If moreover $p \gg 0$, depending only on $\mathrm{End}(A/K)$, the algebra $\mathrm{End}(A/K) \otimes \mathbf{F}_p$ is semisimple, its action on $\mathrm{End}_{\mathbf{F}_p}(A[p])$ is semisimple, and the action of its centraliser $\mathbf{F}_p[U']$ is semisimple. □

We deduce the following, using the well-known relation between Galois representations on the Tate module, and Galois action on isogeny classes in the Siegel modular variety $\mathcal{A}_g$ (see [54]).

Corollary **4.11.** — *Let s_0 be a point in $\mathcal{A}_g = Sh_{GSp(2g,\widehat{\mathbf{Z}})}(\mathfrak{H}_g, GSp(2g))$. Then s_0 satisfies Def. 2.3.*

5. Polynomial Galois bounds

This section is at the heart of this paper. We obtain suitable lower bounds for Galois orbits of points in generalised Hecke orbits under much weaker assumptions than those made in [40] (in particular, as seen above, they are satisfied by all Shimura varieties of abelian type).

5.1. *Statement.* — We use the notations $\succcurlyeq$ and $\approx$ of [40, Def. 6.1] for polynomial domination and polynomial equivalence of functions. For the definition of $H_f(\phi)$ we refer to [40, App. B]. For the MT property we refer to [40, §5, Def. 6.1], and refer to [40, §6.4] for the fact that (1) is satisfied for Galois images.

Theorem **5.1.** — *Let $M \leq G$ be connected reductive $\mathbf{Q}$-groups. Let $U \leq M(\mathbf{A}_f)$ be a subgroup satisfying the following.*
(1) The image of U in M^{ab} is MT in M^{ab}.
(2) The group U satisfies the uniform integral Tate conjectures.

Denote by $\phi_0 : \mathrm{M} \to \mathrm{G}$ *the identity homomorphism and* $\mathrm{W} = \mathrm{G} \cdot \phi_0$ *its conjugacy class. Then as* ϕ *varies in* $\mathrm{W}(\mathbf{A}_f)$, *we have, for any compact open subgroup* $\mathrm{K} \leq \mathrm{G}(\mathbf{A}_f)$, *where* $\mathrm{K}_\mathrm{M} = \mathrm{K} \cap \mathrm{M}(\mathbf{A}_f)$,

$$(\mathbf{11}) \qquad \left[\phi(\mathrm{U}) : \phi(\mathrm{U}) \cap \mathrm{K}\right] \approx \left[\phi(\mathrm{K}_\mathrm{M}) : \phi(\mathrm{K}_\mathrm{M}) \cap \mathrm{K}\right] \succcurlyeq \mathrm{H}_f(\phi).$$

as functions $\mathrm{W}(\mathbf{A}_f) \to \mathbf{Z}_{\geq 1}$.

We note that, by [40, Th. C1], we have

$$\left[\phi(\mathrm{U}) : \phi(\mathrm{U}) \cap \mathrm{K}\right] \leq \left[\phi(\mathrm{K}_\mathrm{M}) : \phi(\mathrm{K}_\mathrm{M}) \cap \mathrm{K}\right] \preccurlyeq \mathrm{H}_f(\phi).$$

In (11) is thus enough to prove

$$(\mathbf{12}) \qquad \left[\phi(\mathrm{U}) : \phi(\mathrm{U}) \cap \mathrm{K}\right] \succcurlyeq \mathrm{H}_f(\phi).$$

5.1.1. *Reduction to the case* $\mathrm{U} = \prod_p \mathrm{U}_p$. — Let $\mathrm{U}' := \prod_p (\mathrm{U} \cap \mathrm{M}^{der}(\mathbf{Z}_p)) \cdot (\mathrm{U} \cap \mathrm{Z}(\mathrm{M})(\mathbf{Z}_p))$. As $\mathrm{U} \geq \mathrm{U}'$, we have that $[\phi(\mathrm{U}) : \phi(\mathrm{U}) \cap \mathrm{K}] \geq [\phi(\mathrm{U}') : \phi(\mathrm{U}') \cap \mathrm{K}]$ and thus it is enough to prove (12) with U′ instead of U.

Let W and W′ be the image of U and U′ in $\mathrm{M}^{ab}(\widehat{\mathbf{Z}})$. The assumption (1) of Th. 5.1 implies that there exists $f \in \mathbf{Z}_{\geq 1}$ such that $\prod_p \mathrm{W}_p \geq \{u^f | u \in \mathrm{M}^{ab}(\widehat{\mathbf{Z}})\} \geq \{w^f | w \in \mathrm{W}\}$. Thus W satisfies (100), and by Cor. B.2 we may apply Cor. B.11 to U′ and thus U′ satisfies the assumption (2) of Theorem 5.1. From Cor B.2 we deduce $\forall w \in \mathrm{W}_p$, $w^e \in \mathrm{W}'_p := \mathrm{W}' \cap \mathrm{M}^{ab}(\mathbf{Q}_p)$. By Lem. B.10 we have $[\mathrm{M}^{ab}(\mathbf{Z}_p) : \mathrm{W}'_p] \leq [\mathrm{M}^{ab}(\mathbf{Z}_p) : \mathrm{W}_p] \cdot k(n, e)$. Thus U′ satisfies the assumption (1) of Theorem 5.1.

We may thus substitute U with U′ in Theorem 5.1 and thus assume that $\mathrm{U} = \prod_p \mathrm{U}_p$.

5.1.2. *Reduction to the case* $\mathrm{U}_p = \mathrm{U}_p^0$. — Let $\mathrm{V} = \prod_p \mathrm{U}_p^0$, where U_p^0 is the neutral component of U_p for the Zariski topology of $\mathrm{M}(\mathbf{Q}_p)$. By Proposition 5.2 below, there exists $\mathrm{D} \in \mathbf{Z}_{\geq 1}$ such that $\forall p$, $[\mathrm{U}_p : \mathrm{U}_p^0] \leq \mathrm{D}$.

It follows that the assumption (1) of Theorem 5.1 is satisfied for U = V.

By remarks (7) and (9) of §2.1.3, it follows that the assumption (2) of Theorem 5.1 is satisfied for U = V.

As $\mathrm{V} \leq \mathrm{U}$, we have $[\phi(\mathrm{U}) : \phi(\mathrm{U}) \cap \mathrm{K}] \geq [\phi(\mathrm{V}) : \phi(\mathrm{V}) \cap \mathrm{K}]$ and thus it is enough to prove (12) with V instead of U.

It follows that, in proving Theorem 5.1, we may assume

(**13**) For every prime p, U_p is Zariski connected.

Proposition **5.2.** — *Let* U, M *and* G *be as in Definition 2.1.*
Assume moreover that the image of U *in* M^{ab} *is MT in* M^{ab} *as in Th.5.1 (1).*
Define $\mathrm{H}_p := \overline{\mathrm{U}_p}^{Zar} \leq \mathrm{M}_{\mathbf{Q}_p}$.
Then:

- *we have* $Z(M)^0_{\mathbf{Q}_p} \leq H_p$ *for every* p,
- *and we have*

$$\sup_p \#\pi_0(H_p) < +\infty.$$

Proof. — By Definition 2.1, the algebraic group H_p^0 is reductive. We write

$$H_p^0 = S_p \cdot T_p$$

where S_p, the derived subgroup of ${H_p}^0$, is semisimple, and $T_p = Z(H_p^0)^0$, the neutral component of the centre of H_p^0, is a torus.

We have $T_p = Z(H_p^0)^0 \leq Z_{M_{\mathbf{Q}_p}}(H_p^0) = Z_{M_{\mathbf{Q}_p}}(U_p^0) = Z(M_{\mathbf{Q}_p})$, by Definition 2.1.

Denote by $ab_M : M \to M^{ab}$ the abelianisation map. As S_p is semisimple, we have $S_p \leq \ker(ab_M)$. Thus $ab_M(H_p^0) = ab_M(S_p \cdot T_p) = ab_M(T_p)$.

Since the image of U in M^{ab} is MT in M^{ab}, the subgroup $ab_M(U_p^0) \leq M^{ab}(\mathbf{Q}_p)$ is open in the p-adic topology. It follows that $ab_M(H_p^0) \leq M^{ab}$ is open in the Zariski topology.

Thus ${M^{ab}}^0 \leq ab_M(H_p^0) = ab_M(T_p)$.

As M is reductive, the map $Z(M) \to M^{ab}$ is an isogeny. As $ab_M(T_p)$ is of finite index in M^{ab}, the algebraic subgroup $T_p \leq Z(M_{\mathbf{Q}_p})$ is of finite index. That is, $Z(M_{\mathbf{Q}_p})^0 \leq T_p \leq H_p$.

This proves the first claim.

Denote by $N := N_{M_{\mathbf{Q}_p}}(H_p^0)$ the normaliser. The adjoint action induces an homomorphism

$$H_p \to N \to Aut\big(H_p^0\big).$$

As H_p^0 is reductive, we have $N^0 = H_p^0 \cdot Z_{M_{\mathbf{Q}_p}}(H_p)^0$. The first claim implies $H_p^0 = N^0$.

We thus have an inclusion $\pi_0(H_p) = H_p/H_p^0 \to N/N^0 = \pi_0(N)$. Let K and I denote the kernel and the image of the map

$$N/H_p^0 \to Out\big(H_p^0\big).$$

A coset $kH_p^0 \in N/H_p^0$ is in K if and only if there exists $h \in H_p^0$ such that $k \cdot h \in Z_{M_{\mathbf{Q}_p}}(H_p^0)$. But $Z_{M_{\mathbf{Q}_p}}(H_p^0) = Z(M_{\mathbf{Q}_p})$ and, by the first claim, $H_p^0 \geq Z(M_{\mathbf{Q}_p})^0$. Thus the epimorphism $Z(M_{\mathbf{Q}_p}) \to Z(M_{\mathbf{Q}_p})H_p^0/H_p^0 = K$ factors through $Z(M_{\mathbf{Q}_p})/Z(M_{\mathbf{Q}_p})^0$. Thus $\#K \leq \#\pi_0(Z(M_{\mathbf{Q}_p}))$. For $p \gg 0$, the number $\#\pi_0(Z(M_{\mathbf{Q}_p}))$ does not depend on p.

As S_p is semisimple, we have $\#Out(S_p) < +\infty$. As there are only finitely many conjugacy classes of semisimple groups in $M_{\mathbf{C}}$, there exists C(M) such that, for all p, we have $\#Out(S_p) < C(M)$.

For $t \in T_p$, we have $t \in Z(M_{\mathbf{Q}_p})^0$, by the first claim. Thus, for $m \in N$, we have $m \in Z_M(T_p)$. This implies that the image of N in $Out(T_p) = Aut(T_p)$ is trivial. Thus the map $Out(H_p^0) \to Out(S_p)$ is injective on I.

Thus $\#I \leq C(M)$.
The second claim follows. □

5.1.3. *Reduction to a local problem.* — From [40, Th. B.1, Cor. B.2] we already have

$$\left[\phi(K_M):\phi(K_M)\cap K\right] \succcurlyeq H_f(\phi),$$

and because $U \leq K_M$, we have

$$\left[\phi(U):\phi(U)\cap K\right] \leq \left[\phi(K_M):\phi(K_M)\cap K\right].$$

By [40, Th. C1], we have

$$\left[\phi(K_M):\phi(K_M)\cap K\right] \preccurlyeq H_f(\phi).$$

Thus we have $[\phi(K_M):\phi(K_M)\cap K] \preccurlyeq H_f(\phi)$, and (12) is equivalent to

$$\left[\phi(U):\phi(U)\cap K\right] \succcurlyeq \left[\phi\big(M(\widehat{\mathbf{Z}})\big):\phi\big(M(\widehat{\mathbf{Z}})\big)\cap K\right]. \tag{14}$$

Since they are commensurable groups, we may replace K by $G(\widehat{\mathbf{Z}})$.
By §5.1.1, we have $U = \prod_p U_p$.
Then the required inequality (14) can be rewritten in the product form

$$\prod_p \left[\phi(U_p):\phi(U_p)\cap G(\mathbf{Z}_p)\right] \succcurlyeq \prod_p \left[\phi\big(M(\mathbf{Z}_p)\big):\phi\big(M(\mathbf{Z}_p)\big)\cap G(\mathbf{Z}_p)\right]$$

and thus the problem can be studied prime by prime.

More precisely, it will be enough to prove

- that there exists $c \in \mathbf{R}_{>0}$ such that, for almost all primes

$$\forall \phi \in W(\mathbf{Q}_p),\quad \left[\phi(U_p):\phi(U_p)\cap G(\mathbf{Z}_p)\right] \geq \left[\phi\big(M(\mathbf{Z}_p)\big):\phi\big(M(\mathbf{Z}_p)\big)\cap G(\mathbf{Z}_p)\right]^c. \tag{15}$$

- and, for the finitely remaining primes, that we have the polynomial domination, as functions $W(\mathbf{Q}_p) \to \mathbf{R}_{\geq 0}$,

$$\left[\phi(U_p):\phi(U_p)\cap G(\mathbf{Z}_p)\right] \succcurlyeq \left[\phi\big(M(\mathbf{Z}_p)\big):\phi\big(M(\mathbf{Z}_p)\big)\cap G(\mathbf{Z}_p)\right].$$

  Namely, there exist $a(p), c(p) \in \mathbf{R}_{>0}$ such that

$$\forall \phi \in W(\mathbf{Q}_p),\quad \left[\phi(U_p):\phi(U_p)\cap G(\mathbf{Z}_p)\right] \geq a(p)\cdot\left[\phi\big(M(\mathbf{Z}_p)\big):\phi\big(M(\mathbf{Z}_p)\big)\cap G(\mathbf{Z}_p)\right]^{c(p)}. \tag{16}$$

It will be sufficient, instead of (15), to prove: there exist $a, c \in \mathbf{R}_{>0}$ such that, for almost all primes

$$(\mathbf{17}) \qquad \forall \phi \in W(\mathbf{Q}_p), \quad \left[\phi(U_p) : \phi(U_p) \cap G(\mathbf{Z}_p)\right] \geq a \cdot \left[\phi\big(M(\mathbf{Z}_p)\big) : \phi\big(M(\mathbf{Z}_p)\big) \cap G(\mathbf{Z}_p)\right]^c.$$

Proof that (17) and (16) imply (14). — By [40, Th. B4], we may substitute $[\phi(M(\mathbf{Z}_p)) : \phi(M(\mathbf{Z}_p)) \cap G(\mathbf{Z}_p)]$ with $H_p(\phi)$, possibly changing a and c. Using the following Lemma with $n_p = [\phi(U_p) : \phi(U_p) \cap G(\mathbf{Z}_p)]$ and $h_p = H_p(\phi)$, we deduce that (12) (and thus (14)) is a consequence of (17) and (16). □

Lemma **5.3.** — *Let $h \in \mathbf{Z}_{\geq 1}$ and let $h = \prod_p h_p$ be its primary factorisation with $h_p \in p^{\mathbf{Z}_{\geq 1}}$ for every prime p. Let $(n_p)_p$ be a sequence in $\mathbf{Z}_{\geq 1}$ indexed by primes p. Let $a, b \in \mathbf{R}_{>0}$ be such that*

$$\forall p, \quad n_p \geq a \cdot h_p^b.$$

Then there exists $\alpha(a, b) \in \mathbf{R}_{>0}$, depending only on a and b, such that

$$\prod_p n_p \geq \alpha(a, b) \cdot h^{b/2}.$$

Proof. — For p such that $h_p = 1$, we have

$$n_p \geq 1 = h_p = h_p^{b/2}.$$

If $p^{b/2} \leq 1/a$, we have

$$n_p \geq a \cdot h_p^b \geq a \cdot h_p^{b/2}.$$

If $p^{b/2} \geq 1/a$ and $h_p \neq 1$, we have $h_p \geq p$, and $a \cdot h_p^b \geq h_p^{b/2}$, and

$$n_p \geq a \cdot h_p^b \geq h_p^{b/2}.$$

Let $d(a, b)$ be the number of primes p such that $p^{b/2} \leq 1/a$. Multiplying the above inequalities prime by prime we deduce

$$\prod_p n_p \geq a^{d(a,b)} \prod_p h_p^{b/2} = h^{b/2}.$$

□

By the previous discussion, we are reduced to proving the following.

Theorem **5.4** *(Local Galois bounds).* — *In the setting of Th. 5.1, there exist $a, c \in \mathbf{R}_{>0}$, and for each p, there exists $b(p) \in \mathbf{R}_{>0}$ such that*

$$\left[\phi(U_p) : \phi(U_p) \cap K_p\right] \geq b(p) \cdot \left[\phi\big(M(\mathbf{Z}_p)\big) : \phi\big(M(\mathbf{Z}_p)\big) \cap K_p\right]^c$$

and such that $b(p) \geq a$ for almost all p.

We prove (16) in 5.2. It is deduced from the functoriality of heights.

We prove, for almost all primes, (15) in 5.3. It requires new tools developed in this article.

For reference, we rephrase "The image of U in M^{ab} is MT in M^{ab}" as follows. We denote $ab_M : M \to M^{ab} := M/M^{der}$ the abelianisation map. Denote by $M^{ab}(\mathbf{Z}_p)$ the maximal compact subgroup of the torus M^{ab}. Then there exists $C_{MT} \in \mathbf{Z}_{\geq 1}$ such that

$$\forall p, \quad \left[M^{ab}(\mathbf{Z}_p) : ab_M(U_p)\right] \leq C_{MT}. \tag{18}$$

Because $\exp(p\mathfrak{m}^{ab}_{\mathbf{Z}_p})$ is a pro-p-group, its action on $M^{ab}(\mathbf{Z}_p)/ab_M(U_p)$ is trivial when $p > C_{MT}$: we have

$$\forall p > C_{MT}, \quad \exp\left(p\mathfrak{m}^{ab}_{\mathbf{Z}_p}\right) \leq ab_M(U_p). \tag{19}$$

5.2. *For every prime.* — We fix a prime p. Let $f_1, \ldots, f_k$ be a basis of the $\mathbf{Q}_p$-Lie algebra $\mathfrak{u}_p$ of U_p. Replacing each f_i by a sufficiently small scalar multiple, we may assume that each $u_i = \exp(f_i)$ converges and belongs to U_p. By (2) of Th. 5.1 and (1a) of Def. 2.1, we have[7]

$$Z_{G_{\mathbf{Q}_p}}(U_p) = Z_{G_{\mathbf{Q}_p}}\left({U_p}^0\right) = Z_{G_{\mathbf{Q}_p}}(\mathfrak{u}_p) = Z_{G_{\mathbf{Q}_p}}\left(\{f_1, \ldots, f_k\}\right).$$

Let $\mathfrak{g}$ be the Lie $\mathbf{Q}_p$-algebra of $G_{\mathbf{Q}_p}$. We define, for some faithful linear representation $\rho : G \to GL(V)$ defined over $\mathbf{Q}$,

$$v = (f_1, \ldots, f_k) \in \mathfrak{g}^k \overset{d\rho}{\hookrightarrow} E := \mathrm{End}(V_{\mathbf{Q}_p})^k.$$

For the induced action of G on E we have

$$Z_{G_{\mathbf{Q}_p}}(U_p) = Stab_{G_{\mathbf{Q}_p}}(v).$$

By our assumption,

$$Z_{G_{\mathbf{Q}_p}}(U_p) = Stab_{G_{\mathbf{Q}_p}}(\phi_0) = Z_G(M)_{\mathbf{Q}_p}.$$

As a consequence, we have a well defined isomorphism $W \to G \cdot v$ defined over $\mathbf{Q}_p$ of homogeneous varieties, given by

$$g \cdot Z_G(M)_{\mathbf{Q}_p} \mapsto g \cdot v$$

---

[7] We observe the following facts. For a compact subgroup $U \leq GL(n, \mathbf{Q}_p)$, any open subgroup $V \leq U$ is of finite index. If U is connected for the Zariski topology, then V is Zariski dense in U. By construction, we have $\mathfrak{u} \subseteq \mathbf{Q}_p[V]$ in $\mathfrak{gl}(n, \mathbf{Q}_p)$, and, with $V \leq U$ generated by $U \cap \exp(\mathfrak{u} \cap p^2\mathfrak{gl}(\mathfrak{n}, \mathbf{Z}_\mathfrak{p}))$, we have $V \subseteq \mathbf{Q}_p[\mathfrak{u}]$. From these facts, it follows that

$$Z_{GL(n)_{\mathbf{Q}_p}}(\mathfrak{u}) = Z_{GL(n)_{\mathbf{Q}_p}}\left(\mathbf{Q}_p[\mathfrak{u}]\right) = Z_{GL(n)_{\mathbf{Q}_p}}\left(\mathbf{Q}_p[V]\right) = Z_{GL(n)_{\mathbf{Q}_p}}(V) = Z_{GL(n)_{\mathbf{Q}_p}}\left(U^0\right).$$

From (1) of Def. 2.1, the Zariski closure $\overline{\mathrm{U}_p}^{Zar}$ is reductive. We may thus apply [42, Th. 3.6], and deduce that the induced map

$$\iota : \mathrm{W} \to \mathrm{E}$$

is a closed affine embedding.

We use the standard norm on $\mathrm{E} \simeq \mathbf{Q}_p^{\dim(\mathrm{V})^2 \cdot k}$. We denote by H_ι the local Weil height associated to this embedding, which is given by

$$\begin{aligned} \mathrm{H}_\iota : \phi \mapsto \mathrm{H}_p(g \cdot v) &:= \max\left\{1; \|g \cdot v\|\right\} \\ &= \max\left\{1; \|g \cdot f_1\|; \ldots; \|g \cdot f_k\|\right\}. \end{aligned} \tag{20}$$

By functoriality properties of height functions, the function H_ι and $\phi \mapsto \mathrm{H}_p(\phi)$ are polynomially equivalent.

Namely, there are $a(p)$ and $c(p)$ such that

$$\mathrm{H}_\iota \geq a(p) \cdot \mathrm{H}_p(\phi)^{c(p)}.$$

We denote $\mathrm{U}' \leq \mathrm{U}_p$ the p-adic Lie subgroup generated by

$$\left\{\exp(f_1); \ldots; \exp(f_k)\right\}.$$

We have

$$\left[\phi(\mathrm{U}_p) : \phi(\mathrm{U}_p) \cap \mathrm{G}(\mathbf{Z}_p)\right] \geq \left[\phi\left(\mathrm{U}'\right) : \phi\left(\mathrm{U}'\right) \cap \mathrm{G}(\mathbf{Z}_p)\right].$$

Using [40, Th. A3], we also have, with $\mathrm{U}_i := \exp(f_i)^{\mathbf{Z}} \leq \mathrm{U}'$, and denoting by $\|\ \| : \mathrm{End}(\mathrm{V}_{\mathbf{Q}_p}) \simeq \mathbf{Q}_p^{\dim(\mathrm{V})^2} \to \mathbf{R}_{\geq 0}$ the standard norm,

$$\begin{aligned} \left[\phi\left(\mathrm{U}'\right) : \phi\left(\mathrm{U}'\right) \cap \mathrm{K}_p\right] &\geq \max_{1 \leq i \leq k}\left[\phi(\mathrm{U}_i) : \phi(\mathrm{U}_i) \cap \mathrm{K}_p\right] \\ &\geq \max_{1 \leq i \leq k} \max\left\{1; \left\|d\phi(f_i)\right\|\right\} = \mathrm{H}_\iota(\phi)/\dim(\mathrm{V}). \end{aligned} \tag{21}$$

By [40, App. Th. C2 (102)], we have $[\phi(\mathrm{M}(\mathbf{Z}_p)) : \phi(\mathrm{M}(\mathbf{Z}_p)) \cap \mathrm{K}_p] \preccurlyeq \mathrm{H}_\iota(\phi)$. We deduce (16).

5.2.1. *Remark.* — Note that the above bound already implies the André-Pink-Zannier conjecture for S-Hecke orbits. This is more general than the result of Orr ([33]) for Shimura varieties of abelian type and less precise than [39] which proves a strong topological form under a weaker hypothesis. The method of Orr relies on Masser-Wüstholz bounds, and [39] relies ultimately on S-adic Ratner theorems through the work of [41].

5.3. *For almost all primes.*

5.3.1. *Construction of tuples.* — We denote by

$$\mathrm{Y} \mapsto \overline{\mathrm{Y}} : \mathfrak{m}_{\mathbf{Z}_p} \to \mathfrak{m}_{\mathbf{F}_p} \text{ and } \pi_p : \mathrm{U}_p \to \mathrm{M}_{\mathbf{F}_p}(\mathbf{F}_p)$$

the reduction modulo p maps, and define

$$\mathrm{U}(p) := \pi_p(\mathrm{U}_p)$$

the image of U_p in $\mathrm{M}_{\mathbf{F}_p}(\mathbf{F}_p)$. We denote the subgroup of $\mathrm{U}(p)$ generated by its unipotent elements by

$$\mathrm{U}(p)^{\dagger} \tag{22}$$

and its inverse image in U_p by

$$\mathrm{U}_p^{\dagger} := \overset{-1}{\pi_p}(\mathrm{U}_p).$$

Let $\mathfrak{m}^{der}$ and $\mathfrak{z}(\mathfrak{m})$ be the derived and centre subalgebras of $\mathfrak{m}$, and define $\mathfrak{m}^{der}_{\mathbf{Z}_p} = \mathfrak{m}^{der} \cap \mathfrak{m}_{\mathbf{Z}_p}$ and $\mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p} = \mathfrak{z}(\mathfrak{m}) \cap \mathfrak{m}_{\mathbf{Z}_p}$. We define

$$\nu = \mathfrak{m}^{der}_{\mathbf{Z}_p} + p \cdot \mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p}. \tag{23}$$

Proposition **5.5.** — *We consider the setting of Theorems 5.4 and 5.1. For almost all p, there exists*

$$\mathrm{X}_1, \ldots, \mathrm{X}_k, \mathrm{Y}_1, \ldots, \mathrm{Y}_l \in \mathfrak{m}_{\mathbf{Z}_p}$$

satisfying the following

(1) The exponentials $\exp(\mathrm{X}_1), \ldots, \exp(\mathrm{X}_k)$ *converge and topologically generate* $\mathrm{U}_p^{\dagger}$.

(2) We have

$$\mathrm{Y}_1, \ldots, \mathrm{Y}_l \in \mathfrak{u} := \mathbf{Z}_p \cdot \mathrm{X}_1 + \cdots + \mathbf{Z}_p \cdot \mathrm{X}_k.$$

(3) We have

$$\frac{1}{p} \cdot \mathrm{Y}_1 \cdot \mathbf{Z}_p + \cdots + \frac{1}{p} \cdot \mathrm{Y}_l \cdot \mathbf{Z}_p = \mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p} \pmod{p \cdot \mathfrak{m}_{\mathbf{Z}_p}}.$$

(4) We have

$$\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}(\mathrm{M}_{\mathbf{F}_p}) = \mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}\left(\left\{\overline{\mathrm{X}_1}; \ldots; \overline{\mathrm{X}_k}; \overline{\frac{1}{p}\mathrm{Y}_1}; \ldots; \overline{\frac{1}{p}\mathrm{Y}_l}\right\}\right). \tag{24}$$

Proof. — Let $u_1, \ldots, u_i$ be unipotent generators of $\mathrm{U}(p)^{\dagger}$. Because $\mathrm{U}(p)^{\dagger} \leq \mathrm{U}(p) = \pi_p(\mathrm{U}_p)$, we may write

$$u_1 = \pi_p(x_1), \ldots, u_i = \pi_p(x_i)$$

with $x_1, \ldots, x_i \in \mathrm{U}_p$. By definition of $\mathrm{U}_p^\dagger$, we have $x_1, \ldots, x_i \in \mathrm{U}_p^\dagger$.

The compact group $\ker(\pi_p) \leq \mathrm{U}_p \leq \mathrm{M}(\mathbf{Z}_p)$ is topologically of finite type. We choose $x_{i+1}, \ldots, x_k$ a topologically generating family of $\ker(\pi_p)$. By construction

$$x_1, \ldots, x_k \text{ topologically generate } \mathrm{U}_p^\dagger. \tag{25}$$

Moreover, the $\pi_p(x_i)$ are unipotent. By [40, Prop. A.1], the series $\mathrm{X}_1 = \log(x_1), \ldots, \mathrm{X}_k = \log(x_k)$ converge, and, for $p > d$, we have $\mathrm{X}_1, \ldots, \mathrm{X}_k \in \mathfrak{m}_{\mathbf{Z}_p}$. Using [43, §6.1.5] (cf. [40, Proof of Lem. A.2]), we can deduce, for $p > d + 1$, that

$$x_1 = \exp(\mathrm{X}_1), \ldots, x_k = \exp(\mathrm{X}_k). \tag{26}$$

We define

$$\mathfrak{u} = \mathbf{Z}_p \cdot \mathrm{X}_1 + \cdots + \mathbf{Z}_p \cdot \mathrm{X}_k \qquad u = (\mathrm{X}_1, \ldots, \mathrm{X}_k). \tag{27}$$

For $\mathrm{X} \in \mathfrak{m}_{\mathbf{Z}_p} \smallsetminus \nu$, its reduction in $\mathfrak{m}_{\mathbf{F}_p}$ is not nilpotent. Thus $\mathrm{X}^n/n!$ does not converge to 0, and the series $\exp(\mathrm{X})$ does not converge. Consequently we have

$$\mathrm{X}_1, \ldots, \mathrm{X}_k \in \nu \text{ and thus } \mathfrak{u} \leq \nu.$$

We define

$$\pi_\nu : \nu \to \overline{\nu} := \nu \otimes \mathbf{F}_p,$$

and denote the image of $\mathfrak{u}$ by

$$\overline{\mathfrak{u}} \leq \overline{\nu}.$$

From (23), we have

$$\overline{\nu} = \mathfrak{m}_{\mathbf{Z}_p}^{der} \pmod{p\nu} + p \cdot \mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p} \pmod{p\nu}. \tag{28}$$

We notice that

$$\mathfrak{m}_{\mathbf{F}_p} = \mathfrak{m}_{\mathbf{F}_p}^{der} + \mathfrak{z}(\mathfrak{m})_{\mathbf{F}_p} \text{ and } \overline{\nu} \simeq \mathfrak{m}_{\mathbf{F}_p}^{der} + \frac{p \cdot \mathfrak{z}(\mathfrak{m})}{p^2 \cdot \mathfrak{z}(\mathfrak{m})} \tag{29}$$

are isomorphic $\mathbf{F}_p$-linear representation of $\mathrm{M}(\mathbf{Z}_p)$ and $\mathrm{M}(\mathbf{F}_p)$, and thus as representations of U_p and $\mathrm{U}(p)$ as well.

We consider

$$ab_{\mathfrak{m}} : \mathfrak{m}_{\mathbf{Z}_p} \to \mathfrak{m}_{\mathbf{Z}_p}^{ab} := \mathfrak{m}_{\mathbf{Z}_p}/\mathfrak{m}_{\mathbf{Z}_p}^{der}.$$

Let us prove the claim

$$p \cdot \mathfrak{m}_{\mathbf{Z}_p}^{ab} \leq ab_{\mathfrak{m}}(\mathfrak{u}). \tag{30}$$

Proof. — Let $Z \in p \cdot \mathfrak{m}^{ab}_{\mathbf{Z}_p}$. Let $z = \exp(Z) \in M^{ab}(\mathbf{Z}_p)$.

From (19), when $p > C_{MT}$, there exists $y \in U_p$ with $ab_M(y) = z$.

Assume $p \gg 0$, so that the algebraic tori $Z(M)_{\mathbf{Z}_p}$ and M^{ab} have good reduction, and assume furthermore $p > \# \ker(Z(M) \to M^{ab})$ so that the differential of the isogeny $Z(M) \to M^{ab}$ induces a $\mathbf{Z}_p$-isomorphism $\mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p} \to \mathfrak{m}^{ab}_{\mathbf{Z}_p}$. Thus, there exists $Z' \in p\mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p}$ with $ab_{\mathfrak{m}}(Z') = Z$.

Let $z' = \exp(Z')$. As $Z' \in \mathfrak{z}(\mathfrak{m})$, we have $z' \in Z(M)(\mathbf{Q}_p)$. As $Z' \in p \cdot \mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p}$, we have, for $p > 2$, $z' \in Z(M)(\mathbf{Z}_p)$. Moreover $\lim_{n\to\infty} z'^{p^n} = \lim_{n\to\infty} \exp(p^n Z') = 1$. Thus $\overline{z'} \in Z(M)(\mathbf{F}_p)$ is unipotent.

We have $\overline{z'} \in Z(M)(\mathbf{F}_p)^\dagger$ and, because $Z(M)(\mathbf{F}_p)$ has good reduction, we have $Z(M)(\mathbf{F}_p)^\dagger = \{1\}$. We also have $y \in M^{der}(\mathbf{Z}_p) \cdot z'$. Thus

$$\bar{y} \in U(p) \cap M^{der}(\mathbf{F}_p).$$

Let $\gamma = [\hat{U} : \hat{U}^\dagger] \leq c(\dim(G))$ from Lem. 5.11. Then $\mathrm{ad}_M(\bar{y})^\gamma \in \mathrm{ad}_M(U(p))^\dagger = \mathrm{ad}_M(U(p)^\dagger)$. Let U' be the inverse image of $\mathrm{ad}_M(U(p))^\dagger$ by $\mathrm{ad}_{M^{der}} : M^{der}(\mathbf{F}_p) \to M^{ad}(\mathbf{F}_p)$. Since $\mathrm{ad}_M(U(p))^\dagger = \mathrm{ad}_M(U(p)^\dagger)$ and $\ker(\mathrm{ad}_{M^{der}}) = Z(M^{der})(\mathbf{F}_p)$, we have $U' = U(p)^\dagger \cdot Z(M^{der})(\mathbf{F}_p)$. There are $u \in U(p)^\dagger$ and $z \in Z(M^{der})(\mathbf{F}_p)$ such that

$$\bar{y}^\gamma = u \cdot z.$$

We use that $f := \sup_p Z(M^{der})(\mathbf{F}_p) < +\infty$. Since z commutes with u, we have

$$\left(\bar{y}^\gamma\right)^{f!} = u^{f!} \cdot z^{f!} = u^{f!} \in U(p)^\dagger.$$

Thus $y^{\gamma \cdot f!} \in U_p^\dagger$. Assume $p > c(\dim(G)) \geq \gamma' := \gamma \cdot f!$, so that $\gamma' \in \mathbf{Z}_p^\times$. Because Z is arbitrary, we have

$$ab_M\left(U_p^\dagger\right) \geq \exp\left(p \cdot \mathfrak{m}^{ab}_{\mathbf{Z}_p}\right)^{\gamma'} = \exp\left(\gamma' \cdot p \cdot \mathfrak{m}^{ab}_{\mathbf{Z}_p}\right) = \exp\left(p \cdot \mathfrak{m}^{ab}_{\mathbf{Z}_p}\right).$$

Conversely $ab_M(U_p^\dagger) \leq \exp(p \cdot \mathfrak{m}^{ab}_{\mathbf{Z}_p}) = \ker M^{ab}(\mathbf{Z}_p) \to M^{ab}(\mathbf{F}_p)$ because $ab_{M_{\mathbf{F}_p}}(U(p)^\dagger) \leq ab_{M_{\mathbf{F}_p}}(M^{der}(\mathbf{F}_p)^\dagger) = \{1\}$.

The group $U_p^\dagger$ is topologically generated by $\exp(X_1), \ldots, \exp(X_k)$, and thus $ab_M(U_p^\dagger)$ is topologically generated by $\exp(Z_1), \ldots, \exp(Z_k)$ with $Z_i = ab_{\mathfrak{m}}(X_i)$.

Thus the logarithms

$$Z_i = \log(z_i) = ab_{\mathfrak{m}}(X_i)$$

topologically generate $\log(\exp(p \cdot \mathfrak{m}^{ab})) = p \cdot \mathfrak{m}^{ab}$. The conclusion follows. □

We let

(**31**) $$Z_1, \ldots, Z_l \text{ be a basis of } \frac{p\mathfrak{m}^{ab}}{p^2\mathfrak{m}^{ab}} \simeq \nu / \left(\mathfrak{m}^{der}_{\mathbf{Z}_p} + p^2 \mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p}\right) \simeq \overline{\nu} / \mathfrak{m}^{der}_{\mathbf{F}_p}.$$

Pick an arbitrary $Z \in \{Z_1; \ldots; Z_l\}$, and define

$$A = \left\{\overline{Y} \in \overline{\mathfrak{u}} | \overline{Y} \equiv Z \pmod{\mathfrak{m}^{der}_{\mathbf{F}_p}}\right\}.$$

From (30), this A is non empty. It is thus an affine subspace of $\overline{\nu}$, and, for any $\overline{Y}_0 \in A$, we have

$$A = \overline{Y}_0 + V,$$

where $V = \overline{\mathfrak{u}} \cap \mathfrak{m}^{der}_{\mathbf{F}_p}$ is the "direction" of A. The $\mathbf{F}_p$-linear vector subspace $V \leq \overline{\nu}$ is invariant under $U(p)$, and because the action of $U(p)$ is semisimple on $\mathfrak{m}_{\mathbf{F}_p}$, and thus, by (29), on $\overline{\nu}$, there exists a supplementary $U(p)$-invariant $\mathbf{F}_p$-linear subspace

$$W \leq \overline{\nu}.$$

The following intersection is an affine space of dimension 0, hence it is a singleton

$$A \cap W = \{\overline{Y}\}.$$

It is also invariant under $U(p)$. Thus the line

$$\mathbf{F}_p \cdot \overline{Y}$$

is fixed by $U(p)$. But the centraliser of $U(p)$ and $M_{\mathbf{F}_p}$ in $M_{\mathbf{F}_p}$ are the same. For $p \gg 0$, these centralisers are smooth as group schemes (cf. Lem. 5.6), and thus have the same Lie algebra

$$\mathfrak{z}_{\mathfrak{m}_{\mathbf{F}_p}}\big(U(p)\big) = \mathfrak{z}(\mathfrak{m})_{\mathbf{F}_p}.$$

Thus

$$\overline{Y} \in p \cdot \mathfrak{z}(M) \pmod{p\nu}. \tag{32}$$

We finally choose a representative $Y' \in \mathfrak{u}$ of $\overline{Y} \in \overline{\mathfrak{u}}$.

Thus $Y' \in p \cdot \mathfrak{z}(\mathfrak{m}) + p\nu = p\mathfrak{m}_{\mathbf{Z}_p}$ and $Y' \in \mathfrak{u}$. We define

$$\widetilde{\mathfrak{m}} := \frac{p\mathfrak{m}_{\mathbf{Z}_p}}{p^2\mathfrak{m}_{\mathbf{Z}_p}} = (p\mathfrak{m}_{\mathbf{Z}_p}) \otimes \mathbf{F}_p,$$

and denote the image of $Y' \in \mathfrak{u} \cap p\mathfrak{m}_{\mathbf{Z}_p} \leq p\mathfrak{m}_{\mathbf{Z}_p}$ by

$$\widetilde{Y} \in \widetilde{\mathfrak{u}} := (\mathfrak{u} \cap p\mathfrak{m}_{\mathbf{Z}_p}) \otimes \mathbf{F}_p \leq \widetilde{\mathfrak{m}}.$$

Again $\widetilde{\mathfrak{m}} \simeq \mathfrak{m}_{\mathbf{F}_p}$ as a representation. We define $\widetilde{\mathfrak{m}^{der}} := (\mathfrak{m}^{der} \cap p\mathfrak{m}_{\mathbf{Z}_p})/p^2\mathfrak{m}_{\mathbf{Z}_p}$, and

$$A' = \left\{\widetilde{Y}' \in \widetilde{\mathfrak{u}} | \widetilde{Y}' \equiv \widetilde{Y} \pmod{\widetilde{\mathfrak{m}^{der}}}\right\}.$$

and similarly, there exists $\widetilde{Y}' \in A'$ which is fixed by $U(p)$ and thus is in $\widetilde{\mathfrak{z}(\mathfrak{m})}$.

We choose a lift Y of $\widetilde{Y}'$ in $\mathfrak{u}$.

Repeating the process for each $Z \in \{Z_1; \ldots; Z_l\}$ we define

$$Y_1, \ldots, Y_l \in \mathfrak{u}. \tag{33}$$

The assertion (1) follows from (25), (26), (27).

The assertion (2) follows from (27), (33).

The assertion (3) follows from (32), (31), (23), (28). (We observe that $Y_1, \ldots, Y_l$ generate $\mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p}$ modulo $p \cdot \mathfrak{m}_{\mathbf{Z}_p}$ since their images in $p\mathfrak{m}^{ab}_{\mathbf{Z}_p}$ are congruent to $Z_1, \ldots, Z_l$ modulo $p^2\mathfrak{m}^{ab}_{\mathbf{Z}_p}$. Indeed $Y \equiv \widetilde{Y} \pmod{\widetilde{\mathfrak{m}^{der}}}$ gives

$$Y \in Y' + p^2\mathfrak{m}_{\mathbf{Z}_p} + p\mathfrak{m}^{der}_{\mathbf{Z}_p} = Y' + p\left(p \cdot \mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p} + \mathfrak{m}^{der}_{\mathbf{Z}_p}\right) = Y' + p\nu.$$

Thus $Y \equiv Y' \pmod{p\nu}$, and therefore maps to $Z \in p\mathfrak{m}^{ab}_{\mathbf{Z}_p}$ modulo $p^2\mathfrak{m}^{ab}_{\mathbf{Z}_p}$.)

We will now prove the assertion (4). We define

$$Z := Z_{G_{\mathbf{F}_p}}\left(\left\{\overline{X_1}; \ldots; \overline{X_k}, \overline{\frac{1}{p}Y_1}; \ldots; \overline{\frac{1}{p}Y_l}\right\}\right).$$

and

$$U' := \left(U(p) \cap Z(M)^0_{\mathbf{F}_p}(\mathbf{F}_p)\right) \text{ and } U'' := U(p)^\dagger \cdot U'.$$

We first note that $\pi_p(\exp(X_1)), \ldots, \pi_p(\exp(X_k))$ generates the group $U(p)^\dagger$ and that $\overline{Y_1/p}, \ldots, \overline{Y_l/p}$ generates the Lie algebra $\mathfrak{z}(\mathfrak{m})_{\mathbf{F}_p}$.

Thus

$$Z = Z_{G_{\mathbf{F}_p}}\left(U(p)^\dagger\right) \cap Z_{G_{\mathbf{F}_p}}\left(\mathfrak{z}(\mathfrak{m})_{\mathbf{F}_p}\right) \tag{34}$$

We have[8]

$$Z_{G_{\mathbf{F}_p}}\left(\mathfrak{z}(\mathfrak{m})_{\mathbf{F}_p}\right) = Z_{G_{\mathbf{F}_p}}\left(Z(M)^0\right).$$

Applying Lem. 5.11 with $\delta = C_{MT}$ from (18), we have,

$$\left[U(p) : U(p)^\dagger \cdot U'\right] \leq D := C_{MT} \cdot \gamma\left(\dim(G)\right).$$

With $p > M(D)$, with $M(D)$ as in Def. 2.1, we have

$$Z_{G_{\mathbf{F}_p}}(M_{\mathbf{F}_p}) = Z_{G_{\mathbf{F}_p}}\left(U(p)^\dagger \cdot U'\right) = Z_{G_{\mathbf{F}_p}}\left(U(p)^\dagger\right) \cap Z_{G_{\mathbf{F}_p}}\left(U'\right).$$

[8] We use that $Z(M_{\mathbf{F}_p})^0$ is connected and, for $p \gg 0$ smooth as a group scheme.

From Cor. 5.10, we have $[\mathrm{Z}(\mathrm{M})^0_{\mathbf{F}_p}(\mathbf{F}_p):\mathrm{U}']\leq\gamma(n)\cdot\mathrm{C}_{\mathrm{MT}}$ for some n. We may thus apply Lemma 5.14, with $\lambda=\gamma(n)\cdot\mathrm{C}_{\mathrm{MT}}$ and deduce

$$\forall p\gg 0,\quad \mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}(\mathrm{U}')=\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}\left(\mathrm{Z}(\mathrm{M})^0_{\mathbf{F}_p}\right).$$

From $\mathrm{U}'':=\mathrm{U}(p)^{\dagger}\cdot\mathrm{U}'\leq\mathrm{U}(p)^{\dagger}\cdot\mathrm{Z}(\mathrm{M})^0_{\mathbf{F}_p}\leq\mathrm{M}_{\mathbf{F}_p}$ we get

$$\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}(\mathrm{M}_{\mathbf{F}_p})=\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}\left(\mathrm{U}''\right)\leq\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}\left(\mathrm{U}(p)^{\dagger}\right)\cap\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}\left(\mathrm{Z}(\mathrm{M})^0_{\mathbf{F}_p}\right)\leq\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}(\mathrm{M}_{\mathbf{F}_p})$$

Finally

$$\mathrm{Z}=\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}\left(\mathrm{U}(p)^{\dagger}\right)\cap\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}\left(\mathrm{Z}(\mathrm{M})^0_{\mathbf{F}_p}\right)=\mathrm{Z}_{\mathrm{G}_{\mathbf{F}_p}}(\mathrm{M}_{\mathbf{F}_p}).$$ □

5.3.2. *Conjugacy classes of tuples.* — The following will be used to check, for almost all primes, one of the hypotheses of Th. 6.1.

Lemma **5.6.** — *Let p be a prime, let $\mathrm{G}\leq\mathrm{GL}(n)_{\mathbf{F}_p}$ be a reductive algebraic subgroup, and consider $v_1,\ldots,v_k\in\mathrm{G}(\mathbf{F}_p)$. Denote by U the group generated by $\{v_1;\ldots;v_k\}$, and define $v=(v_1,\ldots,v_k)$.*

Assume that

(**35**) *the action of U on $\mathfrak{g}_{\mathbf{F}_p}$ is semisimple.*

If $p>2\cdot\dim(\mathrm{G})$ then the simultaneous conjugacy class $\mathrm{G}\cdot v$ is Zariski closed in G^k.
If $p>c_3(\dim(\mathrm{G}))$ then the centraliser of v in G, as a group scheme over $\mathbf{F}_p$, is smooth.

The quantity c_3 is from [32, §4. Th. E].

Proof. — From [46, §5.1], we have $h(\mathrm{G})\leq\dim(\mathrm{G})$, where $h(\mathrm{G})$ is defined in loc. cit. From [46, Cor. 5.5], if $p>2h(\mathrm{G})-2$, the assumption (35) implies that U is G-cr, or "strongly reductive" in G in the sense of Richardson. The first assertion follows from [46, Th. 3.7] (cf. [42, §16]).

Thanks to (35) and the condition $p>c_3(\dim(\mathrm{G}))$ we may apply [32, §4. Th. E] (cf. also [48, 137. p. 40]). Thus the hypothesis of [12, II, §5.2, 2.8, p. 240] is satisfied and we conclude. (cf. also [2] and [23] on the subject, beyond the semi-simplicity assumption.) □

5.3.3. *Consequences for heights bounds.* — We denote $\|\ \|:\mathfrak{g}_{\mathbf{Q}_p}\to\mathbf{R}_{\geq 0}$ the p-adic norm $\|\mathrm{X}\|=\min(\{0\}\cup\{p^k\in p^{\mathbf{Z}}|p^k\cdot\mathrm{X}\in\mathfrak{g}_{\mathbf{Z}_p}\})$ associated to the $\mathbf{Z}_p$-structure $\mathfrak{g}_{\mathbf{Z}_p}$. We denote $\|\Sigma\|=\max\{\|s\|:s\in\Sigma\}$ for a bounded subset $\Sigma\subseteq\mathfrak{g}_{\mathbf{Q}_p}$. We recall that $\mathrm{H}_f(\phi)=\prod_p\mathrm{H}_p(\phi)$ with $\mathrm{H}_p(\phi)$ given by

$$\mathrm{H}_p(\phi)=\max\left\{1;\ \left\|\phi(\mathfrak{m}_{\mathbf{Z}_p})\right\|\right\}.$$

Proposition **5.7.** — *Define*

$$v=(X_1,\dots,X_k,Y_1,\dots,Y_l) \text{ and } v'=\left(X_1,\dots,X_k,\frac{1}{p}Y_1,\dots,\frac{1}{p}Y_l\right)$$

and

$$H_v(\phi)=\max\{1;\|g\cdot v\|\}=H_p(g\cdot v) \text{ and } H_{v'}(\phi)=H_p(g\cdot v').$$

Then, for $p\gg 0$, we have

$$[\phi(U):\phi(U)\cap G(\mathbf{Z}_p)]\geq H_v(\phi)\geq H_{v'}(\phi)/p \tag{36}$$

and

$$H_{v'}(\phi)\geq H_p(\phi)^{c(\rho)}. \tag{37}$$

Proof. — Let $U_i:=\exp(X_i\cdot\mathbf{Z}_p)$. We have

$$[\phi(U):\phi(U)\cap G(\mathbf{Z}_p)]\geq\max_{1\leq i\leq k}[\phi(U_i):\phi(U_i)\cap G(\mathbf{Z}_p)]$$

and

$$H_v(\phi)=\max\{1;\|g\cdot v\|\}=\max_{1\leq i\leq k}\max\{1;\|g\cdot X_i\|\}.$$

The inequality $[\phi(U_i):\phi(U_i)\cap G(\mathbf{Z}_p)]\geq\max\{1;\|g\cdot X_i\|\}$ follows from the Lemma of the exponential ([40, Th. A.3]). We deduce

$$[\phi(U):\phi(U)\cap G(\mathbf{Z}_p)]\geq H_v(\phi)=\max\{1;\|g\cdot v\|\}=H_p(g\cdot v)$$

The inequality $H_v(\phi)\geq H_{v'}(\phi)/p$ follows from the definitions.

We prove (37). Let $m_1,\dots,m_d$ be a generating set for $\mathfrak{m}_{\mathbf{Z}_p}$ and define $w=(m_1,\dots,m_d)$. We recall that by construction, we have

$$H_p(\phi)=\max\{1;\|g\cdot m_1\|;\dots;\|g\cdot m_d\|\}.$$

We want to apply Th. 6.1. The assumptions over $\mathbf{Q}_p$ are satisfied by Def. 2.1–(1) and [42, Th. 3.6].

Let $\widehat{U}=\mathrm{ad}_M(U(p)^\dagger)$, and let V be the inverse image of $\widehat{U}^\dagger$ by $\mathrm{ad}_M:U(p)\to\widehat{U}$. From Cor. 5.17, we deduce $V^\dagger=U(p)^\dagger$ and $\mathrm{ad}_M(V^\dagger)=\widehat{U}^\dagger$. Thus $V\leq V^\dagger\cdot Z(M)_{\mathbf{F}_p}$. By Lem. 5.13, we have $[\widehat{U}:\widehat{U}^\dagger]\leq c'(n)$. It follows $[U(p):V]\leq c'(n)$. Using 2.1–(2), we deduce that, for $p\gg 0$, the action of V on $\mathfrak{g}_{\mathbf{F}_p}$ is semisimple and $Z_{G_{\mathbf{F}_p}}(V)=Z_{G_{\mathbf{F}_p}}(M_{\mathbf{F}_p})$. We may thus apply Prop. A.4 and Cor. A.5, for $(x_1,\dots,x_k,y_1,\dots,y_l)=(\overline{X_1},\dots,\overline{X_k},\overline{Y_1/p},\dots,\overline{Y_l/p})$.

It follows that the assumptions over $\mathbf{F}_p$ of Th. 6.1 are satisfied (applied to the vectors $(v', 0)$ and $(0, w)$ in $\mathfrak{g}^{k+l+d}$, the sum of $k + l + d$ copies of the adjoint representations).

We may thus apply Theorem 6.1, and we deduce

$$\mathrm{H}_p(g \cdot v') \geq \mathrm{H}_p(\phi)^{\mathrm{C}(\Sigma(\rho))},$$

where $\Sigma(\rho)$ is the set of roots of G and does not depend on p. This proves (37) with $c(\rho) := \mathrm{C}(\Sigma(\rho))$. □

Corollary **5.8.** — *In particular, if* $\mathrm{H}_{v'}(\phi) \notin \{1; p\}$ *we have*

$$[\phi(\mathrm{U}) : \phi(\mathrm{U}) \cap \mathrm{G}(\mathbf{Z}_p)] \geq \mathrm{H}_p(\phi)^{c(\rho)/2}.$$

Proof. — We recall that, because $\mathrm{H}_{v'}(\phi) \in p^{\mathbf{Z}}$, we have $\mathrm{H}_{v'}(\phi) \geq p^2$ as soon as $\mathrm{H}_p(\phi) \notin \{1; p\}$. It follows

$$\mathrm{H}_v(\phi) \geq \mathrm{H}_{v'}(\phi)/p \geq \mathrm{H}_{v'}(\phi)^{1/2} \geq \mathrm{H}_p(\phi)^{1/(2\cdot c(\rho))}.$$ □

In proving Th. 5.4 we may now assume that $\mathrm{H}_{v'}(\phi) \leq p$.

We define

$$\mathrm{H}_{\mathrm{X}}(\phi) = \max\{1; \|\phi(\mathrm{X}_1)\|; \ldots; \|\phi(\mathrm{X}_k)\|\}$$

and

$$\mathrm{H}_{\mathrm{Y}}(\phi) = \max\{1/p; \|\phi(\mathrm{Y}_1)\|; \ldots; \|\phi(\mathrm{Y}_k)\|\},$$

so that

$$\mathrm{H}_{v'}(\phi) = \max\{\mathrm{H}_{\mathrm{X}}(\phi); p \cdot \mathrm{H}_{\mathrm{Y}}(\phi)\}$$

and

$$\mathrm{H}_v(\phi) = \max\{\mathrm{H}_{\mathrm{X}}(\phi); \mathrm{H}_{\mathrm{Y}}(\phi)\}.$$

If $\mathrm{H}_{\mathrm{X}}(\phi) = p$, we have, by (36),

$$[\phi(\mathrm{U}) : \phi(\mathrm{U}) \cap \mathrm{G}(\mathbf{Z}_p)] \geq \mathrm{H}_v(\phi) = \mathrm{H}_{\mathrm{X}}(\phi) = \mathrm{H}_{v'}(\phi) \geq \mathrm{H}_p(\phi)^{c(\rho)}.$$

We now assume $\mathrm{H}_{\mathrm{X}}(\phi) = 1$. We have $p \cdot \mathrm{H}_{\mathrm{Y}}(\phi) = \mathrm{H}_{v'}(\phi) \in \{1; p\}$.

5.3.4. *The case* $p \cdot \mathrm{H}_{\mathrm{Y}}(\phi) = \mathrm{H}_{\mathrm{X}}(\phi) = 1$. — In this case we have $\mathrm{H}_{v'}(\phi) = 1$, and by (37),

$$\mathrm{H}_p(\phi) = 1.$$

Obviously

$$[\phi(\mathrm{U}_p) : \phi(\mathrm{U}_p) \cap \mathrm{G}(\mathbf{Z}_p)] \geq \mathrm{H}_p(\phi).$$

5.3.5. *The case $p \cdot H_Y(\phi) = H_{v'}(\phi) = p$.* — From (3) of Prop. 5.5, for every Y_i, there exists $Z_i \in \mathfrak{z}(\mathfrak{m})_{\mathbf{Z}_p}$ such that

$$\frac{1}{p} Y_i \equiv Z_i \pmod{p \cdot \mathfrak{m}_{\mathbf{Z}_p}}.$$

Define $v'' = (X_1, \ldots, X_k, Z_1, \ldots, Z_l)$. Then the reductions modulo p are equal

$$\overline{v'} = \overline{v''} \quad \text{in } \mathfrak{m}_{\mathbf{F}_p}{}^{k+l} \leq \mathfrak{g}_{\mathbf{F}_p}{}^{k+l}.$$

Thus

- The orbit $G_{\mathbf{F}_p} \cdot \overline{v'}$ is equal to $G_{\mathbf{F}_p} \cdot \overline{v''}$ and is closed;
- and $Stab_{G_{\mathbf{F}_p}}(\overline{v'}) = Stab_{G_{\mathbf{F}_p}}(\overline{v''}) = Z_{G_{\mathbf{F}_p}}(M_{\mathbf{F}_p})$ (cf (4) of Prop. 5.5).

Applying Th. 7.1[9]

$$\phi\big(v'\big) \in \mathfrak{g}_{\mathbf{Z}_p}{}^{k+l} \text{ if and only if } \phi\big(v''\big) \in \mathfrak{g}_{\mathbf{Z}_p}{}^{k+l}.$$

We have, as functions of ϕ,

$$H_{v''} = \max\{H_X; H_Z\} \quad \text{with } H_Z : \phi \mapsto \max\big\{1; \|\phi(Z_1)\|; \ldots; \|\phi(Z_l)\|\big\}.$$

Because $H_X(\phi) = 1$ and $H_{v'}(\phi) = p \neq 1$ in §5.3.5, we have

$$H_Z(\phi) \neq 1. \tag{38}$$

Denote by $\pi : G(\mathbf{Z}_p) \to G(\mathbf{F}_p)$ the reduction modulo p map. Let $N := Z(M)^0(\mathbf{Z}_p) \cap \ker(\pi)$. For p large enough, the torus $Z(M)^0$ has good reduction over $\mathbf{Z}_p$, and, for p large enough, $N = \exp(2p\mathfrak{z}(\mathfrak{m}))$. Thanks to (38) we can apply[10] [17, 4.3.9] and [40, Th. B5 (3)] with the torus $Z(M)^0$: we have, for some $c \in \mathbf{R}_{>0}$ that does not depend on p,

$$\big[\phi\big(Z(M)^0(\mathbf{Z}_p)\big) : \phi\big(Z(M)^0(\mathbf{Z}_p)\big) \cap \big(\phi(N) \cdot G(\mathbf{Z}_p)\big)\big] \geq p/c. \tag{39}$$

Define $\Gamma := Z(M)^0(\mathbf{Z}_p) \cdot U_p$. As $Z(M)^0(\mathbf{Z}_p) \leq \Gamma$, we have

$$\begin{aligned}&\big[\phi(\Gamma) : \phi(\Gamma) \cap \big(\phi(N) \cdot G(\mathbf{Z}_p)\big)\big] \\ &\quad \geq \big[\phi\big(Z(M)^0(\mathbf{Z}_p)\big) : \phi\big(Z(M)^0(\mathbf{Z}_p)\big) \cap \big(\phi(N) \cdot G(\mathbf{Z}_p)\big)\big]\end{aligned}$$

Let $K := \Gamma \cap \overset{-1}{\phi}(G(\mathbf{Z}_p))$. Then $\Gamma \cap (N \cdot \overset{-1}{\phi}(G(\mathbf{Z}_p))) = N \cdot K$, and

$$[\Gamma : N \cdot K] \geq p/c.$$

---

[9] If the adjoint representation is not faithful, we apply Th. 7.1 to the representation $\mathfrak{g}^{k+l} \oplus W$, where $G \to GL(W)$ is a faithful representation, and vectors $v = (v', 0) \in \mathfrak{g}^{k+l} \oplus W$ and $v = (v'', 0) \in \mathfrak{g}^{k+l} \oplus W$. This ensures that the representation $G \to GL(\mathfrak{g}^{k+l} \oplus W)$ is faithful, as assumed in Th. 7.1.

[10] The reference [17, 4.3.9] is phrased in terms of orbits of lattices. See [40, §B.1, proof of conclusion 3 after Th. B.5] for the precise relation with the indices (39).

We have

(**40**) $$\#\phi(U_p)\cdot G(\mathbf{Z}_p)/G(\mathbf{Z}_p) = \left[\phi(U_p):\phi(U_p)\cap G(\mathbf{Z}_p)\right] = [U_p : U_p\cap K] \\ \geq \left[U_p : U_p\cap(K\cdot N)\right] = [U_p\cdot N\cdot K : N\cdot K].$$

The formula $[G_1 : G_3] = [G_1 : G_2]\cdot[G_2 : G_3]$ with $G_1 = \Gamma$, and $G_2 = U_p\cdot N\cdot K$ and $G_3 = N\cdot K$ gives

(**41**) $$[\Gamma : N\cdot K] = [\Gamma : U_p\cdot N\cdot K]\cdot[U_p\cdot N\cdot K : N\cdot K].$$

We deduce

(**42**) $$\#\phi(U_p)\cdot G(\mathbf{Z}_p)/G(\mathbf{Z}_p) \geq \frac{p}{c\cdot[\Gamma : U_p\cdot N\cdot K]}.$$

Recall that $g\in G(\mathbf{Z}_p)$ is "topologically p-nilpotent" if and only if the order of $\pi(g)\in G(\mathbf{F}_p)$ is a power of p. For $H\leq G(\mathbf{Z}_p)$ we denote by $H^\dagger$ the subgroup generated by the topologically p-nilpotent elements contained in H. Note that $\ker(\pi)\cap H\leq H^\dagger$.

Recall that $Z(M)^0(\mathbf{Z}_p)$ commutes with $U_p\leq M(\mathbf{Z}_p)$. The product map $Z(M)^0(\mathbf{Z}_p)\times U_p\to\Gamma$ is thus a surjective group homomorphism. It follows from 5.16 with $U = Z(M)^0(\mathbf{Z}_p)\times U_p$ and $U' = \Gamma$ that we have

$$\Gamma^\dagger = Z(M)^0(\mathbf{Z}_p)^\dagger\cdot U_p^\dagger.$$

Thus $\ker(\pi)\cap\Gamma\leq\Gamma^\dagger\leq Z(M)^0(\mathbf{Z}_p)^\dagger\cdot U_p^\dagger$. As $Z(M)^0$ is a torus, we have $p\nmid\#Z(M)^0(\mathbf{F}_p)$, and thus $Z(M)^0(\mathbf{Z}_p)^\dagger = \ker(\pi)\cap Z(M)^0(\mathbf{Z}_p) = N$. Because $H_X(\phi) = 1$, we have $\phi(X_i)\in\mathfrak{g}_{\mathbf{Z}_p}$. By [40, Proof of Prop. A1, (72)] and the estimate $\forall n, d\in\mathbf{Z}_{\geq 1}, p > 2d+1 \Rightarrow |n!|_p \geq |p|^{\lfloor\frac{n}{d}\rfloor}$, this implies that for $p\gg 0$, we have $\phi(\exp(X_i))\in G(\mathbf{Z}_p)$. Together with (1) of Prop. 5.5, this implies $\phi(U_p^\dagger)\leq G(\mathbf{Z}_p)$. Thus $U_p^\dagger\leq K$.

We deduce

$$\ker(\pi)\cap\Gamma\leq\Gamma^\dagger = Z(M)^0(\mathbf{Z}_p)^\dagger\cdot U_p^\dagger\leq N\cdot K.$$

Let $z\in Z(M)^0(\mathbf{Z}_p)$ be such that $\pi(z)\in Z(M)^0(\mathbf{F}_p)\cap U(p)$. As $\pi(z)\in U(p) = \pi(U_p)$, we have $z\in U_p\cdot(\ker(\pi)\cap\Gamma)\leq U_p\cdot K\cdot N$. Thus $z\in Z(M)^0(\mathbf{Z}_p)\cap(U_p\cdot K\cdot N)$. It follows that the map

$$Z(M)^0(\mathbf{Z}_p)/\big(Z(M)^0(\mathbf{Z}_p)\cap(U_p\cdot K\cdot N)\big) \\ \to Z(M)^0(\mathbf{F}_p)/\big(Z(M)^0(\mathbf{F}_p)\cap U(p)\big)$$

is injective. We deduce

(**43**) $$\left[\Gamma:\Gamma\cap(K\cdot N)\right] = \left[U_p\cdot Z(M)^0(\mathbf{Z}_p) : \big(U_p\cdot Z(M)^0(\mathbf{Z}_p)\big)\cap(K\cdot N)\right]$$

$$\leq \left[Z(M)^0(\mathbf{Z}_p) : Z(M)^0(\mathbf{Z}_p) \cap (U_p \cdot K \cdot N)\right]$$
$$\leq \left[Z(M)^0(\mathbf{F}_p) : (Z(M)^0(\mathbf{F}_p) \cap U(p)\right].$$

By Cor. 5.10 and (42), this implies

$$\#\phi(U_p) \cdot G(\mathbf{Z}_p)/G(\mathbf{Z}_p) \geq \frac{p}{c \cdot \gamma(n) \cdot C_{MT}} = \frac{H_{v'}(\phi)}{c \cdot \gamma(n) \cdot C_{MT}}.$$

Using (37) we conclude

$$\left[\phi(U_p) : \phi(U_p) \cap G(\mathbf{Z}_p)\right] \geq \frac{1}{c \cdot \gamma(n) \cdot C_{MT}} \cdot H_p(\phi)^{c(\rho)}.$$

This proves (17) with $c = c(\rho)$ and $a = 1/(c \cdot \gamma(n) \cdot C_{MT})$. We have proven Th. 5.4 and Th. 5.1.

5.4. *Some structure lemmas.* — We consider the situation of Theorem 5.4. We identify G with its image by a faithful representation in GL(n) such that $G(\mathbf{Z}_p) = GL(n, \mathbf{Z}_p) \cap G$, and we denote by $U(p)$ the image of U_p in $G(\mathbf{F}_p) \leq GL(n, \mathbf{F}_p)$. We denote by $\overline{M} = M_{\mathbf{F}_p}$ the $\mathbf{F}_p$-algebraic group from the model of M over $\mathbf{Z}_p$ induced by $M \leq GL(n)$, and we denote $Z(M_{\mathbf{F}_p})$ the centre of $M_{\mathbf{F}_p}$.

In Lem. 5.11, Prop. 5.9 Cor. 5.10, the quantity depending on n also depends implicitly on the function $D \mapsto M(D)$ in (2) of Def. 2.1.

We recall that the U_p satisfy the uniform integral Tate conjectures as in Def. 2.4 and 2.1. In this section the statements are valid for almost every prime p.

Proposition **5.9.** — *There exists $\gamma(n)$ such that,*

$$\left[Z(M_{\mathbf{F}_p})(\mathbf{F}_p) : Z(M_{\mathbf{F}_p})(\mathbf{F}_p) \cap U(p)\right] \leq \gamma(n) \cdot \left[M^{ab}_{\mathbf{F}_p}(\mathbf{F}_p) : ab_{M_{\mathbf{F}_p}}\big(U(p)\big)\right].$$

By Hypothesis (1) of Th. 5.1 we may use (18) and deduce the following.

Corollary **5.10.** — *With C_{MT} as in (18), we have*

$$\left[Z(M_{\mathbf{F}_p})(\mathbf{F}_p) : Z(M_{\mathbf{F}_p})(\mathbf{F}_p) \cap U(p)\right] \leq \gamma(n) \cdot C_{MT}.$$

We prove Proposition 5.9.

Proof. — We recall $\overline{M} := M_{\mathbf{F}_p}$. We consider the isogeny

$$(ad_{\overline{M}}, ab_{\overline{M}}) : \overline{M} \to \overline{M}^{ad} \times \overline{M}^{ab}$$

We write $U = U(p)$ and denote by $\widetilde{U}$ its image in $\overline{M}^{ad}(\mathbf{F}_p) \times \overline{M}^{ab}(\mathbf{F}_p)$.

We denote $\widetilde{U}_1$ and $\widetilde{U}_2$ its images by the projections on the two factors.

From Lemma 5.11, we have

$$\left[\widetilde{\mathrm{U}}_1 : \widetilde{\mathrm{U}} \cap \overline{\mathrm{M}}^{ad}(\mathbf{F}_p)\right] \leq \left[\widetilde{\mathrm{U}}_1 : \widetilde{\mathrm{U}}^{\dagger}\right] \leq c(n).$$

By Goursat's Lemma 5.20

$$\left[\widetilde{\mathrm{U}}_2 : \widetilde{\mathrm{U}} \cap \overline{\mathrm{M}}^{ab}(\mathbf{F}_p)\right] = \left[\widetilde{\mathrm{U}}_1 : \widetilde{\mathrm{U}} \cap \overline{\mathrm{M}}^{ad}(\mathbf{F}_p)\right] \leq c(n).$$

Let $\mathrm{U}' \leq \mathrm{U}$ be the inverse image of $\widetilde{\mathrm{U}} \cap \overline{\mathrm{M}}^{ab}(\mathbf{F}_p)$ in U, and $\mathrm{U}'' \leq \overline{\mathrm{M}}(\mathbf{F}_p)$ be the inverse image of $\widetilde{\mathrm{U}} \cap \overline{\mathrm{M}}^{ab}(\mathbf{F}_p)$ in $\overline{\mathrm{M}}(\mathbf{F}_p)$. Because $\mathrm{Z}(\overline{\mathrm{M}})$ is the $(ad_{\overline{\mathrm{M}}}, ab_{\overline{\mathrm{M}}})$-inverse image of $\overline{\mathrm{M}}^{ab}$ in $\overline{\mathrm{M}}$, we have

$$\mathrm{U}' \leq \mathrm{U}'' \leq \mathrm{Z}(\overline{\mathrm{M}})(\mathbf{F}_p).$$

Define $\mathrm{F} := \mathrm{Z}(\overline{\mathrm{M}}) \cap \overline{\mathrm{M}}^{der}$, which is a finite $\mathbf{F}_p$-algebraic group of degree at most $c_2(\dim(\mathrm{M})) \leq c_2(n)$.

As we have $\mathrm{U}'' \leq \mathrm{F}(\overline{\mathbf{F}}_p) \cdot \mathrm{U}'$, we have

$$\left[\mathrm{U}'' : \mathrm{U}'\right] \leq \#\mathrm{F} \leq c_2(n).$$

On the other hand, we have

$$\left[\mathrm{Z}(\overline{\mathrm{M}})(\mathbf{F}_p) : \mathrm{U}''\right] \leq \left[\overline{\mathrm{M}}^{ab}(\mathbf{F}_p) : \widetilde{\mathrm{U}} \cap \overline{\mathrm{M}}^{ab}(\mathbf{F}_p)\right].$$

It follows

$$\begin{aligned} \left[\mathrm{Z}(\overline{\mathrm{M}})(\mathbf{F}_p) : \mathrm{U}'\right] &\leq c_2(n) \cdot \left[\overline{\mathrm{M}}^{ab}(\mathbf{F}_p) : \widetilde{\mathrm{U}} \cap \overline{\mathrm{M}}^{ab}(\overline{\mathbf{F}}_p)\right] \\ &\leq c(n) \cdot c_2(n) \cdot \left[\overline{\mathrm{M}}^{ab}(\mathbf{F}_p) : \widetilde{\mathrm{U}}_2\right] \end{aligned} \tag{44}$$

□

Lemma **5.11.** — *Let* $\hat{\mathrm{U}} := ad_{\mathrm{M}_{\mathbf{F}_p}}(\mathrm{U}(p))$ *be the image of* $\mathrm{U}(p)$ *in* $\mathrm{M}^{ad}_{\mathbf{F}_p}(\mathbf{F}_p)$. *Then* $\hat{\mathrm{U}}^{\dagger} = ad_{\mathrm{M}_{\mathbf{F}_p}}(\mathrm{U}(p)^{\dagger})$, *and there exists* $c(n)$ *such that*

$$\left[\hat{\mathrm{U}} : \hat{\mathrm{U}}^{\dagger}\right] \leq c(n).$$

Proof. — The equality $\hat{\mathrm{U}}^{\dagger} = ad_{\overline{\mathrm{M}}}(\mathrm{U}(p)^{\dagger})$ follows from Cor. 5.17.

From Jordan's theorem [47, Th. 3'] (applied to $\mathrm{G} = \hat{\mathrm{U}}/\hat{\mathrm{U}}^{\dagger}$) there exists $\hat{\mathrm{U}}^{\dagger} \leq \hat{\mathrm{U}}' \leq \hat{\mathrm{U}}$ of index $[\hat{\mathrm{U}} : \hat{\mathrm{U}}'] \leq \mathrm{C}(n) := d(n)$ such that

$$\hat{\mathrm{U}}'/\hat{\mathrm{U}}^{\dagger}$$

is abelian, where $d(n)$ is as in [47, §§4–5].Without loss of generality, we may assume $\hat{\mathrm{U}} = \hat{\mathrm{U}}'$.

Let $M^{ad} \to GL(n')$ be the adjoint representation in some $\mathbf{Q}$-linear basis of $\mathfrak{m}^{ad}$. For $p \gg 0$, the $\mathbf{F}_p$-fiber of the induced model of M^{ad} is $\overline{M}^{ad}$. For $p \gg 0$ the following holds: for any subgroup $V \leq M(\mathbf{F}_p)$ whose action on $\mathbf{F}_p{}^n$ is semisimple, the action on $\mathbf{F}_p{}^{n'}$ is semisimple (cf. [46, Th. 5.4]).

Let $c_1(n)$ be the $c(n)$ of Lem. 5.12 applied with $N = n$. Let $c_2(n)$ be the $c(n')$ of Lem. 5.13, where $M \leq GL(n)$ is our $\overline{M}^{ad} \leq GL(n')_{\mathbf{F}_p}$. Let $c_3(n) = c_1(n) \cdot c_2(n)$.

Let us prove that $\widehat{U}$ satisfies the assumptions of 5.13, where $M \leq GL(n)$ is our $M^{ad} \leq GL(n')$, and $p \geq M(c_3(n))$.

By construction $\hat{U}/\hat{U}^{\dagger}$ is abelian. To simplify notations, we write U for $U(p)$, in the rest of the proof.

Let $U' \leq \widehat{U}$ be such that $[\widehat{U} : U'] \leq c_2(n)$. Let V be the inverse image of U' in U. We have $[U : V] \leq [\widehat{U} : U'] \leq c_2(n)$. We apply Lem. 5.12 to V: there is $V' \leq V$ such that $[V : V'] \leq c_1(n)$ and

$$Z_{\overline{M}^{ad}}\left(ad_{\overline{M}}(V')\right) = Z_{\overline{M}}(V')/Z(\overline{M}).$$

We have

$$\left[U : V'\right] = [U : V] \cdot \left[V : V'\right] \leq c_2(n) \cdot c_1(n) = c_3(n).$$

Since $p \geq M(c_3(n))$, we may apply (2) of the Tate hypothesis Def. 2.1, and we have

$$Z_{\overline{M}}(V') = Z(\overline{M}).$$

It follows that

$$Z_{\overline{M}^{ad}}(U) \leq Z_{\overline{M}^{ad}}\left(ad_{\overline{M}}(V')\right) = \{1\}.$$

We can thus apply Lemma 5.13 to $\hat{U} \leq \overline{M}^{ad}(\mathbf{F}_p)$, and the conclusion follows. ☐

Lemma **5.12.** — *For every* $N \in \mathbf{Z}_{\geq 1}$ *there exists* $c(N)$ *such that, for any prime* p *and any reductive algebraic subgroup* $M \leq GL(N)_{\mathbf{F}_p}$, *and any* $U \leq M(\mathbf{F}_p)$, *there exists* $U' \leq U$ *with*

$$\left[U : U'\right] \leq c(N)$$

and

$$Z_{M^{ad}}\left(ad_M(U')\right) = Z_M(U')/Z(M)$$

where $ad_M : M \to M^{ad}$ *is the quotient map to the adjoint group.*

Proof. — Let M^{der} be the derived subgroup of M and $F = Z(M^{der}) = Z(M) \cap M^{der}$ be its centre.

We claim that there exists $e \in \mathbf{Z}_{\geq 1}$ such that for every reductive subgroup $M \leq G$, we have

$$\# Z\left(M^{der}\right)(\overline{\mathbf{F}}_p) \,\Big|\, e\,.$$

Proof. — We have $\# F(\mathbf{F}_p) \leq \# F(\overline{\mathbf{F}}_p) \leq \# \pi_1(M^{ad})$. As in [55, Proof of Lem. 2.4], one can classify M^{ad} in terms of its Dynkin diagram, and, since $\dim(M^{ad})$ is bounded, deduce a bound on $\# \pi_1(M^{ad})$. □

Define

$$U[e] := \bigcap_{\phi \in \mathrm{Hom}(U, \mathbf{Z}/(e))} \ker(\phi).$$

According to [40, Prop. 6.7], the group U is generated by $k(N)$ elements. It follows that

$$\left[U : U[e]\right] \leq e^{k(N)}.$$

Let $m \in M(\overline{\mathbf{F}}_p)$ such that $\widehat{m} := ad_M(m)$ belongs to $Z_{M^{ad}}(ad_M(U))$. Denote by ϕ_m the map

$$u \mapsto mum^{-1}u^{-1} : U \to M(\overline{\mathbf{F}}_p).$$

We claim that ϕ_m is a morphism $U \to F(\overline{\mathbf{F}}_p)$.

Proof. — Consider $u \in U$ and let $\widehat{u} = ad_M(u)$. We have $\widehat{\widehat{m}\widehat{u}\widehat{m}}^{-1} = \widehat{u}$, and this implies $mum^{-1}u^{-1} \in \ker(ad_M)$. It follows

$$\phi_m(U) \subseteq Z(M)(\overline{\mathbf{F}}_p).$$

Since $mum^{-1}u^{-1}$ is a commutator, we have $mum^{-1}u^{-1} \in [M(\overline{\mathbf{F}}_p), M(\overline{\mathbf{F}}_p)] = M^{der}(\overline{\mathbf{F}}_p)$. It follows

$$\phi_m(U) \subseteq M^{der}(\overline{\mathbf{F}}_p).$$

Thus

$$\phi_m(U) \subseteq F(\overline{\mathbf{F}}_p) = M^{der}(\overline{\mathbf{F}}_p) \cap Z(M)(\overline{\mathbf{F}}_p)$$

For $u, u' \in U$, we have

$$\phi_m\left(uu'\right) = muu'm^{-1}\left(uu'\right)^{-1} = mum^{-1}mu'm^{-1}u'^{-1}u^{-1} = mum^{-1}\phi_m\left(u'\right)u^{-1}.$$

Since $\phi_m(u') \in Z(M)(\overline{\mathbf{F}}_p)$, we have $\phi_m(u')u^{-1} = u^{-1}\phi_m(u')$, and

$$\phi_m\left(uu'\right) = mum^{-1}u^{-1}\phi_m\left(u'\right) = \phi_m(u)\phi_m\left(u'\right).$$

obviously, $\phi_m(1) = 1$. The claim follows. □

Since $\#F|e$, by construction of $U[e]$, we have

$$\phi_m\big(U[e]\big) = \{1\}.$$

Equivalently $m \in Z_M(U[e])$.

This proves the Lemma with $U' = U[e]$ and $c(N) = e^{k(N)}$. □

Lemma **5.13.** — *For every $n \in \mathbf{Z}_{\geq 1}$, there exists $c(n)$, $c'(n)$, $m(n)$ such that the following holds.*

Let $p > m(n)$ be a prime, let $M \leq GL(n)$ be adjoint over $\mathbf{F}_p$, and let

$$\hat{U} \leq M(\mathbf{F}_p)$$

be a subgroup

- *such that $\hat{U}/\hat{U}^{\dagger}$ is abelian*
- *and such that for every $U' \leq \hat{U}$ of index at most $c(n)$:*
  - *(1) we have $Z_M(U') = 1$;*
  - *(2) the action of U' is semisimple;*

Then we have $[\hat{U} : \hat{U}^{\dagger}] \leq c'(n)$.

5.4.1. *Remark.* — In proving the Lemma, we may substitute $\hat{U}$ with U' if $\hat{U}^{\dagger} \leq U' \leq \hat{U}$ and $[\hat{U} : U'] \leq f(n)$, where $f(n)$ depends only on n. We then have to change $c(n)$ into $c(n) \cdot f(n)$ accordingly.

Proof. — We assume $p > p(n)$, for some $p(n)$ depeding only on n, so that we can apply Nori theory [32].

We denote by $S \leq M$ the $\mathbf{F}_p$-algebraic group associated by Nori to $\hat{U}^{\dagger}$, and denote by $N = N_M(S)$ the normaliser of S in M.

We deduce from (2) that S is semisimple, and thus $N^0 = S \cdot Z_M(S)^0$.

We recall that the semisimple Lie subalgebras contained in $\mathfrak{gl}(n)_{\overline{\mathbf{F}_p}}$ can assume finitely many types, independently from p. We deduce uniform bounds

$$\#N/N^0 \leq c_1(n) \text{ and } \#Z(S) \leq c_2(n). \tag{45}$$

As $\widehat{U}^{\dagger}$ is a characteristic subgroup of $\widehat{U}$ it is normalised by $\widehat{U}$. It follows that $\widehat{U}$ normalises the associated semisimple subgroup S by Nori. We have $\hat{U} \leq N$. Let $N^{\dagger} \leq N$ be the algebraic subgroup generated by one parameter unipotent subgroups. Then $N^{\dagger}$ is connected, and we have $U^{\dagger} \leq N^{\dagger} \leq N^0$. If $U' = \hat{U} \cap N^0$ we have $[\hat{U} : U'] \leq c_1(n)$ and $U^{\dagger} \leq U'$. Using the Remark 5.4.1, we may replace $\hat{U}$ by $U' = \hat{U} \cap N^0$.

We denote $Z(S) = Z_M(S) \cap S$ and we consider

$$N^0 \to S^{ad} \times Z_M(S)/Z(S).$$

We denote $\widetilde{\mathrm{U}}$ the image of $\hat{\mathrm{U}}$, by

$$\widetilde{\mathrm{U}}_1 \leq \mathrm{S}^{ad}(\mathbf{F}_p) \text{ and } \widetilde{\mathrm{U}}_2 \leq \big(\mathrm{Z}_\mathrm{M}(\mathrm{S})/\mathrm{Z}(\mathrm{S})\big)(\mathbf{F}_p)$$

the projections of $\widetilde{\mathrm{U}}$ and define

$$\mathrm{U}'_1 := \widetilde{\mathrm{U}} \cap \big(\mathrm{S}^{ad}(\mathbf{F}_p) \times \{1\}\big) \text{ and } \mathrm{U}'_2 := \widetilde{\mathrm{U}} \cap \big(\{1\} \times \mathrm{Z}_\mathrm{M}(\mathrm{S})/\mathrm{Z}(\mathrm{S})\big).$$

From Cor. 5.17, the image, in $\mathrm{S}^{ad}(\mathbf{F}_p) \times \mathrm{Z}_\mathrm{M}(\mathrm{S})/\mathrm{Z}(\mathrm{S})$, of $\mathrm{S}(\mathbf{F}_p)^\dagger \leq \hat{\mathrm{U}}$ is $\mathrm{S}^{ad}(\mathbf{F}_p)^\dagger$. Thus

$$\mathrm{S}^{ad}(\mathbf{F}_p)^\dagger \times \{1\} \leq \mathrm{U}'_1 \leq \widetilde{\mathrm{U}}_1 \times \{1\} \leq \mathrm{S}^{ad}(\mathbf{F}_p) \times \{1\}.$$

With $r(n)$ given by [32, 3.6(iv-v) and p. 270], we have

$$\big[\widetilde{\mathrm{U}}_1 \times \{1\} : \mathrm{U}'_1\big] \leq \big[\mathrm{S}^{ad}(\mathbf{F}_p) : \mathrm{S}^{ad}(\mathbf{F}_p)^\dagger\big] \leq r(n) = 2^{n-1}. \tag{46}$$

By Goursat's Lemma 5.20 and (46) we have

$$\big[\widetilde{\mathrm{U}}_1 \times \{1\} : \mathrm{U}'_1\big] = \big[\{1\} \times \widetilde{\mathrm{U}}_2 : \mathrm{U}'_2\big] \leq r(n).$$

Thus, with $\mathrm{U}' := \mathrm{U}'_1 \cdot \mathrm{U}'_2 \simeq \mathrm{U}'_1 \times \mathrm{U}'_2$, we have

$$\big[\widetilde{\mathrm{U}} : \mathrm{U}'\big] \leq \big[\widetilde{\mathrm{U}}_1 \times \widetilde{\mathrm{U}}_2 : \mathrm{U}'\big] \leq r(n)^2.$$

Because $\widehat{\mathrm{U}}^\dagger = \mathrm{S}(\mathbf{F}_p)^\dagger$ is sent to $\mathrm{S}^{ad}(\mathbf{F}_p)^\dagger \times \{1\}$ (cf. Cor. 5.17) and because $\mathrm{S}^{ad}(\mathbf{F}_p)^\dagger \times \{1\} \leq \mathrm{U}'_1 \leq \mathrm{U}'$ we may use the Remark 5.4.1, and replace $\hat{\mathrm{U}}$ by the inverse image of U'. We denote by

$$\hat{\mathrm{U}}_1, \quad \hat{\mathrm{U}}_2$$

the inverse images of U'_1 and U'_2.

Because $\hat{\mathrm{U}}_2 \leq \mathrm{Z}_\mathrm{M}(\mathrm{S})$ and $\hat{\mathrm{U}}_1 \leq \mathrm{S}$, the groups $\hat{\mathrm{U}}_1$ and $\hat{\mathrm{U}}_2$ commute with each other.

We reduce the situation to the case where $\hat{\mathrm{U}}_2$ is abelian.

We know that $\hat{\mathrm{U}}/\hat{\mathrm{U}}^\dagger$ is abelian and that $\hat{\mathrm{U}}^\dagger \leq \mathrm{S}(\mathbf{F}_p)$. It follows that $\hat{\mathrm{U}}_2/(\hat{\mathrm{U}}_2 \cap \mathrm{S}(\mathbf{F}_p))$ is abelian. We have $\mathrm{F} := \hat{\mathrm{U}}_2 \cap \mathrm{S}(\mathbf{F}_p) \leq \mathrm{Z}_\mathrm{M}(\mathrm{S}) \cap \mathrm{S} = \mathrm{Z}(\mathrm{S})$, and thus $|\mathrm{F}| \leq c_2(n)$, and $\hat{\mathrm{U}}_2$ is an extension of the abelian group $\mathrm{U}'_2 = \hat{\mathrm{U}}_2/(\hat{\mathrm{U}}_2 \cap \mathrm{S}) = \widetilde{\mathrm{U}} \cap \mathrm{Z}_\mathrm{M}(\mathrm{S})/\mathrm{Z}(\mathrm{S})$ by a finite group F of order at most $c_2(n)$. Moreover, U'_2 is of order prime to p, and thus is diagonalisable in $\mathrm{Z}_\mathrm{M}(\mathrm{S})/\mathrm{Z}(\mathrm{S})(\overline{\mathbf{F}_p})$. It follows we can find a monomorphism $\mathrm{U}'_2 \leq (\overline{\mathbf{F}_p}^\times)^{rk(\mathrm{Z}_\mathrm{M}(\mathrm{S}))}$ where $rk(\mathrm{Z}_\mathrm{M}(\mathrm{S}))$ is the rank of $\mathrm{Z}_\mathrm{M}(\mathrm{S})$. We have $rk(\mathrm{Z}_\mathrm{M}(\mathrm{S})) \leq rk(\mathrm{M}) \leq n$. From Cor. 5.19, there exists an abelian subgroup $\mathrm{U}''_2 \leq \hat{\mathrm{U}}_2$ of index at most

$$c_3(n) = c_2(n) \cdot e\big(c_2(n)\big)^n.$$

Using the remark we may replace $\hat{U} = \hat{U}_1 \cdot \hat{U}_2$ by $U' = \hat{U}_1 \cdot U''_2$. Then U''_2 commutes with $\hat{U}_1$ and with itself: U'_2 is in the centre of $\hat{U}$. By Hypothesis (1), we have $U''_2 = 1$.

Thus $\hat{U} = \hat{U}_1 \leq S(\mathbf{F}_p)$ and

$$\left[\hat{U} : \hat{U}^{\dagger}\right] \leq \left[S(\mathbf{F}_p) : S(\mathbf{F}_p)^{\dagger}\right] \leq r(n). \qquad \square$$

5.4.2. *Other lemmas.*

Lemma **5.14.** — *Let* $H \leq G \leq GL(d)$ *be algebraic groups over* $\mathbf{Q}$ *with* H *Zariski connected. For every* $\lambda \in \mathbf{Z}_{\geq 1}$, *there exists* $N \in \mathbf{Z}_{\geq 0}$ *such that: for all prime* $p \geq N$, *and for all subgroups* $U \leq H_{\mathbf{F}_p}(\mathbf{F}_p)$ *such that*

$$\left[H_{\mathbf{F}_p}(\mathbf{F}_p) : U\right] \leq \lambda$$

we have

$$Z_{G_{\mathbf{F}_p}}(U) = Z_{G_{\mathbf{F}_p}}(H_{\mathbf{F}_p}).$$

Proof. — Without loss of generality we may assume $G = GL(d)$. We define a scheme $X \leq Y \times Y$, with $Y = GL(d)_{\mathbf{Z}}$ by

$$X = \left\{(g, h) \in GL(d)_{\mathbf{Z}} \times GL(d)_{\mathbf{Z}} | [g, h] = 1, h \in H\right\}$$

and denote by $\phi : X \to Y$ the first projection. According to the Lemma 5.15, for every prime p, and every $g \in G(\overline{\mathbf{F}_p})$,

$$\pi_0\left(H \cap Z_{G_{\overline{\mathbf{F}_p}}}(\{g\})\right) \leq \gamma.$$

For $p \gg 0$, the $\mathbf{F}_p$-group $H_{\mathbf{F}_p}$ will be Zariski connected. ([51, Lem. 37.28.5. (055H)] with $y = \mathrm{Spec}(\mathbf{Q}) \in Y = \mathrm{Spec}(\mathbf{Z})$ and X the schematic closure of H in $GL(d)_{\mathbf{Z}}$).

From [32, Lem. 3.5], we have, for any $g \in GL(d, \mathbf{F}_p)$ and $H' := X_g = H \cap Z_{G_{\overline{\mathbf{F}_p}}}(\{g\})$,

$$\#H'(\mathbf{F}_p) \leq (p+1)^{\dim(H')} \cdot \gamma$$

and

$$\#H(\mathbf{F}_p) \geq (p-1)^{\dim(H)}.$$

Let $\lambda(p) = \frac{1}{\gamma} \cdot (p-1) \cdot (\frac{p-1}{p+1})^{\dim(H)-1}$. Let λ be given. Since $\lim_p \lambda(p) = +\infty$, there exists N such that for $p \geq N$, we have $\lambda(p) > \lambda$.

Let

$$U \leq H(\mathbf{F}_p)$$

be such that

$$Z_{G_{\mathbf{F}_p}}(U) \neq Z_{G_{\mathbf{F}_p}}(H_{\mathbf{F}_p}).$$

We have $Z_{G_{\mathbf{F}_p}}(U) \geq Z_{G_{\mathbf{F}_p}}(H_{\mathbf{F}_p})$.

Recall that $G = GL(d)$. We remark that $Z_{G(\overline{\mathbf{F}_p})}(U)$ and $Z_{G(\overline{\mathbf{F}_p})}(H_{\overline{\mathbf{F}_p}})$ are Zariski connected because they are non empty Zariski open subsets $A \cap GL(d, \overline{\mathbf{F}_p})$ in a subalgebra $A \leq End(\overline{\mathbf{F}_p}^d)$. For $p - 1 > (\frac{p+1}{p-1})^{\dim(G)}$ (cf. [32, Lem. 3.5]), we will have

$$\#Z_{G_{\mathbf{F}_p}}(U)(\mathbf{F}_p) > \#Z_{G_{\mathbf{F}_p}}(H_{\mathbf{F}_p})(\mathbf{F}_p)$$

and there exists

$$g \in Z_{G_{\mathbf{F}_p}}(U)(\mathbf{F}_p) \smallsetminus Z_{G_{\mathbf{F}_p}}(H_{\mathbf{F}_p})(\mathbf{F}_p).$$

We have then, with $H' = X_g = Z_{G_{\mathbf{F}_p}}(\{g\}) \cap H_{\mathbf{F}_p}$, which is defined over $\mathbf{F}_p$,

$$H' < H.$$

Because H is connected, we have $\dim(H') < \dim(H)$ and

$$\begin{aligned}\#U \leq \#H'(\mathbf{F}_p) \leq \#\pi_0\big(H'\big) \cdot \#H'^0(\mathbf{F}_p) \leq \gamma \cdot (p+1)^{\dim(H)-1} \\ \leq \frac{1}{\lambda} \cdot (p-1)^{\dim(H)} \leq \frac{1}{\lambda} \cdot \#H(\mathbf{F}_p).\end{aligned}$$

The Lemma follows. □

Lemma **5.15.** — *Let $\phi : X \to Y$ be a morphism of schemes of finite type over* $\mathbf{Z}$.

Then there exists γ such that: for every field K*, and every $y \in Y(K)$, the number of geometric connected components $\#\pi_0(X_y)$ of the fibre X_y satisfies*

$$\#\pi_0(X_y) \leq \gamma.$$

If ϕ is flat, then $y \mapsto \dim(X_y)$ is lower semicontinuous on Y.

Proof. — If Y is non-empty, there exists a proper closed subset outside of which the function $y \mapsto \#\pi_0(X_y)$ is constant, according to[11] [51, Lemma 37.28.5. (055H)]. We conclude by noetherian induction.

The second assertion, on dimensions, is [51, Lemma 37.28.4. (0D4H)]. □

Lemma **5.16.** — *Let $\phi : U \to U'$ be a continuous epimorphism of profinite groups. Let p be a prime number.*

[11] Applied to the generic point of an irreducible component of Y. The latter exists because Y is noetherian.

Recall that an element $u \in \mathrm{U}$ is said to be "topologically p-nilpotent" if $\lim_{n\to\infty} u^{p^n} = 1$. *Equivalently, for every continuous map $q : \mathrm{U} \to \mathrm{F}$ to a finite group F, the order of $q(u)$ is a power of p.*

Then every "topologically p-nilpotent" $u' \in \mathrm{U}'$ is of the form $u' = \phi(u)$ for some "topologically p-nilpotent" $u \in \mathrm{U}$.

Proof. — Let $u' \in \mathrm{U}'$ be arbitrary. As ϕ is surjective, there exists $v \in \mathrm{U}$ such that $\phi(v) = u'$. The homomorphism $\pi : k \to v^k : \mathbf{Z} \to \mathrm{U}$ extends uniquely to a continuous homomorphism $\widehat{\mathbf{Z}} \to \mathrm{U}$ on the profinite completion $\widehat{\mathbf{Z}}$ of $\mathbf{Z}$.

Let $1 \in \widehat{\mathbf{Z}}$ be the multiplicative unit, and, for a prime number ℓ, let $1_\ell \in \mathbf{Z}_\ell \subseteq \widehat{\mathbf{Z}}$ be the multiplicative unit of the ℓ-adic factor $\mathbf{Z}_\ell$. We note that $1 = \lim_{n\to+\infty} \sum_{\ell\leq n} 1_\ell$. It follows that $v = \lim_{n\to+\infty} \sum_{\ell\leq n} \pi(1_\ell)$.

As u' is "topologically p-nilpotent", the map $\phi \circ \pi : k \to u'^k : \mathbf{Z} \to \mathrm{U}'$ factors through $\mathbf{Z} \to \mathbf{Z}_p \to \mathrm{U}'$. The closed subgroup $\overline{u'^{\mathbf{Z}}} \leq \mathrm{U}'$ generated by u' is thus a quotient of $\mathbf{Z}_p$.

Note that, for $\ell \neq p$, the only continuous additive homomorphism $\mathbf{Z}_\ell \to \mathbf{Z}_p$ or $\mathbf{Z}_\ell \to \mathbf{Z}/(p^k)$ maps 1_ℓ to 0. We deduce that for $\ell \neq p$, the morphism $\phi \circ \pi : \widehat{\mathbf{Z}} \to \mathrm{U} \to \mathrm{U}'$ maps 1_ℓ to $u'^0 = 1$. It follows that, for $n \geq p$, we have $\phi \circ \pi(1_p) = \phi \circ \pi(\sum_{\ell\leq n} 1_\ell)$. But $\lim_{n\to\infty} \phi \circ \pi(\sum_{\ell\leq n} 1_\ell) = \phi \circ \pi(1) = u'$. We deduce that

$$\phi \circ \pi(1_p) = u'.$$

Since $1_p \in \mathbf{Z}_p$, it is topologically p-nilpotent in $\mathbf{Z}_p$. Thus $u := \pi(1_p)$ is topologically p-nilpotent in U. We have $u' = \phi(u)$ where u is topologically p-nilpotent in U. This concludes the proof. □

Corollary **5.17.** — *Let $\phi : \mathrm{U} \to \mathrm{U}'$ be an epimorphism of finite groups and, for some prime p, denote by $\mathrm{U}^\dagger$, resp. $\mathrm{U}'^\dagger$ be the subgroup generated by elements of order a power of p.*

Then

$$\phi\left(\mathrm{U}^\dagger\right) = \mathrm{U}'^\dagger.$$

Proof. — This follows immediately from Lemma 5.16. □

Lemma **5.18.** — *For every $n \in \mathbf{Z}_{\geq 1}$, there exists $e(n)$ such that for every short exact sequence of finite groups*

$$1 \to \mathrm{N} \to \mathrm{G} \xrightarrow{\pi} \mathrm{H} \to 1$$

such that $\#\mathrm{N} \leq n$ and H is abelian, there exists an abelian subgroup $\mathrm{H}'' \leq \mathrm{G}$ such that

$$\forall\, h \in \mathrm{H}, \quad e(n) \cdot h \in \pi\left(\mathrm{H}''\right). \qquad (\mathbf{47})$$

Proof. — Let $\rho : G \to \mathrm{Aut}(N)$ be the adjoint action on its normal subgroup N. Then $G' = \ker \rho$ has index at most $\#\mathrm{Aut}(N) \leq (\#N)!$ and we may replace G with G' and H with $\pi(H)$, that is: we assume the extension of H by N is a central extension.

Because H is abelian, the commutator $n = [a, b]$ of any two $a, b \in G$ is in N. We have $ab = nba$, and, for $i \in \mathbf{Z}_{\geq 0}$,

$$a \cdot b^{i+1} = n \cdot b \cdot a \cdot b^i = \cdots = (n \cdot b)^{i+1} \cdot a.$$

Because N is central, we deduce

$$a \cdot b^i = n^i b^i \cdot a.$$

In, particular, when $i = \gamma := \#N$, we have

$$[a, b^\gamma] = 0. \tag{48}$$

Let $H'' \leq G$ be the subgroup generated by the $\{g^\gamma | g \in G\}$. Then $H' = \pi(H'')$ is the subgroup generated by $\{\gamma \cdot h | h \in H\}$. Thus (47) is satisfied.

By (48), the subgroup H'' has a generating sets made of elements which are central elements of G. Thus H'' is contained is the centre of G. In particular H'' is abelian.

Lemma 5.18 is proved. □

Corollary **5.19.** — *If* H *is generated by* k *elements, we have*

$$[H : \pi(H'')] \leq e(\#N)^k.$$

We used the following form of Goursat's Lemma.

Lemma **5.20** *(Goursat's Lemma).* — *Let* $U \leq G_1 \times G_2$ *be a subgroup, and* U_1, U_2 *be its projections, and define* $U_1' = U \cap (G_1 \times \{1\})$ *and* $U_2' = U \cap (\{1\} \times G_2)$. *Then* $(U_1 \times \{1\})/U_1'$ *and* $(\{1\} \times U_2)/U_2'$ *are isomorphic, and hence*

$$\left|(U_1 \times \{1\})/U_1'\right| = \left|(\{1\} \times U_2)/U_2'\right|.$$

6. Reductive norm estimates from residual stability

6.1. *Standing hypotheses.* — Let $F \leq G \leq GL(d)$ be reductive groups over $\mathbf{Q}_p$. The ultrametric absolute value is denoted by $| \, | : \mathbf{C}_p \to \mathbf{R}_{\geq 0}$ and the norm on ${\mathbf{C}_p}^d$ is denoted by

$$\left\| (v_i)_{i=1}^d \right\| = \max\{|v_1|; \ldots; |v_d|\}.$$

The $\mathbf{Q}_p$-algebraic group $GL(d)$ has a model $GL(d)_{\mathbf{Z}_p}$, which induces models $F_{\mathbf{Z}_p}$ and $G_{\mathbf{Z}_p}$ over $\mathbf{Z}_p$. We denote by $F_{\mathbf{F}_p}$ and $G_{\mathbf{F}_p}$ their special fibres, which are algebraic groups over $\mathbf{F}_p$.

We assume that, in the sense[12] of [52, §3.8]

$$F_{\mathbf{Z}_p} \text{ and } G_{\mathbf{Z}_p} \text{ are “hyperspecial” }. \tag{49}$$

6.1.1. *Some consequences.* — We review some constructions and some properties that hold under hypotheses (49), and will be needed later.

We consider a maximal torus $T \leq G_{\overline{\mathbf{Q}_p}}$, a basis $\mathbf{Z}^d \simeq X(T)$. We denote the set of weights of the representation $\rho : T \to G_{\overline{\mathbf{Q}_p}} \to GL(d)_{\overline{\mathbf{Q}_p}}$ by

$$\Sigma(\rho) \subseteq X(T)$$

and the weight decomposition of $V = \overline{\mathbf{Q}_p}^d$ under the action of T, by

$$\overline{\mathbf{Q}_p}^d = \bigoplus_{\chi\in\Sigma(\rho)} V_\chi \quad \text{where } V_\chi := \left\{ v \in \overline{\mathbf{Q}_p}^d \middle| \forall t \in T(\overline{\mathbf{Q}_p}), t\cdot v = \chi(t)\cdot v \right\}. \tag{50}$$

6.1.2. *Remark.* — For any other maximal torus T' there is a conjugation $t \mapsto gtg^{-1} : T \mapsto T'$ in $G(\overline{\mathbf{Q}_p})$. We deduce a set $\Sigma(T')$ corresponding to $\Sigma(T)$. The resulting set $\Sigma(T)$ does not depend on the choice of the conjugating element. The weight spaces V_χ in the decomposition (50) depend on T.

From[13] [52, §3.5] we know that the induced model $T_{\overline{\mathbf{Z}_p}}$ has good reduction, i.e. $T_{\overline{\mathbf{F}_p}}$ is a torus, and that we have

$$X(T) \simeq X(T_{\overline{\mathbf{F}_p}}).$$

This also implies, c.f. e.g. [49, Prop. 5] that (50) is compatible with integral structures:

$$\overline{\mathbf{Z}_p}^d = \bigoplus \Lambda_\chi \quad \text{where } \Lambda_\chi := \overline{\mathbf{Z}_p}^d \cap V_\chi;$$

and that we have a corresponding weight decomposition

$$\overline{\mathbf{F}_p}^d = \bigoplus \overline{V}_\chi \quad \text{where } \overline{V}_\chi := \Lambda_\chi \otimes \overline{\mathbf{F}_p}.$$

There is a Cartan decomposition [6, 4.4.3] (see also[14] [52, 3.3.3]), for $L/\mathbf{Q}_p$ a finite extension, and T_L a maximally[15] split torus of G/L,

$$G(L) = G_{\mathbf{Z}_p}(O_L) T_L(L) G_{\mathbf{Z}_p}(O_L). \tag{51}$$

[12] An equivalent property is that $F_{\mathbf{F}_p}$ and $G_{\mathbf{F}_p}$ are connected reductive algebraic groups.
[13] With $\Omega = \{x_0\}$ if $x_0 \in \mathcal{BT}(G/L)$ is the fixed point of $G_{\mathbf{Z}_p}(\mathbf{Z}_p)$.
[14] See [52, §3 and §3.3] for assumptions of [52, 3.3.3].
[15] A maximal torus containing a maximal split torus.

and consequently over $\overline{\mathbf{Q}_p}$, when T is a maximal torus,

$$G(\overline{\mathbf{Q}_p}) = G_{\mathbf{Z}_p}(\overline{\mathbf{Z}_p})T(\overline{\mathbf{Q}_p})G_{\mathbf{Z}_p}(\overline{\mathbf{Z}_p}). \tag{52}$$

6.2. *Main statement.* — The following theorem can be seen as a refined more precise version of the functoriality of local heights used in §5.2.

Theorem **6.1** *(Local relative stability estimates). — Under hypotheses from §6.1, let* $v, v' \in {\mathbf{Z}_p}^d$ *be non zero vectors, denote by* $\overline{v}, \overline{v'} \in {\mathbf{F}_p}^d$ *their reduction, and assume that*

(1) the orbits $G_{\mathbf{Q}_p} \cdot v$, $G_{\mathbf{Q}_p} \cdot v' \subseteq \mathbf{A}^d_{\mathbf{Q}_p}$ *are closed subvarieties;*
(2) the stabiliser groups $F_v := \mathrm{Stab}_G(v)$, $F_{v'} := \mathrm{Stab}_G(v')$ *satisfy*

$$F_v = F_{v'} = F;$$

and that

(1) the orbits $G_{\mathbf{F}_p} \cdot \overline{v}$, $G_{\mathbf{F}_p} \cdot \overline{v'} \subseteq \mathbf{A}^d_{\mathbf{F}_p}$ *are closed subvarieties;*
(2) the stabiliser groups $F_{\overline{v}} := \mathrm{Stab}_G(v)$, $F_{\overline{v'}} := \mathrm{Stab}_G(v')$ *satisfy, as group schemes,*[16]

$$F_{\overline{v}} = F_{\overline{v'}} = F_{\mathbf{F}_p}. \tag{53}$$

We define two functions $G(\mathbf{C}_p) \to \mathbf{R}$ *given by*

$$H_v : g \mapsto \max\{1; \|g \cdot v\|\} \text{ and } H_{v'} : g \mapsto \max\{1; \|g \cdot v'\|\}.$$

Then the functions $h_v = \log H_v$ *and* $h_{v'} = \log H_{v'}$ *satisfy*

$$h_v \leq C \cdot h_{v'} \text{ and } h_{v'} \leq C \cdot h_v, \tag{54}$$

in which $C = C(\Sigma(\rho))$ *depends only on the set of weights of* ρ *(cf. 6.1.2).*

In our proof, the quantity $C(\Sigma)$ will depend upon the choice of an invariant euclidean metric "in the root system" of G, and there are canonical choices of such metrics. The hypothesis (53) can be replaced by the weaker hypothesis in (61).

Several features that are important to our strategy.

- The quantity C only depends on the weights of ρ. Thus, when ρ comes from a representation defined over $\mathbf{Q}$, this C does not depend on the prime p.
- The inequality does not need an additive constant: we have

$$H_v \leq A \cdot {H_{v'}}^C$$

  with $A = 1$. Thus, when we multiply the inequalities over infinitely many primes, we don't accumulate an uncontrolled multiplicative factor $\prod_p A(p)$.
- The estimate (54) depends upon v only through its stabiliser group F. This is precisely information about the stabilisers that we deduce from Tate conjecture.

[16] It amounts to the property that $F_{\overline{v}}$ and $F_{\overline{v'}}$ are smooth.

6.3. *Proof.* — We use the notions and notations of §6.4. Because $G_{\mathbf{Z}_p}(\overline{\mathbf{Z}_p}) \leq GL(d, \overline{\mathbf{Z}_p})$ acts isometrically on $\overline{\mathbf{Z}_p}^d$, the functions h_v and $h_{v'}$ are left $G_{\mathbf{Z}_p}(\overline{\mathbf{Z}_p})$-invariant.

Choose an arbitrary $g \in G(\overline{\mathbf{Q}_p})$. It is sufficient to prove (55) (see below) with this element g, as the other inequality in (54) can be deduced after swapping v and v'.

Let $T \leq G$ be a maximal torus defined over $\overline{\mathbf{Q}_p}$. We endow A_T, defined in (56), with a canonical euclidean distance $d(\ ,\ ) = d_G(\ ,\ )$, invariant under $N_G(T)$ and depending only on G (using e.g. [4, LIE VI.12]). We denote $\Sigma(\rho)$ the set of weights of the action $T \to G \xrightarrow{\rho} GL(n)$. We denote $\gamma(\Sigma(\rho))$ the quantity from Prop. 8.3, which does not depend on the maximal torus T up to conjugation, and only on the weights of ρ.

Because $G_{\mathbf{Z}_p}$ is hyperspecial, there is a Cartan decomposition (51). Thus there are some $t' \in T(\overline{\mathbf{Q}_p})$ and $k \in G_{\mathbf{Z}_p}(\overline{\mathbf{Z}_p})$ such that

$$G_{\mathbf{Z}_p}(\overline{\mathbf{Z}_p}) \cdot g = G_{\mathbf{Z}_p}(\overline{\mathbf{Z}_p}) \cdot t$$

with $t = kt'k^{-1}$.

We may thus assume $g = t$. We may write, as in (60)

$$h_v \restriction_{T(\mathbf{C}_p)} = h_\mu \circ a_T \text{ and } h_{v'} \restriction_{T(\mathbf{C}_p)} = h_{\mu'} \circ a_T.$$

According to Proposition 8.2, we have

$$c\big(\Sigma(v)\big) \cdot d(a, C_\mu) \leq h_\mu(a) \text{ and } h_{\mu'}(a) \leq c'\big(\Sigma\big(v'\big)\big) \cdot d(a, C_{\mu'}).$$

Thanks to hypotheses of Theorem 6.1 we may apply Theorem 7.1 and (67) (we note that $h_v(t) = 0$ if and only if $t \cdot v \in \overline{\mathbf{Z}_p}^d$). Thus,

$$\begin{aligned} &\left\{ t \in T(\overline{\mathbf{Q}_p}) \;\middle|\; h_v(t) = 0 \right\} \\ &\quad = \left\{ t \in T(\overline{\mathbf{Q}_p}) \;\middle|\; tF \in (G/F)(\overline{\mathbf{Z}_p}) \right\} \\ &\quad = T(\overline{\mathbf{Q}_p}) \cap \big(G(\overline{\mathbf{Z}_p}) \cdot F(\overline{\mathbf{Q}_p})\big) \\ &\quad = \left\{ t \in T(\overline{\mathbf{Q}_p}) \;\middle|\; h_{v'}(t) = 0 \right\}. \end{aligned}$$

Let $C_\mu^{\mathbf{Q}}$ and $C_{\mu'}^{\mathbf{Q}}$ be defined as in Lemma 6.2. As the valuation group $\Gamma(\overline{\mathbf{Q}_p})$ is $\mathbf{Q}$, we deduce $C_\mu^{\mathbf{Q}} = C_{\mu'}^{\mathbf{Q}}$, and, by Lemma 6.2, we deduce

$$C := C_\mu = C_{\mu'}.$$

Applying Corollary 8.3, we conclude

$$\begin{aligned} h_{v'}(g) &= h_{v'}(t) = h_{\mu'} \circ a_T(t) \\ &\leq \gamma\big(\Sigma(\rho)\big) \cdot h_\mu \circ a_T(t) = \gamma\big(\Sigma(\rho)\big) \cdot h_v(t) = \gamma\big(\Sigma(\rho)\big) \cdot h_v(g). \end{aligned} \tag{55}$$

6.4. *Norms on toric orbits and the apartment.* — For a torus T over an ultrametric extension $L/\mathbf{Q}_p$, the associated "apartment" is defined as

$$A_T = A_{T/L} = Y(T/L) \otimes \mathbf{R} \simeq \mathrm{Hom}\big(X(T), \mathbf{R}\big) \tag{56}$$

where $Y(T) = Y(T/L) := \mathrm{Hom}(GL(1)_L, T)$ and $X(T) = X(T/L) := \mathrm{Hom}(T, GL(1)_L)$ are the group of cocharacters and characters, and are $\mathbf{Z}$-linear dual to each other.

Then the pairing

$$(t, \chi) \mapsto \log_p \big|\chi(t)\big| : T(L) \times X(T) \to \mathbf{R}.$$

induces a map

$$a_T : T(L) \to A_T, \tag{57}$$

Denote by $\mathbf{Z} \leq \Gamma_L := \log_p |L^\times| \leq \mathbf{R}$ the valuation group of L.

When T has a model over L which is a torus T_{O_L} over O_L, the map a_T factors as

$$T(L) \twoheadrightarrow \frac{T(L)}{T_{O_L}(O_L)} \xrightarrow{\sim} Y(T) \otimes \Gamma_L \hookrightarrow A_T.$$

For a character $\chi \in X(T)$ the function

$$\log_p |\chi| : T(L) \xrightarrow{\chi} L^\times \xrightarrow{|\,|} \mathbf{R}_{>0} \xrightarrow{\log_p} \mathbf{R}$$

passes to the quotient to $\frac{T(L)}{T_{O_L}(O_L)}$ and extends to a $\mathbf{R}$-linear form which we denote by

$$\omega_\chi : A_T \to \mathbf{R},$$

which is also the one deduced from $A_T \simeq \mathrm{Hom}(X(T), \mathbf{R})$.

Assume $T \leq GL(n)$ is a torus over L with good reduction: denoting the eigenspace decomposition of L^n for the action of T by

$$L^n = \sum_{\chi \in X(T)} V_\chi$$

we have (6.1.2, [49, Prop. 5])

$$O_L{}^n = \sum_{\chi \in X(T)} V_\chi \cap O_L{}^n. \tag{58}$$

It follows, denoting by $\|\ \|$ the standard norm on L^n, that, for $v \in L^n$,

$$\|v\| = \max\{0\} \cup \big\{\|v_\chi\| \mid \chi \in X(T)\big\}. \tag{59}$$

We denote by $\Sigma(\mathrm{T}) \subseteq \mathrm{X}(\mathrm{T})$ the set of weights for the action of T, and denote by

$$\Sigma(v) = \left\{\chi \in \mathrm{X}(\mathrm{T}) \mid v_\chi \neq 0\right\} \subseteq \Sigma(\mathrm{T}),$$

and, if $v \in \mathrm{O_L}^n$, we define

$$\overline{\Sigma}(v) = \left\{\chi \in \mathrm{X}(\mathrm{T}) \mid \|v_\chi\| = 1\right\} \subseteq \Sigma(v)$$

and a function $\mu : \Sigma(v) \to \mathbf{R}_{\leq 0}$ given by

$$\mu(\chi) = \log_p \|v_\chi\|.$$

The functions $\mathrm{H}_v : \mathrm{T}(\overline{\mathbf{Q}_p}) \to \mathbf{R}_{\geq 0}$ and $h_v = \log(\mathrm{H}_v)$ defined by

$$\mathrm{H}_v(t) = \max\left\{1; \|t \cdot v\|\right\}$$

can be computed from the formula

$$h_v = h_\mu \circ a_\mathrm{T} \quad \text{with } h_\mu(a) := \max\{0\} \cup \left\{\, \omega_\chi(a) + \mu(\chi) \,\middle|\, \chi \in \Sigma(v) \right\}. \tag{60}$$

Lemma **6.2.** — *Define*

$$\mathrm{C}_\mu = \left\{\, a \in \mathrm{A_T} \,\middle|\, h_\mu(a) = 0 \right\} \textit{ and } \mathrm{A}_\mathrm{T}^{\mathbf{Q}} = \mathrm{Y}(\mathrm{T}) \otimes \mathbf{Q} \subseteq \mathrm{A_T}.$$

Then $\mathrm{C}_\mu^{\mathbf{Q}} := \mathrm{C}_\mu \cap \mathrm{A}_\mathrm{T}^{\mathbf{Q}}$ *satisfies*

$$\mathrm{C}_\mu = \overline{\mathrm{C}_\mu^{\mathbf{Q}}}.$$

Lemma 6.2 holds because the convex set C_μ is constructed from affine forms $\omega_\chi + \mu(\chi)$ on $\mathrm{A_T}$ which are *defined over* $\mathbf{Q}$, with respect to the $\mathbf{Q}$-structure $\mathrm{A}_\mathrm{T}^{\mathbf{Q}}$.

7. Residual stability and *p*-adic Kempf-Ness theorem

The estimates of Th. 6.1 rely on the following result which we believe to be of independent interest. This is an analogue of a theorem of Kempf-Ness ([26, Th. 0.1 b)]) in the context [7] of p-adic Mumford's stability. It relies on a careful analysis of the reduction of models of homogeneous spaces given by the invariant theory [49] or of closed orbits in a linear representation.

Theorem **7.1** *(p-adic Kempf-Ness Theorem). — Let* $\mathrm{F}_{\mathbf{Z}_p} \leq \mathrm{G}_{\mathbf{Z}_p} \leq \mathrm{GL}(n)_{\mathbf{Z}_p}$ *be smooth reductive group schemes, such that* $\mathrm{F}_{\mathbf{Z}_p} \to \mathrm{G}_{\mathbf{Z}_p} \to \mathrm{GL}(n)_{\mathbf{Z}_p}$ *are closed immersions, and* $\mathrm{G}_{\mathbf{Z}_p}$ *is connected.*

Let $v \in {\mathbf{Z}_p}^n$. *Denote by* $\overline{v} \in {\mathbf{F}_p}^n$ *its reduction and assume that*

$$\mathrm{Stab}_{\mathrm{G}_{\mathbf{Q}_p}}(v) = \mathrm{F}_{\mathbf{Q}_p} \textit{ and } \dim\!\left(\mathrm{Stab}_{\mathrm{G}_{\mathbf{F}_p}}(\overline{v})\right) = \dim(\mathrm{F}_{\mathbf{F}_p}), \tag{61}$$

(using Krull dimensions) and assume that the orbits

$$(\mathbf{62})\qquad \mathrm{G}_{\mathbf{Q}_p}\cdot v\subseteq \mathbf{A}^n_{\mathbf{Q}_p} \text{ and } \mathrm{G}_{\mathbf{F}_p}\cdot \overline{v}\subseteq \mathbf{A}^n_{\mathbf{F}_p}$$

are closed.

Then, for all $g\in \mathrm{G}(\overline{\mathbf{Q}_p})$*, we have, denoting by* $\mathbf{Z}_p[\mathrm{G}/\mathrm{F}]:=\mathbf{Z}_p[\mathrm{G}]\cap \mathbf{Q}_p[\mathrm{G}]^{\mathrm{F}}$ *the algebra of* F*-invariant functions* $\mathrm{G}\to\mathbf{A}^1$ *defined over* $\mathbf{Z}_p$*,*

$$(\mathbf{63})\qquad g\cdot v\in \overline{\mathbf{Z}_p}^n \text{ if and only if } \forall f\in \mathbf{Z}_p[\mathrm{G}/\mathrm{F}], f(g)\in \overline{\mathbf{Z}_p}.$$

Moreover, $\mathrm{Spec}(\mathbf{Z}_p[\mathrm{G}/\mathrm{F}])$ *is smooth over* $\mathbf{Z}_p$*, and we have*

$$(\mathbf{64})\qquad \left(\mathrm{G}(\overline{\mathbf{Q}_p})\cdot v\right)\cap \overline{\mathbf{Z}_p}^n=\mathrm{G}(\overline{\mathbf{Z}_p})\cdot v.$$

Remarks. — Some of the assumptions can be rephrased as follows.

The $\mathbf{Q}_p$-algebraic groups F and G are reductive, the compact subgroups $\mathrm{F}_{\mathbf{Z}_p}(\mathbf{Z}_p)\leq \mathrm{F}(\mathbf{Q}_p)$ and $\mathrm{G}_{\mathbf{Z}_p}(\mathbf{Z}_p)\leq \mathrm{G}(\mathbf{Q}_p)$ are hyperspecial subgroups, and we have $\mathrm{F}_{\mathbf{Z}_p}(\mathbf{Z}_p)=\mathrm{F}(\mathbf{Q}_p)\cap \mathrm{GL}(n,\mathbf{Z}_p)$ and $\mathrm{G}_{\mathbf{Z}_p}(\mathbf{Z}_p)=\mathrm{G}(\mathbf{Q}_p)\cap \mathrm{GL}(n,\mathbf{Z}_p)$.

The property (62) is related to semi-stability and residual semi-stability of the vector v in the sense of [7].

In (61), the assumption on dimensions means that $\mathrm{Stab}_{\mathrm{G}_{\mathbf{F}_p}}(\overline{v})^{0,red}$ (the reduced subgroup of the neutral component) is equal to $(\mathrm{F}_{\mathbf{F}_p})^0$. Equivalently $\mathrm{Stab}_{\mathrm{G}_{\mathbf{F}_p}}(\overline{v})^0(\overline{\mathbf{F}_p})=\mathrm{F}^0(\overline{\mathbf{F}_p})$. This is implied by the stronger condition

$$(\mathbf{65})\qquad \mathrm{Stab}_{\mathrm{G}(\overline{\mathbf{F}_p})}(\overline{v})=\mathrm{F}(\overline{\mathbf{F}_p})$$

and the stronger one

$$(\mathbf{66})\qquad \mathrm{Stab}_{\mathrm{G}_{\mathbf{F}_p}}(\overline{v})=\mathrm{F}_{\mathbf{F}_p}\quad \text{as group schemes.}$$

Proof of Theorem 7.1. — Let $\mathrm{G}/\mathrm{F}=\mathrm{Spec}(\mathbf{Z}_p[\mathrm{G}/\mathrm{F}])$ and let $\mathrm{S}=\overline{\mathrm{G}_{\mathbf{Q}_p}\cdot v}^{Zar(\mathbf{A}^n_{\mathbf{Z}_p})}$ be the Zariski closure of the orbit $\mathrm{G}_{\mathbf{Q}_p}\cdot v\subseteq \mathbf{A}^n_{\mathbf{Q}_p}$ in the affine scheme $\mathbf{A}^n_{\mathbf{Z}_p}$. We denote by $g\mathrm{F}\in(\mathrm{G}/\mathrm{F})(\overline{\mathbf{Q}_p})$ the image of $g\in \mathrm{G}(\overline{\mathbf{Q}_p})$.

Then, for $g\in \mathrm{G}(\overline{\mathbf{Q}_p})$, we have $g\cdot v\in \overline{\mathbf{Z}_p}^n\Leftrightarrow g\cdot v\in \mathrm{S}(\overline{\mathbf{Z}_p})$, and we have $g\mathrm{F}\in(\mathrm{G}/\mathrm{F})(\overline{\mathbf{Z}_p})\Leftrightarrow \forall f\in\mathbf{Z}_p[\mathrm{G}/\mathrm{F}], f(g)\in\overline{\mathbf{Z}_p}$. Thus (63) is equivalent to

$$(\mathbf{67})\qquad g\mathrm{F}\in(\mathrm{G}/\mathrm{F})(\overline{\mathbf{Z}_p})\Leftrightarrow g\cdot v\in \mathrm{S}(\overline{\mathbf{Z}_p}).$$

Since $\mathrm{G}_{\mathbf{Z}_p}\leq \mathrm{GL}(n)_{\mathbf{Z}_p}$ is a smooth reductive subgroup scheme, we may apply Lem. 7.3. Using the assumptions (61) and (62), we may apply Lem. 7.4. Since $\mathrm{F}_{\mathbf{Z}_p}\leq \mathrm{G}_{\mathbf{Z}_p}$ is a smooth reductive subgroup scheme, we may apply Lem. 7.6 and 7.10. Thus, the map $\mathrm{G}/\mathrm{F}\to\mathrm{S}$ is integral. We may thus apply Prop. 7.15, and (67) follows. This proves (63).

According to Lem. 7.11, the scheme G/F is smooth over $\mathbf{Z}_p$.

We prove (64) by double inclusion. We trivially have $G(\overline{\mathbf{Z}}_p)\cdot v\subseteq(G(\overline{\mathbf{Q}}_p)\cdot v)\cap\overline{\mathbf{Z}}_p^{\,n}$. We prove the other inclusion. Let $g\in G(\overline{\mathbf{Q}}_p)$ be such that $g\cdot v\in S(\overline{\mathbf{Z}}_p)$. By (63), we have $gF\in(G/F)(\overline{\mathbf{Z}}_p)$. By Prop. 7.9, the map $\pi:G\to G/F$ satisfies $\pi(G(\overline{\mathbf{F}}_p))=(G/F)(\overline{\mathbf{F}}_p)$. As the morphism $\pi:G\to G/F$ is smooth, it is in particular flat. We may thus apply Prop. 7.14 with $y=gF$ and $\bar{y}=\overline{gF}\in(G/F)(\overline{\mathbf{F}}_p)\subseteq\pi(G(\overline{\mathbf{F}}_p))$. Thus there exists $g'\in G(\overline{\mathbf{Z}}_p)$ such that $\pi(g')=gF$. Thus $g\cdot v=g'\cdot v\in G(\overline{\mathbf{Z}}_p)\cdot v$. This proves the other inclusion.

This concludes the proof of Theorem 7.1.

7.1. *Summary of the section.* — In the rest of §7, we develop results used in the proof above.

In §7.2, we consider a smooth reductive group scheme $G_{\mathbf{Z}_p}\leq GL(n)_{\mathbf{Z}_p}$ and we study the Zariski closure $S=\overline{G_{\mathbf{Q}_p}\cdot v}^{Zar(\mathbf{A}^n_{\mathbf{Z}_p})}$ of an orbit of $G_{\mathbf{Q}_p}$.

We prove in Lemma 7.3 that the special fibre $S_{\mathbf{F}_p}$ is connected by reduction to the case $G=GL(1)$. Under an extra assumption of the form (61) and (62), we prove in Lemma 7.4 that the reduced subscheme $(S_{\mathbf{F}_p})^{red.}$ is a single $G_{\mathbf{F}_p}$-orbit.

In §7.3 we consider smooth reductive group schemes $F_{\mathbf{Z}_p}\leq G_{\mathbf{Z}_p}\leq GL(n)_{\mathbf{Z}_p}$, and study the scheme $G/F=\mathrm{Spec}(\mathbf{Z}_p[G/F])$. We prove that $(G/F)_{\mathbf{F}_p}$ is reduced and is a single $G_{\mathbf{F}_p}$-orbit. We use results from Seshadri's [49], we prove that G/F can be written in the form S as in §7.2, and we prove the assumptions of Lem. 7.4 are satisfied.

In §7.4, we consider S as in Lem. 7.4, and G/F as in 7.3. We prove that G/F is the normalisation of S. From §7.2 and §7.3, the morphism $G/F\to S$ is quasi-finite, and, by Zariski's main theorem, G/F is open in the normalisation $\widetilde{S}$ of S in G/F. We prove that $G/F\to\widetilde{S}$ is surjective by applying Lem. 7.4 to $\widetilde{S}$.

In §7.5, we prove that the morphisms $G\to G/F$ and $G/F\to\mathrm{Spec}(\mathbf{Z}_p)$ are smooth. This uses the "critère de lissité fibre par fibre" and, for the $\mathbf{F}_p$-fibre, uses §7.3.

In §7.6 and §7.7, we recall some consequences of flatness and integrality. The §§7.6-7.7 do not depend on the rest of §7.

7.2. *Orbit closure over* $\mathbf{Z}_p$.

Proposition **7.2.** — *Let* $G_{\mathbf{Z}_p}\leq GL(n)_{\mathbf{Z}_p}$ *be a closed smooth reductive subgroup scheme. Let* $A\subseteq\mathbf{Z}_p[G]$ *be a* $G(\mathbf{Z}_p)$*-stable subalgebra of finite type over* $\mathbf{Z}_p$.

Then there exists a closed immersion $G_{\mathbf{Z}_p}\to GL(m)_{\mathbf{Z}_p}$ *and a vector* $\lambda\in\mathbf{Z}_p^{\,m}$ *and a* $G_{\mathbf{Z}_p}$*-equivariant isomorphism*

$$\mathrm{Spec}(A)\to\overline{G_{\mathbf{Q}_p}\cdot\lambda}^{Zar(\mathbf{A}^m_{\mathbf{Z}_p})}.$$

Proof. — As A is of finite type over $\mathbf{Z}_p$, there exists a finite generating family $a_1,\ldots,a_N\in A$. As $G_{\mathbf{Z}_p}$ is flat over $\mathbf{Z}_p$, we have $\mathbf{Z}_p[G]\subseteq\mathbf{Q}_p[G]$. By [50, Prop.

2.3.6], there exists a finite dimensional $G(\mathbf{Q}_p)$-stable subspace $V \leq \mathbf{Q}_p[G]$ containing $\{a_1; \ldots; a_N\}$. As $A \otimes \mathbf{Q}_p \leq \mathbf{Q}_p[G]$ is $G(\mathbf{Q}_p)$-invariant, we may replace V by $V \cap A \otimes \mathbf{Q}_p$. Then $\Lambda := A \cap V$ is a $\mathbf{Z}_p$-lattice in V. As both A and V are $G(\mathbf{Z}_p)$-stable, Λ is $G(\mathbf{Z}_p)$-stable. There exists a finite $\mathbf{Z}_p$-linear basis $v_1, \ldots, v_d \in \Lambda$. As $a_1, \ldots, a_N \in A \cap V$, the family $v_1, \ldots, v_d \in \Lambda$ generates the algebra A. Let $\mathbf{Z}_p[X_1, \ldots, X_d]$ denote the polynomial algebra. The action of $G(\mathbf{Z}_p)$ on Λ induces an action of $G_{\mathbf{Z}_p}$ on $\operatorname{Spec}(\mathbf{Z}_p[X_1, \ldots, X_d])$. The morphism

$$X_i \mapsto v_i : \mathbf{Z}_p[X_1, \ldots, X_d] \to A$$

is surjective. It corresponds to a closed embedding $\operatorname{Spec}(A) \to \mathbf{A}^d_{\mathbf{Z}_p}$. By construction, this embedding is $G_{\mathbf{Z}_p}$ equivariant.

As $A \subseteq \mathbf{Q}_p[G]$, the corresponding morphism $G_{\mathbf{Q}_p} \to \operatorname{Spec}(A)$ is dominant. Let $a \in \operatorname{Spec}(A)$ be the image of $1 \in G(\mathbf{Q}_p)$. It follows that $\operatorname{Spec}(A)$ is the Zariski closure of the orbit of $G(\mathbf{Q}_p) \cdot a$. As $\operatorname{Spec}(A)$ is a closed subscheme of $\mathbf{A}^d_{\mathbf{Z}_p}$, we have $\operatorname{Spec}(A) = \overline{G_{\mathbf{Q}_p} \cdot a}^{Zar(\mathbf{A}^d_{\mathbf{Z}_p})}$.

The direct sum of the action of $G_{\mathbf{Z}_p}$ on $\mathbf{A}^d_{\mathbf{Z}_p}$ with the representation $G_{\mathbf{Z}_p} \to GL(n)_{\mathbf{Z}_p}$ is a closed immersion $G_{\mathbf{Z}_p} \to GL(d+n)_{\mathbf{Z}_p}$. The proposition follows for $m = d + n$, and for $\lambda = a \oplus 0$. □

7.2.1. *Connectedness and special fibre.*

Lemma **7.3.** — *Let* $G_{\mathbf{Z}_p} \leq GL(n)_{\mathbf{Z}_p}$ *be a closed smooth reductive subgroup scheme, let* $v \in \overline{\mathbf{Z}}_p^{\,n}$ *and let*

$$S = \overline{G_{\overline{\mathbf{Q}}_p} \cdot v}^{Zar(\mathbf{A}^n_{\overline{\mathbf{Z}}_p})} \tag{68}$$

be the schematic closure of $G_{\overline{\mathbf{Q}}_p} \cdot v \subseteq \mathbf{A}^n_{\overline{\mathbf{Z}}_p}$ *in* $\mathbf{A}^n_{\overline{\mathbf{Z}}_p}$. *Then* $S_{\overline{\mathbf{F}}_p}$ *is connected.*

We first treat the case $T = G = GL(1)$.

Proof. — If $S_{\overline{\mathbf{F}}_p}$ is a closed orbit of T, then it is connected, as it is the image of GL(1), which is connected. We will show that, otherwise, we can decompose $S_{\overline{\mathbf{F}}_p}$ under the form

$$S_{\overline{\mathbf{F}}_p}(\overline{\mathbf{F}}_p) = S^- \cup \{\overline{v_0}\} \cup S^+ \tag{69}$$

where each of S^- and S^+ is either empty or of the form $X = T(\overline{\mathbf{F}}_p) \cdot \overline{w}$ with $\{\overline{v_0}\} \in \overline{X}^{Zar}$. For every $\overline{w} \in \overline{\mathbf{F}}_p^{\,n}$, because $T = GL(1)$ is connected, so is $T \cdot \overline{w}$, and so is its Zariski closure. It follows that S^- and S^+ are contained in the connected component of $\overline{v_0}$, and finally that $S_{\overline{\mathbf{F}}_p}$ is the connected component of $\overline{v_0}$.

By Prop. 7.13, a point in $S(\overline{\mathbf{F}}_p)$ is of the form

$$\overline{x} \quad \text{with } x = t \cdot v \in \overline{\mathbf{Z}}_p^n \text{ and } t \in T(\overline{\mathbf{Q}}_p).$$

We identify $X(T) := \mathrm{Hom}(GL(1), GL(1))$ with $\mathbf{Z}$ and denote by

$$v = \sum_{k \in \mathbf{Z}} v_k$$

the eigendecomposition of v for the action of T. Then $x = t \cdot v = \sum_{k \in \mathbf{Z}} t^k \cdot v_k$, and, by (59),

$$\|x\| = \max\left\{|t|^k \cdot \|v_k\|\right\} \leq 1. \tag{70}$$

Define

$$c = \max_{k<0} \|v_k\|^{-1/k} \in [0; 1] \text{ and } c' = \min_{k>0} \|v_k\|^{-1/k} \in [1; +\infty].$$

(Observe that, if $v_k = 0$ for all $k < 0$, then $c = 0$, and, if $v_k = 0$ for all $k > 0$, then $c' = +\infty$.)

For $t \in T(\overline{\mathbf{Q}}_p)$ we have $t \cdot v \in \overline{\mathbf{Z}}_p^n$ if and only if $c \leq |t| \leq c'$. We define

$$T_- = \left\{ t \in T(\overline{\mathbf{Q}}_p) \;\middle|\; |t| = c \right\}$$
$$T_0 = \left\{ t \in T(\overline{\mathbf{Q}}_p) \;\middle|\; c < |t| < c' \right\}$$
$$T_+ = \left\{ t \in T(\overline{\mathbf{Q}}_p) \;\middle|\; |t| = c' \right\}.$$

and

$$S^- = \{ \overline{t \cdot v} \mid t \in T_- \} \text{ and } S^+ = \{ \overline{t \cdot v} \mid t \in T_+ \}.$$

and

$$v_- = \sum_{k<0} v_k \text{ and } v_+ = \sum_{k>0} v_k.$$

If $c = 0$, then $T_- = S^- = \emptyset$. Otherwise, let us pick $u \in T_-$. Assume first $c \neq c'$. Thus $c < c'$, and $u \cdot v^+ = v^+ = 0$, or else $c' < +\infty$, and, for some $u' \in T_+ \neq \emptyset$, we have

$$\|u \cdot v_+\| = \max_{k>0} c^k \cdot \|v_k\| < \max_{k \in \mathbf{Z}} c'^k \cdot \|v_k\| = \left\|u' \cdot v\right\| \leq 1.$$

We then have $\overline{u \cdot v_+} = 0$ and

$$\overline{w} := \overline{u \cdot v} = \overline{u \cdot v_- + v_0}$$

We then have

$$S^- = T(\overline{\mathbf{F}}_p) \cdot \overline{w}$$

Because the weights of $\overline{u\cdot v_-}$ are negative we have

$$\lim_{\bar{t}\to+\infty}\bar{t}\cdot\overline{u\cdot v_-+v_0}=\overline{u\cdot 0+v_0}=\overline{v_0},$$

where limits are understood in the sense of the Hilbert-Mumford criterion, as in [25, Lem. 1.3].

Thus $\{v_0\}\in\overline{\mathrm{S}^-}^{Zar}$. The case of S^+ is treated similarly and we have obtained (69) with the desired properties.

We now treat the remaining case $c=c'$. We then have

$$\mathrm{S}_{\overline{\mathbf{F}_p}}(\overline{\mathbf{F}_p})=\mathrm{S}^+=\mathrm{S}^-=\mathrm{T}(\overline{\mathbf{F}_p})\cdot\overline{v}.$$

(This is then a closed orbit of $\mathrm{T}_{\overline{\mathbf{F}_p}}$, as $\mathrm{S}_{\overline{\mathbf{F}_p}}$ is closed). □

We reduce Lemma 7.3 to the case of a torus $\mathrm{GL}(1)\simeq\mathrm{T}\leq\mathrm{G}$.

Proof. — It is enough to prove that for an arbitrary $\overline{x}\in\mathrm{S}(\overline{\mathbf{F}_p})$, this $\overline{x}$ and $\overline{v}$ belong to the same connected component of $\mathrm{S}_{\overline{\mathbf{F}_p}}$.

We may find $x\in\mathrm{S}(\overline{\mathbf{Q}_p})\cap\overline{\mathbf{Z}_p}^n$ with reduction $\overline{x}$. There exists $g\in\mathrm{G}(\overline{\mathbf{Q}_p})$ with $g\cdot v=x$. From a Cartan decomposition $g=k_1t'k_2\in\mathrm{G}(\overline{\mathbf{Z}_p})\cdot\mathrm{T}'(\overline{\mathbf{Q}_p})\cdot\mathrm{G}(\overline{\mathbf{Z}_p})$ as in (52), we construct $k:=k_1k_2\in\mathrm{G}(\overline{\mathbf{Z}_p})$ and a maximal torus $\mathrm{T}:=k_2{}^{-1}\mathrm{T}'k_2\leq\mathrm{G}$ and $t=k_2{}^{-1}t'k_2\in\mathrm{T}(\overline{\mathbf{Q}_p})$ with $k\cdot t=g$. The torus T has good reduction by 6.1.2. There exists $y:\mathrm{GL}(1)\to\mathrm{T}$ defined over $\overline{\mathbf{Q}_p}$ and $u\in\mathrm{T}(\overline{\mathbf{Z}_p})$ and $\lambda\in\overline{\mathbf{Q}_p}^\times$ with $y(\lambda)=u\cdot t$. Because $\mathrm{G}_{\mathbf{F}_p}$ is connected and $\mathrm{S}_{\overline{\mathbf{F}_p}}$ is $\mathrm{G}_{\overline{\mathbf{F}_p}}$-invariant, the orbit $\mathrm{G}_{\mathbf{F}_p}(\overline{\mathbf{F}_p})\cdot\overline{x}$ is connected and contained in $\mathrm{S}_{\overline{\mathbf{F}_p}}$. Thus $\overline{x}$ and $\overline{x'}=\overline{(k\cdot u)}^{-1}\cdot\overline{x}$ lie in the same connected component of $\mathrm{S}_{\overline{\mathbf{F}_p}}$. We may thus replace $\overline{x}$ by $\overline{x'}$ and x by $k\cdot u^{-1}\cdot x$, and g by $y(\lambda)$.

We have $x\in\mathrm{GL}(1)(\overline{\mathbf{Q}_p})\cdot v\cap\overline{\mathbf{Z}_p}^n$ and thus

$$\overline{v},\overline{x}\in\mathrm{S}_\mathrm{T}(\overline{\mathbf{F}_p})\quad\text{with }\mathrm{S}_\mathrm{T}:=\overline{\mathrm{T}\cdot v}^{Zar(\mathbf{A}^n_{\overline{\mathbf{Z}_p}})}\subseteq\mathrm{S}.$$

From the previous GL(1) case, S_T is connected. Thus $\overline{x}$ and $\overline{v}$ lie in the same connected component. □

Lemma **7.4.** — *In the situation of Lemma 7.3, we assume that*

(**71**) *the orbit* $\mathrm{G}_{\mathbf{F}_p}\cdot\overline{v}$ *is Zariski closed in* $\mathbf{A}^n_{\mathbf{F}_p}$

and denote by S' *the corresponding reduced subscheme of* $\mathbf{A}^n_{\mathbf{F}_p}$. *We assume furthermore that, using Krull dimension,*

(**72**) $$\dim\mathrm{Stab}_{\mathrm{G}_{\mathbf{Q}_p}}(v)=\dim\mathrm{Stab}_{\mathrm{G}_{\mathbf{F}_p}}(\overline{v})$$

Then

$$S' = (S_{\mathbf{F}_p})^{red.}.$$

Proof. — By construction S is the Zariski closure of its generic fiber. Hence, by Prop. 7.13, it is flat over $\mathbf{Z}_p$. According to Lemma 5.15 we have

$$\dim S_{\mathbf{F}_p} \leq \dim S_{\mathbf{Q}_p} \tag{73}$$

From (87), we have

$$\dim S_{\mathbf{Q}_p} = \dim G_{\mathbf{Q}_p} - \dim \mathrm{Stab}_{G_{\mathbf{Q}_p}}(v), \tag{74}$$

$$\dim S' \quad = \dim G_{\mathbf{F}_p} - \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{v}). \tag{75}$$

We deduce $\dim(S') \geq \dim(S_{\mathbf{F}_p})$. Because $S' \subseteq S_{\mathbf{F}_p}$ we have actually

$$\dim\big(S'\big) = \dim(S_{\mathbf{F}_p}).$$

Thus, S′ contains a generic point of one irreducible component of S, and thus[17] contains a non empty open subset of S. Because S′ is closed in $\mathbf{A}^n_{\mathbf{F}_p}$, it is closed in $S_{\mathbf{F}_p}$. Thus S′ contains a connected component of $S_{\mathbf{F}_p}$, and because $S_{\mathbf{F}_p}$ is connected and S′ is reduced,

$$S' = (S_{\mathbf{F}_p})^{red}. \qquad \square$$

Corollary **7.5.** — *Let* S *be as in Lemma 7.4. Let* T *be a* $\mathbf{F}_p$*-algebraic variety on which* $G_{\mathbf{F}_p}$ *acts, and let*

$$T \to S_{\mathbf{F}_p}$$

be a $G_{\mathbf{F}_p}$*-equivariant morphism. Then, for every* $t \in T(\overline{\mathbf{F}}_p)$*,*

$$\dim \mathrm{Stab}_{G_{\overline{\mathbf{F}}_p}}(t) \leq \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{v})$$

Proof. — Let $s \in S(\overline{\mathbf{F}}_p)$ be the image of t. We have $\mathrm{Stab}_{G_{\overline{\mathbf{F}}_p}}(t) \leq \mathrm{Stab}_{G_{\overline{\mathbf{F}}_p}}(s)$. By Lemma 7.4, we have $G(\overline{\mathbf{F}}_p) \cdot \overline{v} = S(\overline{\mathbf{F}}_p)$. Thus there exists $g \in G(\overline{\mathbf{F}}_p)$ such that $g \cdot \overline{v} = s$. Thus $\mathrm{Stab}_{G_{\overline{\mathbf{F}}_p}}(s) = g\,\mathrm{Stab}_{G_{\overline{\mathbf{F}}_p}}(\overline{v})g^{-1}$. We deduce

$$\dim \mathrm{Stab}_{G_{\overline{\mathbf{F}}_p}}(t) \leq \dim \mathrm{Stab}_{G_{\overline{\mathbf{F}}_p}}(s) = \dim \mathrm{Stab}_{G_{\overline{\mathbf{F}}_p}}(\overline{v}) = \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{v}). \qquad \square$$

[17] This S′ is of finite type over S and its image will be constructible.

7.3. *Reductive GIT quotient over* $\mathbf{Z}_p$. — Let $F_{\mathbf{Z}_p} \leq G_{\mathbf{Z}_p} \leq GL(n)_{\mathbf{Z}_p}$ be smooth reductive closed subgroup schemes. We define, following [49],

$$\mathbf{Z}_p[G/F] = \mathbf{Z}_p[G] \cap \mathbf{Q}_p[G]^{F(\mathbf{Q}_p)} \text{ and } G/F := \mathrm{Spec}\big(\mathbf{Z}_p[G/F]\big). \tag{76}$$

We have $\mathbf{F}_p[G_{\mathbf{F}_p}] \simeq \mathbf{Z}_p[G_{\mathbf{Z}_p}] \otimes \mathbf{F}_p$. Thus there exists an homomorphism $\mathbf{Z}_p[G_{\mathbf{Z}_p}]^F \otimes \mathbf{F}_p \to \mathbf{F}_p[G_{\mathbf{F}_p}]^{F_{\mathbf{F}_p}}$, and hence a morphism

$$G_{\mathbf{F}_p}/F_{\mathbf{F}_p} \to (G/F)_{\mathbf{F}_p}. \tag{77}$$

Lemma **7.6.** — *Let* $F_{\mathbf{Z}_p} \leq G_{\mathbf{Z}_p} \leq GL(n)_{\mathbf{Z}_p}$ *be smooth reductive closed subgroup schemes. Then* $(G/F)_{\mathbf{F}_p}$ *is reduced: we have*

$$(G/F)_{\mathbf{F}_p} = \big((G/F)_{\mathbf{F}_p}\big)^{red}. \tag{78}$$

The map

$$G_{\mathbf{F}_p}/F_{\mathbf{F}_p} \to G/F$$

from (77) *induces an isomorphism*

$$G_{\mathbf{F}_p}/F_{\mathbf{F}_p} \simeq \big((G/F)_{\mathbf{F}_p}\big)^{red} = (G/F)_{\mathbf{F}_p}. \tag{79}$$

We prove (78) in 7.3.1, and prove (79) in 7.3.3.

7.3.1. *Proof of (78) of Lemma 7.6.* — We deduce (78) from Corollary 7.8 below.

Lemma **7.7.** — *Let* A *be a flat* $\mathbf{Z}_p$*-algebra such that* $A \otimes \mathbf{F}_p$ *is reduced. Let* Γ *be a group of automorphisms of the* $\overline{\mathbf{Q}_p}$*-algebra* $A \otimes \overline{\mathbf{Q}_p}$. *Then* $(A^{\Gamma}) \otimes \mathbf{F}_p$ *is reduced.*

Proof of Lemma 7.7. — Let $a \in pA \cap A^{\Gamma}$. As A is flat over $\mathbf{Z}_p$, there exists a unique b such that $p \cdot b = a$. For $\gamma \in \Gamma$, we have $p \cdot \gamma(b) = \gamma(a) = a$. Thus $b = \gamma(b)$. Thus b is Γ-invariant. We deduce that $pA \cap A^{\Gamma} \subseteq pA^{\Gamma}$. Equivalently the map

$$A^{\Gamma}/pA^{\Gamma} \to A/pA$$

is injective. Thus $A^{\Gamma} \otimes \mathbf{F}_p$ is isomorphic to a subalgebra of the reduced algebra $A \otimes \mathbf{F}_p$. This implies that $A^{\Gamma} \otimes \mathbf{F}_p$ is reduced. □

Corollary **7.8.** — *Let* G *be a smooth linear group scheme over* $\mathbf{Z}_p$, *and let* $F \leq G$ *be a subgroup scheme. Then* $\mathbf{Z}_p[G]^F \otimes \mathbf{F}_p$ *is a reduced algebra.*

Proof of Corollary 7.8. — We apply Lem. 7.7 with $A = \mathbf{Z}_p[G]$ and $\Gamma = F(\overline{\mathbf{Q}_p})$. □

7.3.2. *Results from geometric invariant theory over* $\mathbf{Z}_p$. — Before proving (79) of Lem. 7.6, we recall some results from [49].

Proposition **7.9.** — *Let* $F_{\mathbf{Z}_p} \leq G_{\mathbf{Z}_p} \leq GL(n)_{\mathbf{Z}_p}$ *be smooth reductive closed subgroup schemes. Then, for every algebraically closed extension k of* $\mathbf{Q}_p$ *or* $\mathbf{F}_p$, *the map*

$$G_{\mathbf{Z}_p} \to G/F = \operatorname{Spec}\left(\mathbf{Z}_p[G]^F\right)$$

induces bijections

$$G(k)/F(k) \to (G/F)(k).$$

This is an application of [49, Prop. 6, §II.1, Prop. 9 Cor. 2 (i) §II.3] with $X = G$ and $Y = G/F$. In our case, every geometric point of X is "stable" in the sense of [49, §II.1, Def. 1], every geometric orbit is closed, and the closures of two distinct orbits have an empty intersection.

7.3.3. *Proof of (79) of Lemma 7.6.* — Before proving (79) of Lemma 7.6, let us recall some facts.

We have $\mathbf{Z}_p[G/F] \otimes \mathbf{Q}_p = \mathbf{Q}_p[G]^F$. Thus

$$(G/F)_{\mathbf{Q}_p} = G_{\mathbf{Q}_p}/F_{\mathbf{Q}_p}$$

and, by Lem. 7.12,

$$\dim(G/F)_{\mathbf{Q}_p} = \dim G_{\mathbf{Q}_p} - \dim F_{\mathbf{Q}_p}. \tag{80}$$

As $\mathbf{Z}_p[G/F] = \mathbf{Z}_p[G]^F \subseteq \mathbf{Q}_p[G]$ has no torsion, G/F is flat over $\mathbf{Z}_p$, and, by [51, Lemma 37.30.4. (0D4H)],

$$\dim\left((G/F)_{\mathbf{F}_p}\right) \leq \dim\left((G/F)_{\mathbf{Q}_p}\right) = \dim(G_{\mathbf{Q}_p}) - \dim(F_{\mathbf{Q}_p}). \tag{81}$$

By [51, Prop. 10.162.16. (0335)], we may apply [49, §II.4, Th. 2]. We have

$$\mathbf{Z}_p[G/F] \text{ is of finite type over } \mathbf{Z}_p. \tag{82}$$

Proof of Lemma 7.6. — By (82), we may apply Prop. 7.2 to the algebra $A := \mathbf{Z}_p[G]^F \subseteq \mathbf{Z}_p[G]$, and write

$$\operatorname{Spec}(A) \simeq \overline{G \cdot \lambda}^{Zar(\mathbf{A}^m_{\mathbf{Z}_p})}.$$

In order to apply Lem. 7.4 to $\overline{G \cdot \lambda}^{Zar(\mathbf{A}^m_{\mathbf{Z}_p})}$, we prove the assumptions (71) and (72).
By Prop. 7.9 the map (77) is surjective (on geometric points).
This implies that

$$G(\overline{\mathbf{F}}_p) \to (G_{\mathbf{F}_p}/F_{\mathbf{F}_p})(\overline{\mathbf{F}}_p) \to (G/F)(\overline{\mathbf{F}}_p)$$

is surjective. Thus $(G/F)_{\mathbf{F}_p}(\overline{\mathbf{F}_p})$ is a single $G(\overline{\mathbf{F}_p})$-orbit. By construction $(G/F)_{\mathbf{F}_p} \subseteq \mathbf{A}^m_{\mathbf{F}_p}$ is closed. Thus $G(\overline{\mathbf{F}_p}) \cdot \overline{\lambda}$ is a closed orbit. This proves (71).

We thus have

$$\dim(G/F)_{\mathbf{F}_p} = \dim G_{\mathbf{F}_p} \cdot \overline{\lambda} = \dim G_{\mathbf{F}_p} - \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{\lambda}). \tag{83}$$

By Prop. 7.9 the map (77) is injective (on geometric points) and thus quasi-finite. This implies

$$\begin{aligned}\dim\big((G/F)_{\mathbf{F}_p}\big) &\geq \dim(G_{\mathbf{F}_p}/F_{\mathbf{F}_p}) = \dim(G_{\mathbf{F}_p}) - \dim(F_{\mathbf{F}_p})\\ &= \dim(G_{\mathbf{Q}_p}) - \dim(F_{\mathbf{Q}_p}).\end{aligned}$$

Together with (81), this implies $\dim((G/F)_{\mathbf{F}_p}) = \dim(G_{\mathbf{Q}_p}) - \dim(F_{\mathbf{Q}_p})$. By (83), we have

$$\dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{\lambda}) = \dim(F_{\mathbf{F}_p}) = \dim(F_{\mathbf{Q}_p}).$$

This implies (72).

We may thus apply Lem. 7.4. This implies that

$$(G/F)_{\mathbf{F}_p} = \big((G/F)_{\mathbf{F}_p}\big)^{red.} = G_{\mathbf{F}_p} \cdot \overline{\lambda}.$$

From Lemma 7.12, we know that the morphisms

$$G_{\overline{\mathbf{Q}_p}} \to (G/F)_{\overline{\mathbf{Q}_p}} = G_{\overline{\mathbf{Q}_p}}/F_{\overline{\mathbf{Q}_p}} \text{ and } G_{\overline{\mathbf{F}_p}} \to (G/F)_{\overline{\mathbf{F}_p}} = G_{\overline{\mathbf{F}_p}} \cdot \overline{\lambda}$$

are flat morphisms of algebraic varieties. We know that $G_{\mathbf{Z}_p}$ is flat and smooth over $\mathbf{Z}_p$ by hypothesis.

By the "Critère de platitude par fibre", ([19, Part 2, §5.6, Lem. 5.21, p. 132] or [51, Lem. 37.16.3. (039D)]) the morphism

$$G \to G/F$$

is flat. Consider $\mathrm{Stab}_{G_{\mathbf{Z}_p}}(\lambda) \leq G_{\mathbf{Z}_p}$ a subgroup scheme over $\mathbf{Z}_p$. Let $\xi : \mathrm{Spec}(\mathbf{Z}_p) \to G/F$ be the morphism corresponding to $\lambda \in (G/F)(\mathbf{Z}_p)$. Then $\mathrm{Stab}_{G_{\mathbf{Z}_p}}(\lambda) \to \mathrm{Spec}(\mathbf{Z}_p)$ is the pullback of $\pi : G \to G/F$ by ξ. As $G \to G/F$ is flat, the morphism $\mathrm{Stab}_{G_{\mathbf{Z}_p}}(\lambda) \to \mathrm{Spec}(\mathbf{Z}_p)$ is flat. By (3) of Prop. 7.13, the generic fibre $F_{\mathbf{Q}_p} = \mathrm{Stab}_{G_{\mathbf{Q}_p}}(\lambda)$ is Zariski dense in $\mathrm{Stab}_{G_{\mathbf{Z}_p}}(\lambda)$. Thus

$$\mathrm{Stab}_{G_{\mathbf{Z}_p}}(\lambda) = F_{\mathbf{Z}_p}.$$

Thus $\mathrm{Stab}_{G_{\mathbf{F}_p}}(\lambda) = F_{\mathbf{F}_p}$, and is smooth. Thus

$$(G/F)_{\mathbf{F}_p} = G_{\mathbf{F}_p} \cdot \overline{\lambda} = G_{\mathbf{F}_p}/F_{\mathbf{F}_p}.$$

This proves (79) and concludes the proof of Lemma 7.6. □

7.4. *Normalisation and integrality.*

Lemma **7.10.** — *We keep the situation of Lemma 7.4.*
We assume that $G_{\mathbf{Q}_p}\cdot v\subseteq \mathbf{A}^n_{\mathbf{Q}_p}$ *is Zariski closed.*
We assume that there exists smooth reductive closed subgroup scheme $F_{\mathbf{Z}_p}\leq GL(n)_{\mathbf{Z}_p}$ *such that* $F_{\mathbf{Q}_p}=\mathrm{Stab}_{G_{\mathbf{Q}_p}}(v)$. *We use the notations of §7.3.*
Then the map $G/F\to S$ *identifies* G/F *with the normalisation of* S *(in its fraction field).*

The map $G/F\to S$ is obtained as follow. As $G_{\mathbf{Q}_p}\cdot v$ is closed, and by definition of F, we have $G_{\mathbf{Q}_p}/F_{\mathbf{Q}_p}\simeq S_{\mathbf{Q}_p}$, or equivalently $\mathbf{Q}_p[G]^F=\mathbf{Q}_p[S]$. As S is flat, we have $\mathbf{Z}_p[S]\subseteq \mathbf{Q}_p[S]$. As $v\in\mathbf{Z}_p{}^n$ and $G_{\mathbf{Z}_p}\leq GL(n)_{\mathbf{Z}_p}$ the map $g\to g\cdot v$ is defined over $\mathbf{Z}_p$. Thus $\mathbf{Z}_p[S]\subseteq \mathbf{Z}_p[G]$. We thus have,

$$\mathbf{Z}_p[S]\subseteq\mathbf{Z}_p[G]\cap\mathbf{Q}_p[G]^F=\mathbf{Z}_p[G/F].$$

Proof of Lemma 7.10. — Let us denote by $\widetilde{\mathbf{Z}_p[S]}$ the integral closure of $\mathbf{Z}_p[S]$ in $\mathbf{Z}_p[G/F]$, and define $\widetilde{S}=\mathrm{Spec}(\widetilde{\mathbf{Z}_p[S]})$. We denote by $\pi : G/F\to S$ the morphism from the statement of Lemma 7.10, and denote by

$$\widetilde{\pi} : G/F\to\widetilde{S}\text{ and }\nu:\widetilde{S}\to S$$

the morphisms given by the inclusions $\mathbf{Z}_p[S]\subseteq\widetilde{\mathbf{Z}_p[S]}\subseteq\mathbf{Z}_p[G/F]$.
Let us prove that Lemma 7.10 is the consequence of the following claims.

(1) The scheme G/F is normal.
(2) The morphism $\pi : G/F\to S$ is quasi-finite.
(3) The map $G(\overline{\mathbf{F}}_p)\to\widetilde{S}(\overline{\mathbf{F}}_p)$ is surjective.

By (2) and (82), we may apply Zariski's Main Theorem in the form [51, Theorem 29.55.1 (Algebraic version of Zariski's Main Theorem) (03GT)]. It follows that the map $\widetilde{\pi}$: $G/F\to\widetilde{S}$ is an open immersion. Recall that the morphism $(G/F)_{\mathbf{Q}_p}=G_{\mathbf{Q}_p}/F_{\mathbf{Q}_p}\to S_{\mathbf{Q}_p}$ is an isomorphism. As normalisation commutes with localisation, $(G/F)_{\mathbf{Q}_p}$ is normal, and $\widetilde{S}_{\mathbf{Q}_p}$ is the normalisation of $S_{\mathbf{Q}_p}$ in $(G/F)_{\mathbf{Q}_p}$. Thus, the morphism $(G/F)_{\mathbf{Q}_p}\to\widetilde{S}_{\mathbf{Q}_p}$ is an isomorphism. By (3) the map $G(\overline{\mathbf{F}}_p)\to(G/F)(\overline{\mathbf{F}}_p)\to\widetilde{S}(\overline{\mathbf{F}}_p)$ is surjective. It follows that the map $(G/F)(\overline{\mathbf{F}}_p)\to\widetilde{S}(\overline{\mathbf{F}}_p)$ is surjective. Thus, the open immersion $\widetilde{\pi} : G/F\to\widetilde{S}$ is surjective, and it is thus an isomorphism.

Let $\mathbf{Z}_p[S]^{norm}$ be the integral closure of $\mathbf{Z}_p[S]$ in $\mathbf{Q}_p(S)$. By (1), the scheme G/F is normal. That is, $\mathbf{Z}_p[G/F]$ is integrally closed in $\mathbf{Q}_p(G/F)$. It follows that $\mathbf{Z}_p[S]^{norm}\subseteq\mathbf{Z}_p[G/F]^{norm}=\mathbf{Z}_p[G/F]$. Thus $\mathbf{Z}_p[S]^{norm}$ is integral over $\mathbf{Z}_p[S]$ and $\mathbf{Z}_p[S]^{norm}\subset\mathbf{Z}_p[G/F]$. This proves $\mathbf{Z}_p[S]^{norm}\subseteq\widetilde{\mathbf{Z}_p[S]}$. As $S_{\mathbf{Q}_p}\simeq(G/F)_{\mathbf{Q}_p}$, we have $\mathbf{Q}_p[S]=\mathbf{Q}_p[G/F]$ and $\mathbf{Q}_p(S)=\mathbf{Q}_p(G/F)$. Thus $\widetilde{\mathbf{Z}_p[S]}$ is integral over $\mathbf{Z}_p[S]$ and $\widetilde{\mathbf{Z}_p[S]}\subset\mathbf{Q}_p(S)$. This proves $\widetilde{\mathbf{Z}_p[S]}\subseteq\mathbf{Z}_p[S]^{norm}$.

We conclude that $G/F = \widetilde{S}$ and that $\widetilde{S}$ is the normalisation of S in its fraction field.

We have proved Lemma 7.10 assuming the claims (1), (2) and (3). We now prove each of the claims.

Proof of claim (1). — By assumption, the scheme $G_{\mathbf{Z}_p}$ is smooth over the regular ring $\mathbf{Z}_p$. Thus $G_{\mathbf{Z}_p}$ is regular, and normal, and $\mathbf{Z}_p[G]$ is integrally closed in its fraction field $\mathbf{Q}_p(G)$.

Let $\lambda \in \mathbf{Q}_p(G/F)$ be integral over $\mathbf{Z}_p[G/F]$. Then λ is integral over $\mathbf{Z}_p[G]$: thus $\lambda \in \mathbf{Z}_p[G]$. We also have $\lambda \in \mathbf{Q}_p(G/F) \subseteq \mathbf{Q}_p(G)^F$. Therefore $\lambda \in \mathbf{Z}_p[G]^F$.

This proves that $\mathbf{Z}_p[G/F]$ is integrally closed in $\mathbf{Q}_p(G/F)$, that is, that G/F is normal. □

Proof of claim (2). — By assumption (72), and by definition of $F_{\mathbf{Q}_p}$, and as $F_{\mathbf{Z}_p}$ is smooth over $\mathbf{Z}_p$, we have

$$\dim\big(\mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{v})\big) = \dim\big(\mathrm{Stab}_{G_{\mathbf{Q}_p}}(v)\big) = \dim(F_{\mathbf{Q}_p}) = \dim(F_{\mathbf{F}_p}).$$

By (87), we have $\dim(S_{\mathbf{F}_p}) = \dim(G_{\mathbf{F}_p}) - \dim(\mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{v}))$. By Lem. 7.6, we have $(G/F)_{\mathbf{F}_p} = G_{\mathbf{F}_p}/F_{\mathbf{F}_p}$, and by (87), we have $\dim(G_{\mathbf{F}_p}/F_{\mathbf{F}_p}) = \dim(G_{\mathbf{F}_p}) - \dim(F_{\mathbf{F}_p})$.

Thus $\dim((G/F)_{\mathbf{F}_p}) = \dim(S_{\mathbf{F}_p})$.

Lem. 7.4 implies that the map $G_{\mathbf{F}_p} \to S_{\mathbf{F}_p}$ is dominant and that $S_{\mathbf{F}_p}$ is irreducible. Thus, the map $(G/F)_{\mathbf{F}_p} \to S_{\mathbf{F}_p}$ is dominant. Lem. 7.6 implies that $(G/F)_{\mathbf{F}_p}$ is irreducible. By [22, 3.22 (e), p. 95] (with $e = 0$), there is a non empty Zariski open subset $U \subseteq S_{\mathbf{F}_p}$, such that for $u \in U(\overline{\mathbf{F}}_p)$ the fibre of $(G/F)_{\mathbf{F}_p} \to S$ above u is of dimension 0. As, by Lem. 7.4, the action of $G(\overline{\mathbf{F}}_p)$ on $S(\overline{\mathbf{F}}_p)$ is transitive, we may take $U = S_{\mathbf{F}_p}$.

The claim follows. □

Proof of claim (3). — Our goal is to prove that $\widetilde{S}(\overline{\mathbf{F}}_p)$ is a single $G(\overline{\mathbf{F}}_p)$-orbit.

By [51, Prop. 10.162.16. (0335)], $\mathbf{Z}_p$ is universally Japanese. Thus the algebra $\mathbf{Z}_p[S]$, which is an integral domain of finite type over $\mathbf{Z}_p$, is Japanese. Namely, the integral closure $\widetilde{\mathbf{Z}_p[S]}$, of $\mathbf{Z}_p[S]$ in its field of fractions is of finite type over $\mathbf{Z}_p$.

We apply Proposition 7.2 and write accordingly $\widetilde{S} \simeq \overline{G\cdot\lambda}^{Zar(\mathbf{A}^m_{\mathbf{Z}_p})}$ for some $m \in \mathbf{Z}_{\geq 0}$.

We prove that the assumptions (71) and (72) of Lemma 7.4 are satisfied. The claim will then follow from Lemma 7.4.

By Cor. 7.5 for $T = (\widetilde{S}_{\mathbf{F}_p})^{red}$ and $t = \overline{\lambda}$, and by (72) for $v \in S$, we have

$$\dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{\lambda}) \leq \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{v}) = \dim \mathrm{Stab}_{G_{\mathbf{Q}_p}}(v) = \dim F_{\mathbf{Q}_p}.$$

As $\widetilde{S}$ is integral over S, we have $\dim \widetilde{S}_{\mathbf{F}_p} \leq \dim S_{\mathbf{F}_p}$. Thus

$$\dim G_{\mathbf{F}_p} - \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{\lambda}) = \dim(G_{\mathbf{F}_p}\cdot\overline{\lambda}) \leq \dim \widetilde{S}_{\mathbf{F}_p} \tag{84}$$

$$\leq \dim S_{\mathbf{F}_p} = \dim G_{\mathbf{F}_p} - \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{v})$$
$$= \dim G_{\mathbf{F}_p} - \dim F_{\mathbf{F}_p}.$$

Thus $\dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{\lambda}) = \dim F_{\mathbf{F}_p} = \dim F_{\mathbf{Q}_p} = \dim \mathrm{Stab}_{G_{\mathbf{Q}_p}}(\lambda)$.

This proves (72).

Assume by contradiction that $G_{\mathbf{F}_p} \cdot \overline{\lambda}$ is not closed in $\widetilde{S}_{\mathbf{F}_p}$. Then there exists $t \in \overline{G_{\mathbf{F}_p} \cdot \overline{\lambda}} \smallsetminus G_{\mathbf{F}_p} \cdot \overline{\lambda}$. Let $T = \overline{G_{\mathbf{F}_p} \cdot t}^{red}$. Then T is $G_{\mathbf{F}_p}$-stable. We have $\dim(T) = \dim(G_{\mathbf{F}_p} \cdot t) = \dim(G_{\mathbf{F}_p}) - \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(t)$, and $\dim(T) \leq \dim \overline{G_{\mathbf{F}_p} \cdot \overline{\lambda}} \smallsetminus G_{\mathbf{F}_p} \cdot \overline{\lambda} \leq \dim(G_{\mathbf{F}_p} \cdot \overline{\lambda}) - 1 < \dim G_{\mathbf{F}_p} - \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{\lambda})$. Thus $\dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(t) \geq \dim \mathrm{Stab}_{G_{\mathbf{F}_p}}(\overline{\lambda}) + 1$. This contradicts Cor. 7.5 for $t \in T = \overline{G_{\mathbf{F}_p} \cdot t}^{red}$.

This proves (71).

We may thus apply Lemma 7.4. We deduce $\widetilde{S}_{\mathbf{F}_p} = G_{\mathbf{F}_p} \cdot \overline{\lambda}$. This proves the claim. □

This concludes the proof of Lemma 7.10. □

7.5. *Flatness and smoothness.*

Lemma **7.11.** — *Let* $F_{\mathbf{Z}_p} \leq G_{\mathbf{Z}_p} \leq GL(n)_{\mathbf{Z}_p}$ *be a closed smooth reductive subgroup schemes. Then the maps*

$$G \to G/F \text{ and } G/F \to \mathrm{Spec}(\mathbf{Z}_p) \tag{85}$$

are smooth.

Proof. — We apply the "Critère de lissité par fibre" [16, 17.8.2] with $h = f \circ g : X \to Y \to S$ the morphisms $G \to G/F \to \mathrm{Spec}(\mathbf{Z}_p)$.

By assumption $G \to \mathrm{Spec}(\mathbf{Z}_p)$ is smooth, and in particular flat. For $s = \mathrm{Spec}(\mathbf{Q}_p)$, the morphism $X_s \to Y_s$ is $G_{\mathbf{Q}_p} \to (G/F)_{\mathbf{Q}_p} \simeq G_{\mathbf{Q}_p}/F_{\mathbf{Q}_p}$. By Lem. 7.6, for $\overline{s} = \mathrm{Spec}(\mathbf{F}_p)$, the morphism $g_{\overline{s}} : X_{\overline{s}} \to Y_{\overline{s}}$ is $G_{\mathbf{F}_p} \to (G/F)_{\mathbf{F}_p} \simeq G_{\mathbf{F}_p}/F_{\mathbf{F}_p}$. By assumption $F_{\mathbf{Z}_p}$ is smooth, and thus $F_{\mathbf{Q}_p}$ and $F_{\mathbf{F}_p}$ are smooth. By Lem. 7.12, we deduce that g_s and $g_{\overline{s}}$ are smooth morphisms.

Thus, by [16, 17.8.2], the morphism $G \to G/F$ is smooth.

Recall that $G \to \mathrm{Spec}(\mathbf{Z}_p)$ is smooth. Thus, by [16, 17.11.1 b) ⇒ a)], the morphism $G/F \to \mathrm{Spec}(\mathbf{Z}_p)$ is smooth on the image of $G \to G/F$.

Note that g_s and $g_{\overline{s}}$ are surjective. Thus $G/F \to \mathrm{Spec}(\mathbf{Z}_p)$ is smooth. □

Lemma **7.12.** — *Over a field* κ *let* $G \leq GL(n)_\kappa$ *be a algebraic subgroup (smooth closed group subscheme), and choose* $v \in \kappa^n$. *Then the map "orbit through* v*" map*

$$\omega : G \to G \cdot v \tag{86}$$

is flat, where $G \cdot v \simeq G/\operatorname{Stab}_G(v)$ *is locally closed and endowed with a reduced scheme structure.*

We have, using Krull dimension,

$$\dim(G \cdot v) = \dim(G) - \dim\big(\operatorname{Stab}_G(v)\big). \tag{87}$$

If $\operatorname{Stab}_G(v)$ *is smooth as a group scheme,*[18] *then* ω *is a smooth map and* $G \cdot v$ *is smooth (regular).*

Proof. — According to the Orbit Lemma [3, §I 1.8], the orbit $G \cdot v$ is locally closed.

Because $G \cdot v$ is reduced, by [51, Prop. 29.27.2 (052B)], there exists a non empty open subset $U \subseteq G \cdot v$ such that ω is flat above U. As U is G-invariant, the map ω is flat everywhere.

We deduce (87) from the flat case of [51, Lem. 29.28.2. (02JS)] and [51, Lem. 29.29.3. (02NL)], (using Krull dimension, cf. [51, Def. 5.10.1. (0055)]). (One can also find (87) in [20, p. 7]).)

Concerning smoothness, see [13, $\mathrm{VI_B}$ Prop. 9.2 (xii) (and V Th. 10.1.2)]. □

7.6. *Flatness and lifting of closed points.* — Let $\overline{\mathbf{Q}_p}$ be an algebraically closed algebraic extension of $\mathbf{Q}_p$, and denote by $\overline{\mathbf{Z}_p}$ the integral closure of $\mathbf{Z}_p$ in $\overline{\mathbf{Q}_p}$. We fix a ring homomorphism (the "reduction map")

$$r : \overline{\mathbf{Z}_p} \to \overline{\mathbf{F}_p},$$

and denote the induced morphism $(x_1, \ldots, x_n) \mapsto (r(x_1), \ldots, r(x_n))$ by

$$r^n : \mathbf{A}^n(\overline{\mathbf{Z}_p}) = \overline{\mathbf{Z}_p}^n \to \mathbf{A}^n(\overline{\mathbf{F}_p}) = \overline{\mathbf{F}_p}^n.$$

Proposition **7.13** *(cf. [15, Prop. 14.5.6, Rem. 14.5.7]).* — *Consider a Zariski closed subscheme* $X \subseteq \mathbf{A}^n_{\overline{\mathbf{Z}_p}}$ *of finite presentation, and denote by* X_{red} *its reduced subscheme.*

Then the following properties are equivalent.

(1) The scheme X_{red} *is flat over* $\overline{\mathbf{Z}_p}$.

(2) We have

$$X(\overline{\mathbf{F}_p}) = r^n\big(X(\overline{\mathbf{Z}_p})\big). \tag{88}$$

(3) The generic fibre $X_{\overline{\mathbf{Q}_p}}$ *is Zariski dense in* X.

(4) No irreducible component of X *is fully contained in* $\mathbf{A}^n_{\overline{\mathbf{F}_p}}$.

Proof. — This is a translation of [15, Prop. 14.5.6, Rem. 14.5.7].[19] □

[18] In practice $\dim \operatorname{Stab}_{\mathfrak{g}}(v) = \dim \operatorname{Stab}_G(v)$.

[19] Loc. cit. assumes the base is noetherian. Even though our base $\overline{\mathbf{Z}_p}$ is not noetherian, X can be obtained by base change from a model over a ring of integers O_K of a finite extension $K/\mathbf{Q}_p$. Then Rem. 14.5.7 of loc. cit. applies.

7.6.1. *Remark.* — If X is flat over $\overline{\mathbf{Z}}_p$, then $X \to \mathrm{Spec}(\overline{\mathbf{Z}}_p)$ is universaly open, by [14, Th. (2.4.6)], and thus $X^{red} \to \mathrm{Spec}(\overline{\mathbf{Z}}_p)$ is flat, by [15, Prop. 14.5.6, Rem. 14.5.7].

Proposition **7.14.** — *Let $\pi : X \to Y$ be a morphism of reduced affine schemes of finite presentation over $\overline{\mathbf{Z}}_p$.*

Assume that π is flat. Then,

$$\forall y \in Y(\overline{\mathbf{Z}}_p), \quad \overline{y} \in \pi\left(X(\overline{\mathbf{F}}_p)\right) \Leftrightarrow y \in \pi\left(X(\overline{\mathbf{Z}}_p)\right).$$

Proof. — If $y = \pi(x)$, then $\overline{y} = \pi(\overline{x})$. This proves one implication.

Let $\overline{y} = \pi(\overline{x})$ with $\overline{x} \in X(\overline{\mathbf{F}}_p)$. Let X_y be the fibre of X over y. Then $\overline{x} \in X_y(\overline{\mathbf{F}}_p)$. We have $x \in X_y(\overline{\mathbf{Z}}_p)$ if and only if: $x \in X(\overline{\mathbf{Z}}_p)$ and $\pi(x) = y$.

Let $Z = \mathrm{Spec}(\overline{\mathbf{Z}}_p)$ (viewed as a $\overline{\mathbf{Z}}_p$-scheme) and denote by $z \in Z(\overline{\mathbf{Z}}_p)$ the unique element. Let $\xi : Z \to Y$ be the morphism such that $\xi(z) = y$. Then $X_y \to Z$ is the pullback of $\pi : X \to Y$ by ξ. As π is flat, the morphism $X_y \to Z$ is flat. This implies, by Remark 7.6.1, that $X_y^{red.} \to Z^{red.} = Z$ is flat. As X is affine of finite presentation, the fibre subscheme X_y is affine of finite presentation. By Prop. 7.13, there exists $x \in X_y(\overline{\mathbf{Z}}_p)$ such that its reduction in $X_y(\overline{\mathbf{F}}_p)$ is $\overline{x}$. Thus $y \in \pi(X(\overline{\mathbf{Z}}_p))$. We proved the second implication. □

7.7. *Integrality and lifting.*

Proposition **7.15.** — *Let $\pi : X \to Y$ be a morphism of affine schemes over $\overline{\mathbf{Z}}_p$.*

Assume that π is integral. Then

$$\forall x \in X(\overline{\mathbf{Q}}_p), \quad x \in X(\overline{\mathbf{Z}}_p) \Leftrightarrow \pi(x) \in Y(\overline{\mathbf{Z}}_p).$$

Proof. — If $x \in X(\overline{\mathbf{Z}}_p)$, then $\pi(x) \in Y(\overline{\mathbf{Z}}_p)$. We prove the other implication.

We write $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ and $\pi^* : A \to B$ the morphism corresponding to π. Let $x \in X(\overline{\mathbf{Q}}_p)$, and let $\phi : B \to \overline{\mathbf{Q}}_p$ be the corresponding morphism.

Assume $\pi(x) \in Y(\overline{\mathbf{Z}}_p)$. Then $\phi \circ \pi^*(A) \subseteq \overline{\mathbf{Z}}_p$. As B is integral over A, the algebra $\phi(B)$ is integral over $\phi \circ \pi^*(A)$, and is integral over $\overline{\mathbf{Z}}_p$. As $\overline{\mathbf{Z}}_p$ is integrally closed in $\overline{\mathbf{Q}}_p$, we have $\phi(B) \subseteq \overline{\mathbf{Z}}_p$. Thus $x \in X(\overline{\mathbf{Z}}_p)$. □

8. Slopes weights estimates

We consider an integer $n \in \mathbf{Z}_{\geq 0}$ and an Euclidean distance $d(\ ,\ )$ on $\mathbf{R}^n$. The quantities c, c' and γ will implicitly also depend on $d(\ ,\ )$.

Lemma **8.1.** — *Let Σ be a finite set of linear forms on $\mathbf{R}^n$, let the function $h_\Sigma : \mathbf{R}^n \to \mathbf{R}_{\geq 0}$ be given by*

$$h_\Sigma(x) = \max_{\lambda \in \{0\} \cup \Sigma} \lambda(x),$$

and define $C = C(\Sigma) := \{x \in \mathbf{R}^n | h_\Sigma(x) = 0\}$.

Then there exist $c(\Sigma), c'(\Sigma) \in \mathbf{R}_{>0}$ *such that: for all* $x \in \mathbf{R}^n$ *satisfying*

$$d(x, C) = d(x, 0) \tag{89}$$

we have

$$c(\Sigma) \cdot d(0, x) \leq h_\Sigma(x) \leq c'(\Sigma) \cdot d(0, x). \tag{90}$$

Proof. — Let $x \in \mathbf{R}^n$ be arbitrary such that $d(x, C) = d(x, 0)$. Let $x' = \gamma \cdot x$ with $\gamma \in \mathbf{R}_{>0}$. For $v \in \mathbf{R}^n$ we have $h_\Sigma(\gamma \cdot v) = \gamma \cdot h_\Sigma(v)$. It follows that $\gamma \cdot C = C$. For every $c' \in C$, we have $c := c'/\gamma \in C$. We have

$$d\left(x', c'\right) = d(\gamma \cdot x, \gamma \cdot c) = \gamma \cdot d(x, c) \geq \gamma \cdot d(x, 0) = d(\gamma \cdot x, 0) = d\left(x', 0\right).$$

Thus $d(0, x') = d(C, x')$. Since the inequalities (90) are homogeneous, we may assume $x \neq 0$, and substituting x with $x' := x/d(0, x)$, we may assume that

$$d(0, x) = 1. \tag{91}$$

We can rewrite the condition (89) as

$$\forall c \in C, \quad d(0, x) \leq d(c, x). \tag{92}$$

The set $C^\perp := \{ x \in \mathbf{R}^n \mid d(0, x) = d(C, x) \}$ is an intersection of affine half-spaces, and is a closed set $C^\perp \subseteq \mathbf{R}^n$ (it is the *polar dual cone* to C). The intersection K of $C^\perp$ with the unit sphere $\{ x \in \mathbf{R}^n \mid d(0, x) = 1 \}$ is thus a compact set. We have $x \in K$, by (89) and (91).

The continuous function h_Σ has a minimum value and maximum value on the compact K, which we denote by

$$c(\Sigma) := \min_{k \in K} h_\Sigma(k) \text{ and } c'(\Sigma) := \max_{k \in K} h_\Sigma(k). \tag{93}$$

By definition, (90) is satisfied and we have $0 \leq c(\Sigma) \leq c'(\Sigma) < +\infty$. It will be enough to prove $0 < c(\Sigma)$.

Assume by contradiction that $c(\Sigma) = 0$ and choose $k \in K$ such that $h_\Sigma(k) = 0$. Then $k \in C$. From (92) for $x = c = k$, we deduce $d(0, k) \leq d(k, k) = 0$, contradicting (91). □

Proposition **8.2.** — *We keep the setting of Lemma 8.1.*

Let us fix a map $\mu : \Sigma \to \mathbf{R}_{\leq 0}$ *and let* $h_\mu : \mathbf{R}^n \to \mathbf{R}_{\geq 0}$ *be defined by*

$$h_\mu(x) = \max\left\{0; \max_{\lambda \in \Sigma} \lambda(x) + \mu(\lambda)\right\}.$$

Define $C = C_\mu = \{x \in \mathbf{R}^n | h_\mu(x) = 0\}$.

Then,

(**94**) $$\forall x \in \mathbf{R}^n, \quad c(\Sigma)\cdot d(\mathrm{C},x) \leq h_\mu(x) \leq c'(\Sigma)\cdot d(\mathrm{C},x).$$

Proof. — Define $\overline{\Sigma} := \{\lambda \in \Sigma | \mu(\lambda) = 0\}$ and $\overline{\mathrm{C}} = \{x \in \mathbf{R}^n | h_{\overline{\Sigma}}(x) = 0\}$. We have $h_{\overline{\Sigma}} \leq h_\mu$ and thus

$$\mathrm{C} \subseteq \overline{\mathrm{C}}$$

In a first step we prove (94) with the extra condition

(**95**) $$d(x,\mathrm{C}) = d(x,0) \quad (\text{that is: } \forall c \in \mathrm{C},\ d(x,c) \geq d(x,0)).$$

Let $\| \ \|$ denote the euclidean norms induced by $d(\ ,\ )$ on $\mathbf{R}^n$ and its dual. For $a \in \mathbf{R}^n$ we have

$$\max_{\sigma\in\Sigma} \sigma(a) \leq \|a\| \cdot \max_{\sigma\in\Sigma} \|\sigma\|.$$

Define $\mu_0 := \max\{\mu(\sigma) \,\big|\, \sigma \in \Sigma \smallsetminus \overline{\Sigma}\} < 0$. Then, if $a \in \mathbf{R}^n$ satisfies

(**96**) $$\|a\| \cdot \max_{\sigma\in\Sigma} \|\sigma\| \leq -\mu_0,$$

we have

(**97**) $$h_\mu(a) - h_{\overline{\Sigma}}(a) \leq 0, \text{ and thus } h_\mu(a) = h_{\overline{\Sigma}}(a).$$

Let us prove that $d(x,\overline{\mathrm{C}}) = d(x,0)$.

Proof. — We want to prove that for an arbitrary $b \in \overline{\mathrm{C}}$ we have

(**98**) $$d(x,b) \geq d(x,0).$$

Let $\lambda \in \mathbf{R}_{>0}$ be sufficiently small so that $a := \lambda \cdot b$ satisfies (96). We deduce from (97) that $a \in \mathrm{C}$, and from (95) that

$$d(x,a) \geq d(x,0).$$

Equivalently, denoting by $(\ ,\ )$ the euclidean scalar product, $(a,x) \leq 0$. It follows $(b,x) = \lambda \cdot (a,x) \leq 0$ and, equivalently, (98). □

Applying Lemma 8.1 to $\overline{\Sigma}$, we deduce (94) under the assumption (95).

Note that $\overline{\Sigma}$ is a subset of the finite set Σ. Thus, there are only finitely many possibilities for $\overline{\Sigma}$. As a consequence, we may assume, when applying Lemma 8.1, that $c(\overline{\Sigma})$ and $c'(\overline{\Sigma})$ do not depend on $\overline{\Sigma}$.

We now reduce the general case to the first step, by a translation of the origin of $\mathbf{R}^n$.

Let $x_0 \in \mathrm{C}$ be such that

$$d(x, \mathrm{C}) = d(x, x_0)$$

and define

$$\mu'(\lambda) = \lambda(x_0) + \mu(\lambda),$$

so that

$$h_{\mu'}(y) = h_\mu(y + x_0).$$

and $\mathrm{C}_{\mu'} = \mathrm{C}_\mu - x_0$. Thus

$$d(x - x_0, \mathrm{C}_{\mu'}) = d(x - x_0, \mathrm{C}_\mu - x_0) = d(x, \mathrm{C}_\mu) = d(x, x_0) = d(x - x_0, 0).$$

From $x_0 \in \mathrm{C}_\mu$, we deduce $h_{\mu'}(0) = h_\mu(x_0) = 0$ and $\forall \sigma \in \Sigma$, $\mu'(\sigma) \leq 0$.

Then (94) for x follows from the first step applied to $x - x_0$. □

Defining

$$\gamma(\Sigma_0) = \frac{\max\{\, c'(\Sigma) \mid \Sigma \subseteq \Sigma_0 \,\}}{\min\{\, c(\Sigma) \mid \Sigma \subseteq \Sigma_0 \,\}}$$

we deduce the following.

Corollary **8.3.** — *Let Σ_0 be a finite set of linear forms on $\mathbf{R}^n$. There exists $\gamma(\Sigma_0) \in \mathbf{R}_{>0}$ such that for $\Sigma, \Sigma' \subseteq \Sigma_0$ and $\mu : \Sigma \to \mathbf{R}_{\leq 0}$ and $\mu' : \Sigma' \to \mathbf{R}_{\leq 0}$ such that $\mathrm{C}_\mu = \mathrm{C}_{\mu'}$, we have*

$$\forall x \in \mathbf{R}^n, \quad h_\mu(x) \leq \gamma(\Sigma_0) \cdot h_{\mu'}(x).$$

Appendix A: On complete reducibility and closed orbit of Lie algebras over $\mathbf{F}_p$

Lemma **A.1.** — *Let V be a finite dimensional vector space over a field $\mathrm{K} \neq \mathbf{F}_2$, and let $\mathrm{G}' \leq \mathrm{H} \leq \mathrm{G} \leq \mathrm{GL}(\mathrm{V})$ be subgroups.*

Assume that V is semisimple as a representation of G and as a representation of G', and that

$$\mathrm{Z}_{\mathrm{GL(V)}}(\mathrm{G}) = \mathrm{Z}_{\mathrm{GL(V)}}\left(\mathrm{G}'\right).$$

Then V is semisimple as a representation of H and

$$\mathrm{Z}_{\mathrm{GL(V)}}(\mathrm{H}) = \mathrm{Z}_{\mathrm{GL(V)}}(\mathrm{G}).$$

Proof. — Let $W \leq V$ be a H-stable vector subspace. Then W is G'-invariant. By assumption, there exists $W' \leq V$ a supplementary G'-invariant subspace. Let π be the projector with image W and kernel W', and choose $\lambda \in K \smallsetminus \{0; 1\}$. Define $g = \pi + \lambda \cdot (1_V - \pi) \in \mathrm{End}(V)$. Then

$$g \in Z_{GL(V)}(G') = Z_{GL(V)}(G) \leq Z_{GL(V)}(H).$$

Thus g is H-equivariant, and thus $\pi = \frac{1}{1-\lambda} \cdot (g - \lambda \cdot 1_V)$ is H-equivariant. Thus $W' = \ker(\pi)$ is H-invariant, and is supplementary to W.

Since W is arbitraty, V is semisimple as a representation of H.

Finally, using $G' \leq H \leq G$, we get

$$Z_{GL(V)}(H) \leq Z_{GL(V)}(G') = Z_{GL(V)}(G) \leq Z_{GL(V)}(H). \qquad \square$$

A.1

We recall that (see [46, §3.2]) for an algebraic group G over a field k, an abstract subgroup $\Gamma \leq G(\overline{k})$ *is* G-*cr*, resp. a k-algebraic subgroup $H \leq G$ *is* G-*cr* if and only if: for every k-parabolic subgroup $P \leq G$ such that $\Gamma \leq P(\overline{k})$, resp. such that $H(\overline{k}) \leq P(\overline{k})$, there exists a Levi subgroup $L \leq P$ such that $\Gamma \leq L(\overline{k})$, resp. such that $H(\overline{k}) \leq L(\overline{k})$.

Lemma **A.2.** — *Let* G *be a reductive group over* $\overline{\mathbf{F}_p}$*, and let* $H \leq L \leq M \leq G$ *be* $\overline{\mathbf{F}_p}$*-algebraic subgroups.*

Assume that H *and* M *are* G-*cr and that*

$$Z_G(H) = Z_G(M).$$

Then, for every parabolic subgroup $P \leq G$ *defined over* $\overline{\mathbf{F}_p}$*, we have*

$$H \leq P \Leftrightarrow L \leq P \Leftrightarrow M \leq P.$$

Proof. — It suffices to prove that, for every parabolic subgroup $P \leq G$, we have

$$H \leq P \Rightarrow M \leq P.$$

Let $P \leq G$ be a parabolic subgroup such that $H \leq P$. Since H is G-cr, there exists, by definition, a Levi subgroup $Q \leq G$ such that $H \leq Q$. Let $Z(Q)$ be the center of Q. Then $Z(Q) \leq Z_G(Q) \leq Z_G(H) = Z_G(M)$. Thus $M \leq Z_G(Z(Q))$. Since Q is the Levi subgroup of a parabolic subgroup, we have $Z_G(Z(Q)) = Q$ ([50, proof of 16.1.1., p. 269]. We conclude $M \leq Z_G(Z(Q)) = Q \leq P$. $\square$

Corollary **A.3.** — *The subgroup* L *is* G-*cr.*

Proof. — Let $P \leq G$ be a parabolic subgroup such that $L \leq P$. We have $H \leq P$, and, since H is G-cr, there exists an opposite parabolic subgroup P' such that, with $Q = P \cap P'$, we have $H \leq Q$. By Lem. A.2, we have $M \leq P'$, and thus $L \leq M \leq P'$.

Thus L is contained in the Levi subgroup $Q \leq P$, and since P is arbitrary, L is, by definition, G-cr. □

Proposition **A.4.** — *Let* $M \leq G \leq GL(n)$ *be connected reductive algebraic subgroups defined over* $\mathbf{Q}$. *We denote by* $M_{\mathbf{F}_p} \leq G_{\mathbf{F}_p}$ *the* $\mathbf{F}_p$*-algebraic groups induced by the model* $GL(n)_{\mathbf{Z}}$. *There exists* $c(M, G)$ *such for every* $p \geq c(M, G)$ *the following holds.*

Let $V \leq M(\mathbf{F}_p)$ *and define* $V^\dagger$ *defined as in Rem. 2.1.3 (1). We also view* V *and* $V^\dagger$ *as* 0-*dimensional algebraic subgroups of* $M_{\mathbf{F}_p}$. *We assume*

$$V \leq V^\dagger \cdot Z(M_{\mathbf{F}_p}).$$

and $Z_{G_{\mathbf{F}_p}}(V) = Z_{G_{\mathbf{F}_p}}(M_{\mathbf{F}_p})$, *and that the action of* V *on* ${\mathbf{F}_p}^n$ *is semisimple.*

Let W *be the algebraic group associated to* $V^\dagger$ *by Nori ([32]), and* $L = W \cdot Z(M)$, *and let* $\mathfrak{l} \leq \mathfrak{g}_{\mathbf{F}_p}$ *be the Lie algebra of* L.

Then $L_{\overline{\mathbf{F}_p}}$ *is* $G_{\overline{\mathbf{F}_p}}$*-cr and* $\mathfrak{l} \otimes \overline{\mathbf{F}_p} \leq \mathfrak{g}_{\overline{\mathbf{F}_p}}$ *is* $G_{\overline{\mathbf{F}_p}}$*-cr (in the sense of [29]).*

Proof. — We know that there exists $c'(M, G)$ such that, for $p \geq c'(M, G)$, the algebraic groups $M_{\mathbf{F}_p}$ and $G_{\mathbf{F}_p}$ are reductive. By [46, Th. 5.3.], we have, for $p \geq h(G)$ and $p \geq h(GL(n))$, that $M_{\overline{\mathbf{F}_p}}$ and $G_{\overline{\mathbf{F}_p}}$ are $GL(n)_{\overline{\mathbf{F}_p}}$-cr and $M_{\overline{\mathbf{F}_p}}$ is $G_{\overline{\mathbf{F}_p}}$-cr.

Let n_p be the invariant $n(\Lambda)$ of [46, §5.2.] for the representation $\Lambda = {\mathbf{F}_p}^n$ of $G_{\mathbf{F}_p}$. The definition of n_p is given in terms of weights, and, for $p \gg 0$, does not depend on p. Thus $n_{\max} := \max_p n_p < +\infty$. Recall that the action of $V \leq M(\mathbf{F}_p)$ is semisimple. By (4) of §2.1.3, the $\overline{\mathbf{F}_p}$-linear action of V on $\Lambda \otimes \overline{\mathbf{F}_p}$ is semisimple. By [46, Th. 5.4.] this implies, for $p \geq n_{\max}$, that V is $G_{\overline{\mathbf{F}_p}}$-cr.

We apply Lem. A.2 and Cor. A.3 with $H = V$ (viewed as an $\overline{\mathbf{F}_p}$-algebraic group of dimension 0), $L = L_{\overline{\mathbf{F}_p}}$ and $M = M_{\overline{\mathbf{F}_p}}$ and $G = G_{\overline{\mathbf{F}_p}}$. It follows that $L_{\overline{\mathbf{F}_p}}$ is $G_{\overline{\mathbf{F}_p}}$-cr. According to [29, Th. 1(2)], it follows that $\mathfrak{l}_{\overline{\mathbf{F}_p}} := \mathfrak{l} \otimes \overline{\mathbf{F}_p}$ is $G_{\overline{\mathbf{F}_p}}$-cr.

This proves the Proposition with

$$c(M, G) = \max\left\{c'(M, G); h(G); h\big(GL(n)\big); n(V)\right\}.$$ □

Corollary **A.5.** — *Let* $x_1, \ldots, x_k \in \mathfrak{gl}(n, \mathbf{F}_p)$ *and* $y_1, \ldots, y_l \in \mathfrak{gl}(n, \mathbf{F}_p)$ *be such that* $V^\dagger$ *is generated by* $\{\exp(x_1); \ldots; \exp(x_k)\}$ *and* $\{y_1; \ldots; y_l\}$ *generates* $\mathfrak{z}(\mathfrak{m})$.

Then the orbit (as an algebraic variety over $\mathbf{F}_p$*) of* $(x_1, \ldots, x_k, y_1, \ldots, y_l)$ *in* $\mathfrak{gl}^{k+l}$ *under the action of* $G_{\mathbf{F}_p}$ *by conjugation, is Zariski closed.*

Proof. — Let $F = \mathrm{Hom}(\mathfrak{l}_{\overline{\mathbf{F}_p}}, \mathfrak{g}_{\overline{\mathbf{F}_p}})$ be the vector space of $\overline{\mathbf{F}_p}$-linear maps. Let $Z \subseteq F$ be the subset of Lie algebra homomorphisms. Then Z is a Zariski closed subvariety. Let $E : F \to \mathfrak{g}_{\overline{\mathbf{F}_p}}^{k+l}$ be the evaluation map $\phi \mapsto (\phi(x_1), \ldots, \phi(x_k), \phi(y_1), \ldots, \phi(y_l))$.

Since $\mathfrak{l}$ is generated by $x_1, \dots, x_k, y_1, \dots, y_l$, the map E is injective on Z.

We claim that $Z' := E(Z) \subseteq \mathfrak{g}_{\overline{\mathbf{F}}_p}^{k+l}$ is Zariski closed and that the map $E|_Z : Z \to Z'$ is a homeomorphism.

Proof of the claim. — For $\phi \in Z$ and $a, b \in \mathfrak{l}_{\overline{\mathbf{F}}_p}$, we have $\phi([a, b]) = [\phi(a), \phi(b)]$. We fix a basis of $\mathfrak{g}_{\overline{\mathbf{F}}_p}$. Observe that the map $[\ ,\ ] : \mathfrak{g}_{\overline{\mathbf{F}}_p}^2 \to \mathfrak{g}_{\overline{\mathbf{F}}_p}$ is polynomial. Therefore, the coordinates of $\phi([a, b])$ are polynomials in the coordinates of $\phi(a)$ and $\phi(b)$. The polynomials giving these coordinates depend only on the map $[\ ,\ ] : \mathfrak{g}_{\overline{\mathbf{F}}_p}^2 \to \mathfrak{g}_{\overline{\mathbf{F}}_p}$, and are independent of ϕ, a and b.

We deduce for instance that the coordinates of the $\phi([x_i, x_j])$ are polynomials in the coordinates of $(\phi(x_1), \dots, \phi(x_k), \phi(y_1), \dots, \phi(y_l))$. By induction, for every z in the Lie algebra generated by $\{x_1; \dots; x_k; y_1; \dots; y_l\}$, the coordinates of $\phi(z)$ are polynomials in the coordinates of

$$\big(\phi(x_1), \dots, \phi(x_k), \phi(y_1), \dots, \phi(y_l)\big).$$

Let $z_1, \dots, z_d$ be a basis of $\mathfrak{l}$, and let us identify $F \simeq \mathfrak{g}_{\overline{\mathbf{F}}_p}^d$. We apply the above for $z = z_i$. We deduce that the inverse map $Z' \to Z$ extends to a polynomial map $E^* : \mathfrak{g}_{\overline{\mathbf{F}}_p}^{k+l} \to \mathfrak{g}_{\overline{\mathbf{F}}_p}^d \simeq F$.

We note that $E^*|_{Z'} : Z' \to Z$ is the inverse of $E|_Z : Z \to Z'$. We deduce that $E|_Z : Z \to Z'$ is a homeomorphism.

Since Z is Zariski closed, we have $E^*(\overline{Z'}) \subseteq \overline{Z} = Z$. It follows that $E \circ E^*(\overline{Z'}) = Z'$. Recall that the restriction to Z' of the map $E \circ E^*$ is the identity on Z'. It follows that $E \circ E^*$ is the identity on $\overline{Z'}$. Thus $\overline{Z'} = E \circ E^*(Z') \subseteq Z'$. This implies the claim. □

Let $G \cdot \phi$ be the conjugacy class of the inclusion $\phi : \mathfrak{l} \to \mathfrak{g}$. According to [29, Th. 1(1)], this is a closed subvariety in F. As it is contained in Z, this is a closed subvariety of Z. As $E|Z : Z \to Z'$ is a homeomorphism, its image in Z' is thus a closed subvariety of Z'. This image is $E(G \cdot \phi) = G \cdot (x_1, \dots, x_k, y_1, \dots, y_l)$. Since Z' is closed in $\mathfrak{g}_{\overline{\mathbf{F}}_p}^{k+l}$, the conjugacy class $G \cdot (x_1, \dots, x_k, y_1, \dots, y_l)$ is closed in $\mathfrak{g}_{\overline{\mathbf{F}}_p}^{k+l}$. □

Appendix B: Consequences of uniform integral Tate property for ℓ-independence

Throughout App. B, we fix reductive **Q**-algebraic groups $M \leq G \leq GL(n)$.

The following is proved in §B.4.

Theorem **B.1.** — *Let* $U \leq M(\widehat{\mathbf{Z}})$ *be a compact subgroup satisfying Def. 2.1. Then*

$$\exists e \in \mathbf{Z}_{\geq 1}, \forall u \in U, \quad u^e \in \left(\prod_p U \cap M^{der}(\mathbf{Z}_p)\right) \cdot \big(U \cap Z(M)(\widehat{\mathbf{Z}})\big). \tag{99}$$

We deduce the following.

Corollary **B.2.** — *Let* U *be as in Theorem B.1, let* W *be the image of* U *by* $ab_{\mathrm{M}} : \mathrm{M}(\mathbf{A}_f) \to \mathrm{M}^{ab}(\mathbf{A}_f)$ *with* $\mathbf{A}_f = \widehat{\mathbf{Z}} \otimes \mathbf{Q}$, *let* $\mathrm{W}_p := \mathrm{W} \cap \mathrm{M}^{ab}(\mathbf{Q}_p)$ *and assume that*

$$\exists f, \forall w \in \mathrm{W}, \quad w^f \in \prod_p \mathrm{W}_p. \tag{100}$$

Then $\exists e \in \mathbf{Z}_{\geq 1}, \forall u \in \mathrm{U}, u^e \in \prod_p (\mathrm{U} \cap \mathrm{M}^{der}(\mathbf{Z}_p)) \cdot (\mathrm{U} \cap \mathrm{Z}(\mathrm{M})(\mathbf{Z}_p))$.

Proof of Corollary B.2. — For $d \in \mathbf{Z}_{\geq 1}$ and an abelian group A, let $[d] : \mathrm{A} \to \mathrm{A}$ denote the multiplication-by-d map $a \mapsto d \cdot a := a + \cdots + a$. For any group homomorphism $\mathrm{A} \to \mathrm{B}$ such that $\ker(\phi) \leq \ker([d])$, we have

$$\exists \psi : \phi(\mathrm{A}) \to \mathrm{A}, \quad \psi \circ \phi = [d] \tag{101}$$

because we can factor $[d] : \mathrm{A} \to \phi(\mathrm{A}) \to \mathrm{A}$ using

$$\mathrm{A} \to \phi(\mathrm{A}) \simeq \mathrm{A}/\ker(\phi) \to \mathrm{A}/\ker\big([d]\big) \simeq [d](\mathrm{A}) \hookrightarrow \mathrm{A}.$$

Let us denote by $\phi_p : \mathrm{Z}(\mathrm{M})(\mathbf{Z}_p) \to \mathrm{M}^{ab}(\mathbf{Q}_p)$ and $\phi = (\phi_p)_p : \mathrm{Z}(\mathrm{M})(\widehat{\mathbf{Z}}) \to \mathrm{M}^{ab}(\mathbf{Q} \otimes \widehat{\mathbf{Z}})$ the abelianisation maps. Because $\mathrm{Z}(\mathrm{M}) \cap \mathrm{M}^{der}$ is a finite algebraic group, we have

$$\left|\ker(\phi_p)\right| = \left|\mathrm{Z}\big(\mathrm{M}^{der}\big)(\mathbf{Z}_p)\right| \quad \text{divides } d := \left|\mathrm{Z}\big(\mathrm{M}^{der}\big)(\overline{\mathbf{Q}})\right| < +\infty.$$

We deduce $[d](\ker(\phi_p)) = \{1\}$. By (101) there exists $\psi_p : \phi_p(\mathrm{Z}(\mathrm{M})(\mathbf{Z}_p)) \to \mathrm{Z}(\mathrm{M})(\mathbf{Z}_p)$ such that $\psi_p \circ \phi_p = [d]$. Thus, with $\psi := (\psi_p)_p$, we have $\psi \circ \phi = [d]$.

The kernel of ab_{M} is $\mathrm{M}^{der}(\mathbf{A}_f)$. Thus, with $\mathrm{U}' := \mathrm{U} \cap \mathrm{Z}(\mathrm{M})(\widehat{\mathbf{Z}})$ and $\mathrm{W}' = \phi(\mathrm{U}')$, we have

$$ab_{\mathrm{M}}\left(\left(\prod_p \mathrm{U} \cap \mathrm{M}^{der}(\mathbf{Z}_p)\right) \cdot \left(\mathrm{U} \cap \mathrm{Z}(\mathrm{M})(\widehat{\mathbf{Z}})\right)\right) = ab_{\mathrm{M}}\big(\mathrm{U}'\big) = \mathrm{W}'.$$

Let e be as in Theorem B.1. Then we have $[e](\mathrm{W}) \leq \mathrm{W}'$ and thus $[e](\mathrm{W}_p) \leq [e](\mathrm{W}) \cap \mathrm{M}^{ab}(\mathbf{Q}_p) \leq \mathrm{W}'_p := \mathrm{W}' \cap \mathrm{M}^{ab}(\mathbf{Q}_p)$. We deduce from (100) that

$$[e \cdot f]\big(\mathrm{W}'\big) \leq [e \cdot f](\mathrm{W}) \leq [e]\left(\prod_p \mathrm{W}_p\right) \leq \prod_p \mathrm{W}'_p \leq \mathrm{W}'.$$

Applying ψ, we deduce

$$[e \cdot f \cdot d]\big(\mathrm{U}'\big) \leq \prod_p \psi_p\big(\mathrm{W}'_p\big) \leq [d]\big(\mathrm{U}'\big).$$

As $\psi_p(W'_p) \leq \psi(W) \cap Z(M)(\mathbf{Q}_p) \leq U'_p := U' \cap Z(M)(\mathbf{Q}_p)$, we have

$$[e \cdot f \cdot d](U') \leq \prod_p U'_p. \tag{102}$$

By Theorem B.1, for $u \in U$, we can write $u^e = (m_p)_p \cdot u'$ with $u' \in U'$ and $m_p \in U \cap M^{der}(\mathbf{Z}_p)$. By (102), we can write $u'^{e \cdot f \cdot d} = (u'_p)_p$ with $u'_p \in U'_p \leq U \cap Z(M)(\mathbf{Z}_p)$. We conclude

$$u^{e \cdot e \cdot f \cdot d} = \left(m_p^{e \cdot f \cdot d}\right)_p \cdot \left(u'_p\right)_p \in \prod_p \left(U \cap M^{der}(\mathbf{Z}_p)\right) \cdot \left(U \cap Z(M)(\mathbf{Z}_p)\right).$$

The Corollary B.2 follows. □

B.1

Let us recall the following fact. By [52, §3.9.1, p. 55] there exists $p_0(M, G)$ such that for every prime $p \geq p_0(M, G)$, the induced $\mathbf{F}_p$-algebraic groups $M_{\mathbf{F}_p} \leq G_{\mathbf{F}_p} \leq GL(n)_{\mathbf{F}_p}$ are reductive and that $Z(M)_{\mathbf{Z}_p}$ is smooth over $\mathbf{Z}_p$ and $Z(M)_{\mathbf{F}_p}$ is a torus.

B.2 Some properties over $\mathbf{F}_p$

Definition **B.3.** — *For a subgroup* $H \leq M(\mathbf{F}_p)$ *and* $C \in \mathbf{Z}_{\geq 0}$, *let* $Tate(H, C)$ *be the following property: for every subgroup* $H' \leq H$ *such that* $[H : H'] \leq C$,
(T1) the action of $H' \leq GL(n, \mathbf{F}_p)$ *on* $\mathbf{F}_p{}^n$ *is semisimple*
(T2) and $Z_{G_{\mathbf{F}_p}}(H') = Z_{G_{\mathbf{F}_p}}(M_{\mathbf{F}_p})$.

By (4) of §2.1.3, we have (T1) if and only if the action of H' on $\overline{\mathbf{F}_p}^n$ is semisimple. By [46, Th. 5.4], there exists $n(M, G)$ such that, for $p \geq n(M, G)$, we have (T1) ⇔ H' is $G_{\overline{\mathbf{F}_p}}$-cr ⇔ H' is $M_{\overline{\mathbf{F}_p}}$-cr, in the sense of §A.1. If $p \geq \max\{p_0(M, G); n(M, G)\}$, then, by [46, Th. 5.4], any H' satisfying (T1) and (T2) is not contained in a proper parabolic subgroup $P \leq M_{\overline{\mathbf{F}_p}}$.

Lemma **B.4.** — *Let* $p \geq p_1(M, G) := \max\{p_0(M, G); n(M, G)\}$, *let* $H \leq L \leq M(\mathbf{F}_p)$ *be subgroups and let* $C \in \mathbf{Z}_{\geq 0}$ *be such that the property* $Tate(H, C)$ *holds. Then the property* $Tate(L, C)$ *holds.*

Proof. — We prove $Tate(L, C)$, assuming $p \geq p_1(M, G)$ and $Tate(H, C)$. Let $L' \leq L$ be such that $[L : L'] \leq C$. Then $H' := H \cap L'$ verifies $[H : H'] \leq C$. As $H' \leq L' \leq M(\overline{\mathbf{F}}_p)$, we have $Z_{G_{\mathbf{F}_p}}(H') \geq Z_{G_{\mathbf{F}_p}}(L') \geq Z_{G_{\mathbf{F}_p}}(M_{\mathbf{F}_p})$. Using (T2), we deduce $Z_{G_{\mathbf{F}_p}}(M_{\mathbf{F}_p}) = Z_{G_{\mathbf{F}_p}}(L')$. This proves (T2) for L'. Note that $G_{\mathbf{F}_p}$ is reductive since $p \geq p_0(M, G)$ and that H' is $G_{\overline{\mathbf{F}_p}}$-cr since $p \geq n(M, G)$. We may apply Lemma A.2 for $H' \leq L' \leq M_{\overline{\mathbf{F}_p}} \leq G_{\overline{\mathbf{F}_p}}$. By Cor A.3, the group L is $G_{\overline{\mathbf{F}_p}}$-cr. As $p \geq n(M, G)$, this proves (T1) for L'. As L' is arbitrary, this proves $Tate(L, C)$. □

Lemma **B.5.** — *There exist $a_0, a_1 : \mathbf{Z}_{\geq 1} \to \mathbf{Z}_{\geq 1}$ such that the following holds. Let $p \geq a_1(n)$ and $\Gamma \leq \mathrm{M}(\mathbf{F}_p) \leq \mathrm{GL}(n, \mathbf{F}_p)$ be such that property $\mathrm{Tate}(\Gamma, a_0(n))$ holds, and define $\Gamma^\dagger$ as in §2.1.3(1).*

Then $\Lambda := \Gamma^\dagger \cdot (\Gamma \cap \mathrm{Z}(\mathrm{M})(\mathbf{F}_p))$ satisfies

$$[\Gamma : \Lambda] \leq a_0(n)$$

and $\Gamma^\dagger = \mathrm{H}(\mathbf{F}_p)^\dagger$ for a semisimple $\mathbf{F}_p$-algebraic group H.

The proof will use the following facts.

- By [48, No 137, §1 (p. 38), cf. Rem. 38.3 (p. 667) and No 164], there exists $c_2(n)$ such that for every prime p, there are at most $c_2(n)$ $\mathrm{GL}(n, \overline{\mathbf{F}_p})$-conjugacy classes of semisimple subgroups $\mathrm{S} \leq \mathrm{GL}(n)_{\overline{\mathbf{F}_p}}$, and they "come from characteristic 0" for $p \gg 0$.
- In particular, there exists $c_3(n)$ such that,

$$\text{(103)} \qquad \forall p \geq c_3(n), \quad p \nmid \#\mathrm{Z}(\mathrm{S})(\overline{\mathbf{F}_p}).$$

By Lang's Theorem [36, Prop. 6.3, p. 290], $|\mathrm{S}^{ad}(\mathbf{F}_p)/ad(\mathrm{S}(\mathbf{F}_p))| = |\mathrm{Z}(\mathrm{S})(\mathbf{F}_p)|$. For $p \geq c_3(n)$, this implies that $p \nmid \#\mathrm{S}^{ad}(\mathbf{F}_p)/ad(\mathrm{S}(\mathbf{F}_p))$. By Cor. 5.17 this implies that

$$\forall p \geq c_3(n), \quad ad\big(\mathrm{S}(\mathbf{F}_p)^\dagger\big) = ad\big(\mathrm{S}(\mathbf{F}_p)\big)^\dagger = \mathrm{S}^{ad}(\mathbf{F}_p)^\dagger.$$

- We have $\mathrm{A}ut(\mathrm{S})^0 = \mathrm{S}^{ad}$ and $\pi_0(\mathrm{A}ut(\mathrm{S}))$ is finite. In particular, there exists $c_4(n)$ such that we have $\#\pi_0(\mathrm{A}ut(\mathrm{S})) \leq c_4(n)$.
- We have $\#\mathrm{S}^{ad}(\mathbf{F}_p)/\mathrm{S}^{ad}(\mathbf{F}_p)^\dagger \leq 2^n$. ([32, (3.6(v)), p 270])

Proof. — According to [32, Th. B], there exists $c_1(n)$ such that for $p \geq c_1(n)$, we have $\Gamma^\dagger = \mathrm{S}(\mathbf{F}_p)^\dagger$ where $\mathrm{S} \leq \mathrm{GL}(n, \mathbf{F}_p)$ is a $\mathbf{F}_p$-algebraic group generated by connected unipotent subgroups. From Nori's construction, the algebraic group S is invariant under the adjoint action of Γ. We denote this action by

$$\alpha : \Gamma \to \mathrm{A}ut(\mathrm{S})(\mathbf{F}_p).$$

By definition $\ker(\alpha)$ is the centraliser $\mathrm{Z}_\Gamma(\mathrm{S})$.

By [46, Th. 5.3] and property (T1) for $\mathrm{H}' = \Gamma$, the algebraic group S is semisimple.

From now on, let us assume $p \geq a_1(n) := \max\{c_1(n); c_3(n)\}$. The facts preceding the proof imply that, for $\Lambda' := \overset{-1}{\alpha}(\mathrm{S}^{ad}(\mathbf{F}_p)^\dagger)$, we have

$$\big[\Gamma : \Lambda'\big] \leq \#\mathrm{A}ut(\mathrm{S})(\mathbf{F}_p)/ad\big(\mathrm{S}(\mathbf{F}_p)^\dagger\big) \leq c_5(n) := c_4(n) \cdot 2^n.$$

By construction $\ker(\alpha) \leq \Lambda'$ and $\mathrm{S}(\mathbf{F}_p)^\dagger \leq \Lambda'$. Note that $\Gamma^\dagger = \mathrm{S}(\mathbf{F}_p)^\dagger \to ad(\mathrm{S}(\mathbf{F}_p)^\dagger) = \mathrm{S}^{ad}(\mathbf{F}_p)^\dagger$ is surjective: for $\lambda \in \Lambda'$, there is $s \in \Gamma^\dagger$ such that $\alpha(s) = \alpha(\lambda)$. Equivalently $\lambda \in s \cdot \ker(\alpha)$. This proves

$$\Lambda' = \Gamma^\dagger \cdot \ker(\alpha).$$

From $\ker(\alpha) \leq \Lambda' \leq \Gamma$ we deduce $\ker(\alpha)^{\dagger} \leq \Gamma^{\dagger} \leq S$. Thus $\ker(\alpha)^{\dagger} \leq \ker(\alpha) \cap S \leq Z_{\Gamma}(S) \cap S \leq Z(S)$. By (103) we have $\ker(\alpha)^{\dagger} = \{1\}$. Let $n \mapsto d(n)$ be as in [47, §4]. By Jordan's theorem [47, §5.2.2], there exists an abelian subgroup $K \leq \ker(\alpha)$ such that $[\ker(\alpha) : K] \leq d(n)$. Thus $\Lambda'' := \Gamma^{\dagger} \cdot K$ satisfies $[\Gamma : \Lambda''] \leq a_0(n) := c_5(n) \cdot d(n)$.

Let us assume $\mathrm{T}ate(\Gamma, a_0(n))$. By (T2) for $H' := \Lambda''$, we have

$$Z(M_{\mathbf{F}_p}) \leq Z_{M_{\mathbf{F}_p}}(\Lambda'') = Z_{G_{\mathbf{F}_p}}(\Lambda'') \cap M_{\mathbf{F}_p} = Z_{G_{\mathbf{F}_p}}(M_{\mathbf{F}_p}) \cap M_{\mathbf{F}_p} = Z(M_{\mathbf{F}_p}).$$

As K is abelian and $K \leq \ker(\alpha)$, we have

$$\begin{aligned} K &\leq Z_{M_{\mathbf{F}_p}}(K) \cap Z_{M_{\mathbf{F}_p}}(S) = Z_{M_{\mathbf{F}_p}}(K \cdot S) \leq Z_{M_{\mathbf{F}_p}}(K \cdot \Gamma^{\dagger}) \\ &= Z_{M_{\mathbf{F}_p}}(\Lambda'') = Z(M_{\mathbf{F}_p}). \end{aligned}$$

Thus $\Lambda'' = \Gamma^{\dagger} \cdot K \leq \Lambda := \Gamma^{\dagger} \cdot (\Gamma \cap Z(M))$. Thus $[\Gamma : \Lambda] \leq [\Gamma : \Lambda''] \leq a_0(n)$. □

B.3 Independence properties over $\prod_p \mathbf{F}_p$

Let $R := \prod_p \mathbf{F}_p$, and let $W \leq M(R)$ be a closed subgroup (for the Tychonov product topology). We denote by $V(p) \leq M(\mathbf{F}_p)$ the image of W by the projection $M(R) \to M(\mathbf{F}_p)$, and we define $V = \prod_p V(p)$ and $W(p) := M(\mathbf{F}_p) \cap W \leq V(p)$.

Let $\mathrm{T}ate(W)$ be the following property: there exists $m = m_W : \mathbf{Z}_{\geq 1} \to \mathbf{Z}_{\geq 1}$ such that for every $c \in \mathbf{Z}_{\geq 1}$ and every $p \geq m(c)$ we have $\mathrm{T}ate(W(p), c)$. Then $\mathrm{T}ate(W)$ implies $\mathrm{T}ate(V)$ by Lemma B.4.

Proposition **B.6.** — *Assume* $\mathrm{T}ate(V)$ *as defined above. Then there exists* $p(W) \in \mathbf{Z}_{\geq 1}$ *such that,*

$$\forall p \geq p(W), \quad V(p)^{\dagger} \leq W \tag{104}$$

and

$$\exists e \in \mathbf{Z}_{\geq 1}, \forall w \in W, \quad w^e \in Y := \left(\prod_{p \geq p(W)} V(p)^{\dagger} \right) \cdot \big(W \cap Z(M)(R)\big). \tag{105}$$

Proposition B.9, used below, will be proved in §B.3.1.

Proof. — Let us define $R' := \prod_{\ell \neq p} \mathbf{F}_{\ell}$. We denote by $W^{R'}$ the image of W by the projection $GL(n, R) \to GL(n, R')$ and we define $W_{R'} := W \cap GL(n, R') \leq W^{R'}$. We apply Goursat's Lemma to $W \leq GL(n, R) = GL(n, \mathbf{F}_p) \times GL(n, R')$: there exists an isomorphism

$$V(p)/W(p) \to W^{R'}/W_{R'}.$$

Let us define $p(\mathrm{W}) := \max\{5; d(n); c(n)^{(n^2)}; a_1(n); m_{\mathrm{W}}(a_0(n))\}$ with $c(n)$ as in Prop. B.9. For $p \geq p(\mathrm{W})$, Lemma B.5 implies that we have $\mathrm{V}(p)^{\dagger} = \mathrm{H}(\mathbf{F}_p)^{\dagger}$ for a semisimple $\mathrm{H} \leq \mathrm{GL}(n)_{\mathbf{F}_p}$, and Prop. B.9 implies that the morphism $\mathrm{V}(p)^{\dagger} \to \mathrm{V}(p)/\mathrm{W}(p) \to \mathrm{W}^{\mathrm{R}'}/\mathrm{W}_{\mathrm{R}'}$ is trivial. Thus $\mathrm{V}(p)^{\dagger} \leq \mathrm{W}(p) \leq \mathrm{W}$. We proved (104).

Let us define $\Lambda(p) := \mathrm{V}(p)^{\dagger} \cdot (\mathrm{Z}(\mathrm{M})(\mathbf{F}_p) \cap \mathrm{V}(p))$ and $\Lambda := \prod_p \Lambda(p)$. For $p \geq c_0(n) := \max\{m(a_0(n)); a_1(n)\}$ we apply Lemma B.5 for $\Gamma = \mathrm{V}(p)$ and deduce $[\mathrm{V}(p) : \Lambda(p)] \leq a_0(n)$. For $p < c_0(n)$ we have $[\mathrm{V}(p) : \Lambda(p)] \leq \#\mathrm{GL}(n, \mathbf{F}_p) \leq c_0(n)^{n^2}$. Note that $\Lambda(p) \leq \mathrm{V}(p)$ is a normal subgroup. We deduce, with $e' := \max\{a_0(n); c_0(n)^{n^2}\}$, that $\forall p, \forall v \in \mathrm{V}(p)$, $v^{e'!} \in \Lambda(p)$. It follows that

$$\forall w \in \mathrm{W}, \quad w^{e'!} \in \mathrm{W}' := \mathrm{W} \cap \Lambda. \tag{106}$$

Because $\mathrm{V}(p)^{\dagger} \leq \mathrm{W}$ and $\mathrm{V}(p)^{\dagger} \leq \Lambda(p) \leq \Lambda$, we have $\mathrm{V}(p)^{\dagger} \leq \mathrm{W}'$. As Λ and W are closed, we deduce that

$$\prod_{p \geq p(\mathrm{W})} \mathrm{V}(p)^{\dagger} \leq \mathrm{W}'. \tag{107}$$

Let $\mathrm{X} := \ker(\mathrm{W}' \to \prod_{p \leq p(\mathrm{W})} \mathrm{GL}(n, \mathbf{F}_p))$. Then

$$\left[\mathrm{W}' : \mathrm{X}\right] \leq e'' := \# \prod_{p \leq p(\mathrm{W})} \mathrm{GL}(n, \mathbf{F}_p) \leq \left(p(\mathrm{W})!\right)^{(n^2)} < +\infty. \tag{108}$$

Let $x \in \mathrm{X}$ be arbitrary and let $x = (\lambda_p)_p$ be its coordinates in $\prod_p \Lambda(p)$. For $p \geq p(\mathrm{W})$ we can write $\lambda_p = v_p \cdot z_p$ with $v_p \in \mathrm{V}(p)^{\dagger}$ and $z_p \in \mathrm{Z}(\mathrm{M})(\mathbf{F}_p)$. For $p < p(\mathrm{W})$ we define $v_p = z_p = 1$. Let $v := (v_p)_p$ and $z = (z_p)_p$. By (107) we have $v \in \mathrm{X}$. Thus $z = x \cdot v^{-1} \in \mathrm{X}$. Thus $\mathrm{X} \leq \mathrm{Y}$. Together with (106) and (108), this implies (105) with $e := e'! \cdot e''!$. □

B.3.1

We will prove Prop. B.9, using arguments from [47].

Proposition **B.7.** — *Let* $\mathrm{H} \leq \mathrm{GL}(n)_{\mathbf{F}_p}$ *be a semisimple algebraic group. If* $5 \leq p$, *then* $\Gamma := \mathrm{H}(\mathbf{F}_p)^{\dagger}$ *has no non-trivial abelian quotient.*

Proof. — Let us write $\mathrm{H} = \mathrm{H}_1 \cdot \ldots \cdot \mathrm{H}_c$ as an almost direct product of its quasi-simple factors $\mathrm{H}_1, \ldots, \mathrm{H}_c$. For $p \geq 5$, [21, Th. 2.2.7, p 38] implies that, for $i = 1, \ldots, c$, the group $\Gamma_i := \mathrm{H}_i(\mathbf{F}_p)^{\dagger}$ is quasisimple: in particular Γ_i is its own derived subgroup. We deduce that $\Gamma_1 \cdot \ldots \cdot \Gamma_c$ is its own derived subgroup. By [21, Prop 2.2.11, p. 40], we have $\Gamma = \Gamma_1 \cdot \ldots \cdot \Gamma_c$. □

Lemma **B.8.** — *Let* $\mathrm{G} = \mathrm{G}_1 \times \cdots \times \mathrm{G}_c$ *be a product of groups, let* $\mathrm{H} \leq \mathrm{G}$ *be a subgroup, and let* $q : \mathrm{H} \to \Sigma$ *be a simple quotient. Then there exists* $i \in \{1; \ldots; c\}$ *and* $\mathrm{H}_i \leq \mathrm{G}_i$ *such that* Σ *is a quotient of* H_i.

Proof. — We may assume $c = 2$ by induction. The projection $p : G_1 \times G_2 \to G_2$ induces a short exact sequence $1 \to H_1 \to H \to H_2 \to 1$ where $H_1 := H \cap G_1$ and $H_2 := p(H)$. Because Σ is simple, we have $q(H_1) = \Sigma$ or $q(H_1) = \{1\}$. In the first case, we may take $i = 1$. In the other case, we can factor $H \to p(H) \to \Sigma$, and we take $i = 2$. □

Proposition **B.9.** — *Let* $H \leq GL(n)_{\mathbf{F}_p}$ *be a semisimple algebraic group, let* $\Gamma := H(\mathbf{F}_p)^\dagger$ *and assume* $p > \max\{d(n); c(n)^{(n^2)}; 5\}$ *with* $c(n)$ *be as in [47, §6.3, Th. 4]. Let* $G = GL(n, \mathbf{F}_{\ell_1}) \times \cdots \times GL(n, \mathbf{F}_{\ell_c})$ *where* $p \notin \{\ell_1; \ldots; \ell_c\}$. *Let* S/N *be a quotient of a subgroup* $S \leq G$ *and* $\phi : \Gamma \to S/N$ *be an homomorphism (of abstract groups). Then* $\phi(\Gamma) = \{1\}$.

Proof. — Let $\Lambda := \phi(\Gamma)$ and assume by contradiction that $\Lambda \neq 1$. Then there exists a simple quotient $\Lambda \to \Sigma$. Lemma B.8 implies that Σ is a subquotient of $GL(n, \mathbf{F}_\ell)$ for some $\ell \in \{\ell_1; \ldots; \ell_c\}$.

According to Jordan's theorem [47, §4], there is a sequence of normal subgroups $\Lambda^\ddagger \triangleleft \Lambda' \triangleleft \Lambda$ with $\#\Lambda/\Lambda' \leq d(n)$ and $\Lambda'/\Lambda^\ddagger$ abelian, where $\Lambda^\ddagger$ denotes he subgroup generated by $\{\lambda \in \Lambda | \exists k \in \mathbf{Z}_{\geq 1} \lambda^{(\ell^k)} = 1\}$.

As $\Gamma = \Gamma^\dagger$, every non-trivial quotient $\Gamma \to \Gamma/N$ satisfies $p \mid \#\Gamma/N$. As $p > c(n)^{(n^2)}$, we have $\ell > c(n)$. As $p > d(n)$, we have $\Lambda = \Lambda'$. By Cor. B.7, Γ has no non trivial abelian quotient. Thus $\Lambda = \Lambda^\ddagger$. We deduce that every non-trivial quotient $\Lambda \to \Lambda/N$ satisfies $\ell \mid \#\Lambda/N$.

According to [47, §6, Th. 6.4], Σ is in Σ_ℓ (in the sense of [47, §6]). According to [47, §6, Lem. 6.1], Σ is abelian or in Σ_p. According to Prop. B.7, Σ is not abelian. By assumption $p \neq \ell$ and thus, by [47, §6.4], Σ cannot be in both Σ_p and Σ_ℓ. This leads to a contradiction. □

B.4 Proof of Theorem B.1

Let U be as in Theorem B.1. Let us write $U_p := U \cap M(\mathbf{Z}_p)$. Let $\pi_R : GL(n, \widehat{\mathbf{Z}}) \to GL(n, R)$ be the map induced by $\widehat{\mathbf{Z}} \to R := \prod_p \mathbf{F}_p$, and let us define $U(R) := \pi_R(U)$. Def. 2.1(2) implies $Tate(\pi_R(\prod_p U_p))$ in the sense of §B.3. Using Lemma B.4 we deduce $Tate(U(R))$. By Prop. B.6 for $W = U(R)$ and $p_0 \geq p(W)$ and U' the inverse image of $\prod_{p \geq p_0} V(p)^\dagger \cdot (U(R) \cap Z(M)(R))$ in U, we have $\exists e \in \mathbf{Z}_{\geq 1} \forall e \in U$, $u^e \in U'$. It is thus enough to prove (99) for $u \in U'$. Furthermore, for every $u \in U_p$ we have $u^e \in U'_p := U' \cap M(\mathbf{Q}_p)$. By Cor. B.11, the group U' satisfies Def. 2.1. We may thus substitute U with U' in Theorem B.1, and we can thus assume from now on that

$$U(R) = \prod_{p \geq p_0} V(p)^\dagger \cdot \big(U(R) \cap Z(M)(R)\big). \tag{109}$$

The Theorem B.1 will be a consequence of (118) and (127).

Lemma **B.10.** — *For every $e, n \in \mathbf{Z}_{\geq 1}$, there exists $k(e,n) \in \mathbf{Z}_{\geq 1}$ such that for every prime p and every compact subgroup $K \leq GL(n, \mathbf{Z}_p)$, and every $L \leq K$ such that $\forall k \in K$, $k^e \in L$, we have*

$$[K : L] \leq k(e, n)$$

Proof. — It follows from [40, Prop 6.7 and Lem. 6.8 (2)], there exists $c(n)$ such that K is topologocally generated by at most $c(n)$ elements. From the restricted Burnside problem [56, §1.1 Introduction], we can take $k(e,n) = R(e, c(n))$, with $R(r,n)$ as in loc. cit. □

Corollary **B.11.** — *Let $V \leq U$ be a closed subgroup such that $\exists e, \forall u \in U$, $u^e \in V$. If U satisfies Def. 2.1, then V satisfies Def. 2.1.*

Proof. — Note that $\forall u \in U_p := U \cap M(\mathbf{Q}_p)$, we have $u^e \in V_p := V \cap M(\mathbf{Q}_p)$. By Lemma B.10 we have $[U_p : V \cap M(\mathbf{Q}_p)] \leq k(e,n)$ and (7) and (9) of §2.1.3 implies the conclusion. □

B.4.1

Let p be a prime and let $Y = Y(p)$ be the image of U by $M(\widehat{\mathbf{Z}}) \to M(\mathbf{Q}_p) \to M^{ad}(\mathbf{Q}_p)$. Let $W \leq Y$ be a closed subgroup which is invariant under conjugation by U and such that Y/W is abelian. We note that the conjugation action of U on Y/W is trivial. We claim that

$$(\mathbf{110}) \qquad |Y/W| < +\infty.$$

Proof. — Let $\mathfrak{w} \leq \mathfrak{y} \leq \mathfrak{m}^{ad}_{\mathbf{Q}_p}$ be the Lie $\mathbf{Q}_p$-algebras of the p-adic Lie groups $W \leq Y \leq M^{ad}(\mathbf{Q}_p)$. Then $\mathfrak{y}/\mathfrak{w}$ is a subquotient of $\mathfrak{m}^{ad}_{\mathbf{Q}_p}$ as a representation of U, and U acts trivially on $\mathfrak{y}/\mathfrak{w}$. By (1b), the representation $\mathfrak{m}^{ad}_{\mathbf{Q}_p}$ of $U_p := M(\mathbf{Q}_p) \cap U$ is semisimple, and by (1a) it has no nonzero factor with a trivial U_p-action. We deduce that $\mathfrak{w} = \mathfrak{y}$, that W is open in the compact group Y, and (110) follows. □

We claim that

$$(\mathbf{111}) \qquad \forall p \gg 0, \quad |Y/W| = 1.$$

Proof. — Let $M^{ad} \to GL(m)$ be an embedding and consider the induced $\mathbf{Z}$-structure on M^{ad}. Then we can define, for every prime p and every $i \in \mathbf{Z}_{\geq 1}$,

$$K_i := \ker M(\mathbf{Z}_p) \to M\big(\mathbf{Z}/(p^i)\big).$$

Let Y_i and W_i denote the image of $Y \cap K_i$ and $W \cap K_i$ in K_i/K_{i+1}. Then $|Y/W| = 1$ is equivalent to

$$\forall i \in \mathbf{Z}_{\geq 0}, \quad Y_i = W_i. \tag{112}$$

For $p \gg 0$, there is a map $\alpha : M(\mathbf{F}_p) \to M^{ad}(\mathbf{F}_p)$ such that the composed map $M(\mathbf{Z}_p) \to M^{ad}(\mathbf{Z}_p) \to M^{ad}(\mathbf{F}_p)$ is the same as the composed map $M(\mathbf{Z}_p) \to M(\mathbf{F}_p) \to M^{ad}(\mathbf{F}_p)$. Recall (109). We deduce that the image of U in $M^{ad}(\mathbf{F}_p)$ is

$$Y_0 = \alpha\big(V(p)^{\dagger}\big). \tag{113}$$

By Lem. B.5 and Prop. B.7, we have

(**114**) for $p \gg 0$, the image of $V(p)^{\dagger}$ in an abelian group is trivial.

Thus Y_0/W_0 is trivial. This implies

$$Y_0 = W_0. \tag{115}$$

For $p \gg 0$ and $i \in \mathbf{Z}_{\geq 1}$, we have $K_i = \exp(p^i \cdot \mathfrak{m}^{ad}_{\mathbf{Z}_p})$ and this induces an identification of abelian groups, compatible with the action of U by conjugation,

$$K_i/K_{i+1} \simeq \frac{p^i \cdot \mathfrak{m}^{ad}_{\mathbf{Z}_p}}{p^{i+1} \cdot \mathfrak{m}^{ad}_{\mathbf{Z}_p}} \simeq \mathfrak{m}^{ad}_{\mathbf{F}_p}.$$

We note that Y_i/W_i is a subquotient of $\mathfrak{m}^{ad}_{\mathbf{F}_p}$ as a representation of U, and U acts trivially on Y_i/W_i. Thus $\alpha(U(p)) \leq Y_0$ acts trivially on Y_i/W_i.

For $p \gg 0$, the map $M \to M^{ad}$ induces a bijection $\mathfrak{m}^{der}_{\mathbf{F}_p} \to \mathfrak{m}^{ad}_{\mathbf{F}_p}$. For $p \gg 0$ this bijection is equivariant for the action of $M(\mathbf{Z}_p)$ via $M(\mathbf{Z}_p) \to M(\mathbf{F}_p)$ on $\mathfrak{m}^{der}_{\mathbf{F}_p}$ and via $M(\mathbf{Z}_p) \to M(\mathbf{F}_p) \xrightarrow{\alpha} M^{ad}(\mathbf{F}_p)$ on $\mathfrak{m}^{ad}_{\mathbf{F}_p}$. We deduce a $M(\mathbf{Z}_p)$-equivariant embedding $\mathfrak{m}^{ad}_{\mathbf{F}_p} \simeq \mathfrak{m}^{der}_{\mathbf{F}_p} \leq \mathfrak{m}_{\mathbf{F}_p}$. In particular it is U-equivariant.

By (2b) and [46, Th. 5.4] the $U(p)$-representation $\mathfrak{m}_{\mathbf{F}_p}$ is semisimple for $p \gg 0$. By (2a) and Lem. 5.6, the centraliser of $U(p)$ in $\mathfrak{m}_{\mathbf{F}_p}$ is $\mathfrak{z}(\mathfrak{m})_{\mathbf{F}_p}$. Thus $\mathfrak{m}^{ad}_{\mathbf{F}_p}$ is semisimple as a representation of $U(p)$ and, for $p \gg 0$, its $U(p)$-invariants span $\{0\} \simeq \mathfrak{z}(\mathfrak{m})_{\mathbf{F}_p} \cap \mathfrak{m}^{der}_{\mathbf{F}_p}$.

We deduce

$$Y_i = W_i. \tag{116}$$

For $p \gg 0$, we have (115) and (116). We deduce (112), which implies the claim (111).

□

B.4.2

We denote by $M^{der} \leq M$ the derived subgroup, by $Z(M) \leq M$ the centre, and we write $ab_M : M \to M^{ab} = M/M^{der}$ and $ad_M : M \to M^{ad} = M/Z(M)$ the quotient maps. Let $\mathbf{A}_f = \widehat{\mathbf{Z}} \otimes \mathbf{Q}$, let $G_1 = M^{ab}(\mathbf{A}_f)$ and $G_2 = M^{ad}(\mathbf{A}_f)$. We define

$$\Gamma := (ab_M, ad_M)(U) \leq G_1 \times G_2$$

and $\Gamma_1 = ab_M(U)$ and $\Gamma_2 = ad_M(U)$ the projections of Γ on G_1 and on G_2. According to Goursat's lemma, $\Gamma/((\Gamma \cap G_1) \times (\Gamma \cap G_2))$ is the graph of an isomorphism

$$\Gamma_1/(\Gamma \cap G_1) \to \Gamma_2/(\Gamma \cap G_2). \tag{117}$$

We note that G_1 and G_2 are stable under conjugation by U, and that Γ is a U-stable subgroup of $G_1 \times G_2$. The isomorphism (117) is thus U-equivariant. Note that U acts trivially on G_1. This implies that U acts trivially on $\Gamma_2/(\Gamma \cap G_2)$.

We can thus apply (110) and (111) to $Y = Y(p)$ and $W = Y(p) \cap \Gamma \cap G_2$. We deduce

$$\left|\Gamma_2/(\Gamma \cap G_2)\right| \leq \prod_p \left|Y(p)/\big(Y(p) \cap \Gamma \cap G_2\big)\right| < +\infty.$$

It follows that

$$\big[\Gamma : (\Gamma \cap G_1) \times (\Gamma \cap G_2)\big] < +\infty.$$

The inverse images of Γ_1 and Γ_2 in U are respectively $U \cap M^{der}(\widehat{\mathbf{Z}})$ and $U \cap Z(M)(\widehat{\mathbf{Z}})$. We have proved

$$\big[U : \big(U \cap M^{der}(\widehat{\mathbf{Z}})\big) \cdot \big(U \cap Z(M)(\widehat{\mathbf{Z}})\big)\big] < +\infty. \tag{118}$$

B.4.3

For $u \in GL(n, \widehat{\mathbf{Z}})$, we denote by $\overline{u^{\mathbf{Z}}}$ the closed subgroup generated by u (for the adelic topology). Let us write $\pi_R(u) = (v_p)_p \in \prod_p GL(n, \mathbf{F}_p)$. We claim that if

$$\text{for all } p, \text{ the order of } v_p \text{ is a power of } p \tag{119}$$

then

$$u \in \prod_p \overline{u^{\mathbf{Z}}} \cap GL(n, \mathbf{Z}_p). \tag{120}$$

Proof. — The map $k \to u^k : \mathbf{Z} \to u^{\mathbf{Z}}$ extends by continuity to a map $\widehat{\mathbf{Z}} \to \overline{u^{\mathbf{Z}}}$. For every prime ℓ, we denote by $\gamma_\ell(u)$ the image of $1 \in \mathbf{Z}_\ell$ by $\mathbf{Z}_\ell \to \widehat{\mathbf{Z}} \to \overline{u^{\mathbf{Z}}}$. We have in particular $\gamma_\ell(u) \in \overline{u^{\mathbf{Z}}}$. If we write $u = (u_p)_p \in \prod_\ell GL(n, \mathbf{Z}_p)$, then we have

$$\gamma_\ell(u) = \big(\gamma_\ell(u_p)\big)_p, \tag{121}$$

where $\gamma_\ell(u_p)$ is defined similarly. We note the following: if $u^{p^i} \to 1$ as $i \to \infty$, we have

$$(\mathbf{122}) \qquad \gamma_\ell(u) = 1 \text{ if } \ell \neq p, \qquad \text{and } \gamma_\ell(u) = u \text{ if } \ell = p.$$

The property (119) implies that $\forall p$, $(u_p)^{p^i} \to 1$. From (121) and (122), we deduce that for every ℓ, $\gamma_\ell(u)$ is the image of u by $\mathrm{GL}(n, \widehat{\mathbf{Z}}) \to \mathrm{GL}(n, \mathbf{Z}_p) \to \mathrm{GL}(n, \widehat{\mathbf{Z}})$. It follows that $\gamma_\ell(u) \in \overline{u^{\mathbf{Z}}} \cap \mathrm{GL}(n, \mathbf{Z}_p)$ and that $u = \prod_\ell \gamma_\ell(u)$. The claim follows. □

Let $\Gamma \leq \mathrm{GL}(n, \widehat{\mathbf{Z}})$ be a subgroup such that $\pi_{\mathrm{R}}(\Gamma)$ is generated by elements $v = (v_p)_p$ satisfying (119). We claim that

$$(\mathbf{123}) \qquad \overline{\Gamma} = \prod_p \overline{\Gamma} \cap \mathrm{GL}(n, \mathbf{Q}_p).$$

Proof. — We argue by double inclusion, only one of which is non trivial. Because the right hand-side is a closed group, it is enough to prove that it contains a set of generators of Γ. By (120), we can take the set of $u \in \Gamma$ such that $(v_p)_p := \pi_{\mathrm{R}}(u)$ satisfy (119). □

B.4.4

By (114), there exists p_0 such that

$$(\mathbf{124}) \qquad \forall\, p \geq p_0, \quad \mathrm{V}(p)^\dagger \text{ has no abelian quotient.}$$

We recall that $\Lambda := \oplus_{p \geq p_0} \mathrm{V}(p)^\dagger$ (the subgroup generated by $\bigcup_{p \geq p_0} \mathrm{V}(p)^\dagger$) is dense in $\mathrm{V}_0 := \prod_{p \geq p_0} \mathrm{V}(p)^\dagger$.

Let $\mathrm{U}^* = (\mathrm{U} \cap \mathrm{M}^{der}(\widehat{\mathbf{Z}})) \cdot (\mathrm{U} \cap \mathrm{Z}(\mathrm{M})(\widehat{\mathbf{Z}}))$. By (118), we have $[\mathrm{U} : \mathrm{U}^*] < +\infty$.

Let $\mathrm{U}_0 = \ker(\mathrm{U}^* \to \prod_{p < p_0} \mathrm{GL}(n, \mathbf{F}_p))$. We have

$$(\mathbf{125}) \qquad [\mathrm{U} : \mathrm{U}_0] = \big[\mathrm{U} : \mathrm{U}^*\big] \cdot \big[\mathrm{U}^* : \mathrm{U}_0\big] \leq \big[\mathrm{U} : \mathrm{U}^*\big] \cdot \prod_{p < p_0} \big|\mathrm{GL}(n, \mathbf{F}_p)\big| < +\infty.$$

Let $\mathrm{U}' = \mathrm{U} \cap \mathrm{M}^{der}(\widehat{\mathbf{Z}}) \leq \mathrm{U}^*$ and let $\mathrm{U}'_0 = \mathrm{U}' \cap \mathrm{U}_0$. Recall $\mathrm{U}_0 \leq \mathrm{U}^*$. We have

$$(\mathbf{126}) \qquad \frac{\mathrm{U}_0}{\mathrm{U}'_0} \simeq \frac{\mathrm{U}_0 \cdot \mathrm{U}'}{\mathrm{U}'} \leq \frac{\mathrm{U}^*}{\mathrm{U}'} \simeq \frac{\mathrm{U} \cap \mathrm{Z}(\mathrm{M})(\widehat{\mathbf{Z}})}{\mathrm{U} \cap \mathrm{Z}(\mathrm{M}^{der})(\widehat{\mathbf{Z}})}.$$

These are thus abelian groups. It follows that $\pi_{\mathrm{R}}(\mathrm{U}_0)/\pi_{\mathrm{R}}(\mathrm{U}'_0)$ is abelian.

We may assume that p_0 is chosen big enough, so that for $p \geq p_0$, there is a map $\mathrm{M}(\mathbf{F}_p) \to \mathrm{M}^{ab}(\mathbf{F}_p)$ such that we have $p \nmid \#\mathrm{M}^{ab}(\mathbf{F}_p)$ and $\mathrm{M}^{der}(\mathbf{F}_p) = \ker(\mathrm{M}(\mathbf{F}_p) \to \mathrm{M}^{ab}(\mathbf{F}_p))$. Thus $\mathrm{V}(p)^\dagger \leq \mathrm{M}^{der}(\mathbf{F}_p)$ for $p \geq p_0$, and thus $\mathrm{V}_0 \leq \mathrm{M}^{der}(\mathrm{R})$ and

$$\mathrm{M}^{der}(\mathrm{R}) \cap \big(\mathrm{V}_0 \cdot \mathrm{Z}(\mathrm{M})(\mathrm{R})\big) = \mathrm{V}_0 \cdot \big(\mathrm{M}^{der}(\mathrm{R}) \cap \mathrm{Z}(\mathrm{M})(\mathrm{R})\big) = \mathrm{V}_0 \cdot \mathrm{Z}\big(\mathrm{M}^{der}\big)(\mathrm{R}).$$

We deduce $\pi_{\mathrm{R}}(\mathrm{U}_0') \leq \mathrm{M}^{der}(\mathrm{R}) \cap \mathrm{U}(\mathrm{R}) \leq \mathrm{V}_0 \cdot \mathrm{Z}(\mathrm{M}^{der})(\mathrm{R})$. Let U_0'' be the inverse image of V_0 in U_0'. There exists $f \in \mathbf{Z}_{\geq 1}$ such that $\#\mathrm{Z}(\mathrm{M}^{der}(\mathbf{F}_p))$ divides f for all primes p. We deduce

$$\forall u \in \mathrm{U}_0', \quad u^f \in \mathrm{U}_0''.$$

By (124), the morphism $\mathrm{V}(p)^\dagger \to \mathrm{V}_0 \to \pi_{\mathrm{R}}(\mathrm{U}_0)/\pi_{\mathrm{R}}(\mathrm{U}_0')$ is constant. Thus $\mathrm{V}(p)^\dagger \leq \pi_{\mathrm{R}}(\mathrm{U}_0')$, and thus $\Lambda \leq \pi_{\mathrm{R}}(\mathrm{U}_0'') \leq \mathrm{V}_0$. Recall that U_0'' is compact. By [24, Cor. 2.7], the map $\mathrm{U}_0'' \to \pi_{\mathrm{R}}(\mathrm{U}_0'')$ is open. As Λ is dense in V_0, it is dense in $\pi_{\mathrm{R}}(\mathrm{U}_0'')$. As the map $\mathrm{U}_0'' \to \pi_{\mathrm{R}}(\mathrm{U}_0'')$ is open, the inverse image Γ of Λ in U_0'' is dense in U_0''. As U_0'' is compact, we have $\overline{\Gamma} = \mathrm{U}_0''$.

By definition of $\mathrm{V}(p)^\dagger$, the group Λ is generated by elements $v = (v_p)_p$ satisfying (120). From (123) we deduce

$$\mathrm{U}_0'' = \overline{\Gamma} = \prod_p \overline{\Gamma} \cap \mathrm{M}(\mathbf{Q}_p) \leq \prod_p \mathrm{U}' \cap \mathrm{M}(\mathbf{Q}_p).$$

By (125), we have $[\mathrm{U}' : \mathrm{U}_0'] \leq [\mathrm{U} : \mathrm{U}_0] < \infty$, and we conclude

$$\exists e, \quad \forall u \in \mathrm{U}' = \mathrm{U} \cap \mathrm{M}^{der}(\widehat{\mathbf{Z}}),\ u^e \in \mathrm{U}_0'' \leq \prod_p \mathrm{U}' \cap \mathrm{M}(\mathbf{Q}_p). \tag{127}$$

Acknowledgements

We would like to express our greatest gratitude to Laurent Moret-Bailly for discussions and suggestions regarding the content of Section 7. The first author is grateful to Ahmed Abbes and Emmanuel Ullmo for discussions about §7.6, and Ofer Gabber about Th. 7.1. The authors are especially grateful to the referees, for their thorough reading and their numerous useful comments.

The first author is grateful to the IHÉS for its invitation during the preparation of this article.

Funding information

Both authors were supported by Leverhulme Trust Grant RPG-2019-180. The support of the Leverhulme Trust is gratefully acknowledged. The first named author was also supported by the University of Manchester and Grant EP/Y020758/1, and was supported by IHÉS.

Declarations:

Competing interests

The authors declare no competing interests.

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Manuscrit reçu le 6 juin 2024
Manuscrit accepté le 6 janvier 2025
publié en ligne le 3 février 2025.