THE EQUIVALENCE OF HEEGAARD FLOER HOMOLOGY AND EMBEDDED CONTACT HOMOLOGY III: FROM HAT TO PLUS

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ABSTRACT

Given a closed oriented 3-manifold M, we establish an isomorphism between the Heegaard Floer homology group $HF^+(-M)$ and the embedded contact homology group ECH(M). Starting from an open book decomposition (S, h) of M, we construct a chain map Φ^+ from a Heegaard Floer chain complex associated to (S, h) to an embedded contact homology chain complex for a contact form supported by (S, h). The chain map Φ^+ commutes up to homotopy with the U-maps defined on both sides and reduces to the quasi-isomorphism Φ from (Colin et al. in Publ. Math. Inst. Hautes Études Sci., 2024a, 2024b) on subcomplexes defining the hat versions. Algebraic considerations then imply that the map Φ^+ is a quasi-isomorphism.

1. Introduction

This is the last paper in the series which proves the isomorphism between certain Heegaard Floer homology and embedded contact homology groups. References from [I] (resp. [II]) will be written as "Section I.*x*" (resp. "Section II.*x*") to mean "Section *x*" of [I] (resp. [II]), for example.

Let M be a closed oriented 3-manifold. Let $\widehat{HF}(M)$ and $HF^+(M)$ be the hat and plus versions of Heegaard Floer homology of M and let $\widehat{ECH}(M)$ and ECH(M) be the hat and usual versions of the embedded contact homology of M. As usual, embedded contact homology will be abbreviated as ECH. In [0], we introduced the ECH chain group $\widehat{ECC}(N, \partial N)$ and showed that $\widehat{ECH}(N, \partial N) \simeq \widehat{ECH}(M)$. In the papers [I, II], we defined a chain map

$$\Phi : \widehat{CF}(-M) \to \widehat{ECC}(N, \partial N),$$

which induced an isomorphism

$$\Phi_* : \widehat{HF}(-M) \xrightarrow{\sim} \widehat{ECH}(M).$$

The goal of this paper is to extend the above result and prove the following theorem:

Theorem **1.0.1.** — *If* M *is a closed oriented* 3*-manifold, then there is a chain map*

$$\Phi^+ : CF^+(-M) \xrightarrow{\sim} ECC(M)$$

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which is a quasi-isomorphism and which commutes with the U*-maps up to homotopy. On the level of homology* Φ^+ *maps the contact class to the contact class.*

We use $\mathbf{F} = \mathbf{Z}/2\mathbf{Z}$ coefficients for both Heegaard Floer homology and ECH. As is the case for the hat versions, we expect Theorem 1.0.1 to hold over the integers; see Remark I.1.0.1.

Remark **1.0.2.** — The construction of Φ^+ can be carried out with twisted coefficients as in Sections I.6.4 and I.7.1.

Let (S, h) be an open book decomposition for M, where S is a genus $g \geq 2$ bordered surface with connected boundary and $h \in \mathrm{Diff}(S, \partial S)$.[1] In particular we identify

$$M \simeq (S \times [0, 1])/\sim,$$

where $(x, 1) \sim (h(x), 0)$ for all $x \in S$ and $(x, t) \sim (x, t')$ for all $x \in \partial S$ and $t, t' \in [0, 1]$. We write $S_t = S \times \{t\}$ for $t \in [0, 1]$. Let $\Sigma = S_0 \cup -S_{1/2}$ be the Heegaard surface corresponding to (S, h).

Given a pair (Σ_0, h_0) consisting of a surface Σ_0 and $h_0 \in \mathrm{Diff}(\Sigma_0)$, we write the mapping torus of (Σ_0, h_0) as:

$$N_{(\Sigma_0, h_0)} = (\Sigma_0 \times [0, 2])/(x, 2) \sim (h_0(x), 0).$$

The map Φ, defined in Section I.6.2, is induced by the cobordism W_+ which is an S_0-fibration and which restricts to a half-cylinder over $[0, 1] \times S_0$ at the positive end and to a half-cylinder over the mapping torus $N_{(S_0, h)}$ at the negative end. We say that W_+ is a cobordism "from $[0, 1] \times S_0$ to $N_{(S_0, h)}$."

Remark **1.0.3.** — We will interchangeably write $[0, 1] \times S_0$ and $S_0 \times [0, 1]$. This is partly due to the fact that the open book is usually written as $(S \times [0, 1])/\sim$ and the positive end of W_+ is a "symplectization" $\mathbf{R} \times [0, 1] \times S_0$.

The map Φ^+ is induced by a cobordism X_+ from $[0, 1] \times \Sigma$ to M which extends W_+ and is described below. Although Φ was defined in terms of just one page S_0, we can no longer ignore the $S_{1/2}$ portion of Σ when defining Φ^+, since we do not know how to express $HF^+(-M)$ in terms of S_0.

A symplectic cobordism similar to X_+ is constructed by Wendl in [We].

1.0.1. *The cobordism* X_+. — We give a description of $X_+ = X_+^0 \cup X_+^1 \cup X_+^2$ and $W_+ = W_+^0 \cup W_+^1 \cup W_+^2$ as topological spaces, where $W_+^i \subset X_+^i$ for $i = 0, 1, 2$. See Figure 1. *The description given here is the simplified version of the actual construction, and the notation of Section 1.0.1 is not used outside of Section 1.0.1.*

[1] The condition $g \geq 2$ is a technical condition which will used in the definition of Φ^+.

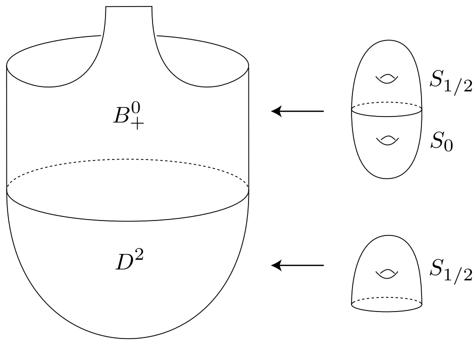


FIG. 1. — Schematic diagram for $X^0_+ \cup X^1_+$ which indicates the fibers over each subsurface

First extend $h \in \mathrm{Diff}(S_0, \partial S_0)$ to $h^+ \in \mathrm{Diff}(\Sigma)$ so that $h^+|_{S_{1/2}} = id$. Let $N_{(\Sigma,h^+)}$ and $N_{(S_0,h)}$ be the mapping tori of h^+ and h and let

$$\pi : [0,\infty) \times N_{(\Sigma,h^+)} \to [0,\infty) \times \mathbf{R}/2\mathbf{Z}$$

be the projection $(s,x,t) \mapsto (s,t)$. Then define $B^0_+ = ([0,\infty) \times \mathbf{R}/2\mathbf{Z}) - B^c_+$, where B^c_+ is the subset $[2,\infty) \times [1,2]$ with the corners rounded. We then set

$$X^0_+ := \pi^{-1}(B^0_+), \quad W^0_+ := \pi^{-1}(B^0_+) \cap ([0,\infty) \times N_{(S_0,h)}).$$

Observe that W^0_+ is the "top half" of W_+ defined in Section I.5.1. Next we set

$$X^1_+ := S_{1/2} \times D^2, \quad W^1_+ := \varnothing$$

and identify $\{0\} \times S_{1/2} \times \mathbf{R}/2\mathbf{Z} \subset \partial X^0_+$ with $S_{1/2} \times \partial D^2 \subset \partial X^1_+$ via the map $(0,x,t) \mapsto (x, e^{\pi i t})$. Then one component of $\partial(X^0_+ \cup X^1_+)$ is given by $M = (\{0\} \times N_{(S_0,h)}) \cup (\partial S_0 \times D^2)$.

Finally we set

$$X^2_+ := (-\infty, 0] \times M, \quad W^2_+ := (-\infty, 0] \times (\{0\} \times N_{(S_0,h)}),$$

where $\{0\} \times M$ is identified with M.

1.0.2. *Sketch of proof.* — The proof of Theorem 1.0.1 proceeds as follows:
Step 1. Express the U-map on $HF^+(-M)$ as a count of $I_{HF} = 2$ curves that pass through a point, in analogy with the definition of U in ECH. This is given by Theorem 3.1.4.
Step 2. Construct a symplectic cobordism (X_+, Ω_{X_+}) from $[0,1] \times \Sigma$ to M, together with stable Hamiltonian and contact structures on $[0,1] \times \Sigma$ and M. This is the goal of Section 4.
Step 3. Define the chain map Φ^+ as a count of $I_{X_+} = 0$ curves in X_+ and show that Φ^+ commutes with the U-maps on both sides up to a chain homotopy K. This is done in Section 5.

Step 4. By an algebraic theorem (Theorem 6.1.5), Φ^+ is a quasi-isomorphism if a map

$$\Phi_{alg} : \widehat{CF}(-M) \to \widehat{ECC}(M),$$

defined using Φ^+ and K, is a quasi-isomorphism.
Step 5. By Theorem 6.4.1, the map Φ_{alg} is a quasi-isomorphism. This is proved by relating Φ_{alg} to the quasi-isomorphism Φ from [I, II].

2. Heegaard Floer chain complexes

The goal of this section is to introduce some notation and recall the definition of the chain complex $CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f, J)$, whose homology is $HF^+(-M)$.

2.1. *Heegaard data.* — Let M be a closed oriented 3-manifold and let (S, h) be an open book decomposition for M.

We use the following notation, which is similar to that of Section I.4.9.1:

- $\Sigma = S_0 \cup -S_{1/2}$ is the associated genus $2g$ Heegaard surface of M;
- $\mathbf{a} = \{a_1, \dots, a_{2g}\}$ is a basis of arcs for S and $\mathbf{b}$ is a small pushoff of $\mathbf{a}$ as given in Figure I.1;
- x_i and x_i' are the endpoints of a_i in ∂S_0 that correspond to the coordinates of the contact class and x_i'' is the unique point of $a_i \cap b_i \cap int(S_{1/2})$;
- $\boldsymbol{\alpha} = (\mathbf{a} \times \{\frac{1}{2}\}) \cup (\mathbf{a} \times \{0\})$ and $\boldsymbol{\beta} = (\mathbf{b} \times \{\frac{1}{2}\}) \cup (h(\mathbf{a}) \times \{0\})$ are the collections of compressing curves on the Heegaard surface Σ;
- z^f is a point in the large (i.e., non-thin-strip) component of $S_{1/2} - \boldsymbol{\alpha} - \boldsymbol{\beta}$ and $(z')^f$ is a point which is close but not equal to z^f.

We say that the pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f)$ is *compatible with* (S, h). We let $\mathbf{x} = \{x_1, \dots, x_{2g}\}$ and consider the *contact element* $[\mathbf{x}, 0]$. In the definition of $\mathbf{x}$ we could replace any component x_i with x_i'.

Remark **2.1.1.** — *The orientation for Σ is opposite to that of Section I.4.9.1.* This is done so that the triple $(S, \mathbf{a}, h(\mathbf{a}))$, used in [I, II], embeds in $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ in an orientation-preserving manner.

2.2. *Symplectic data.* — The stable Hamiltonian structure on $[0, 1] \times \Sigma$ with coordinates (t, x) is given by (λ, ω), where $\lambda = dt$ and ω is an area form on Σ which makes $(\boldsymbol{\alpha}, \boldsymbol{\beta}, z^f)$ *weakly admissible with respect to* ω, i.e., each periodic domain has zero ω-area. The plane field $\xi = \ker \lambda$ is equal to the tangent plane field of $\{t\} \times \Sigma$ and the Hamiltonian vector field is $R = \frac{\partial}{\partial t}$.

We introduce the "symplectization"

$$(X, \Omega) = (\mathbf{R} \times [0, 1] \times \Sigma, ds \wedge dt + \omega),$$

where (s, t) are coordinates on $\mathbf{R}\times[0,1]$. Let $\pi_B : X \to B = \mathbf{R}\times[0,1]$ be the projection along the fibers $\{(s,t)\}\times\Sigma$.

Let J be an Ω_X-admissible almost complex structure on X; we assume that J is regular (cf. Lemma I.4.7.2 and [Li, Proposition 3.8]). We also define the Lagrangian submanifolds

$$L_{\boldsymbol\alpha} = \mathbf{R}\times\{1\}\times\boldsymbol\alpha, \quad L_{\boldsymbol\beta} = \mathbf{R}\times\{0\}\times\boldsymbol\beta.$$

2.3. *The chain complex* $CF^+(\Sigma, \boldsymbol\alpha, \boldsymbol\beta, z^f, J)$. — In this subsection we recall the definition of the chain complex $CF^+(\Sigma, \boldsymbol\alpha, \boldsymbol\beta, z^f, J)$, whose homology group

$$HF^+(\Sigma, \boldsymbol\alpha, \boldsymbol\beta, z^f, J)$$

is isomorphic to $HF^+(-M)$. This definition is due to Lipshitz [Li], with one modification: we are using the ECH index I_{HF} from Definition I.4.5.11. We will often suppress J from the notation.

Let $\mathcal{S} = \mathcal{S}_{\boldsymbol\alpha,\boldsymbol\beta}$ be the set of $2g$-tuples $\mathbf{y} = \{y_1, \dots, y_{2g}\}$ of intersection points of $\boldsymbol\alpha$ and $\boldsymbol\beta$ for which there exists some permutation $\sigma \in \mathfrak{S}_{2g}$ such that $y_j \in \alpha_j \cap \beta_{\sigma(j)}$ for all j. Then $CF^+(\Sigma, \boldsymbol\alpha, \boldsymbol\beta, z^f, J)$ is generated over $\mathbf{F}$ by pairs $[\mathbf{y}, i]$, where $\mathbf{y}\in\mathcal{S}$ and $i \in \mathbf{N}$, with the French convention that $0 \in \mathbf{N}$.

The differential $\partial = \partial_{HF}$ is given by

$$\partial[\mathbf{y}, i] = \sum_{[\mathbf{y}', j]\in\mathcal{S}\times\mathbf{N}} \langle \partial[\mathbf{y}, i], [\mathbf{y}', j]\rangle \cdot [\mathbf{y}', j],$$

where the coefficient $\langle \partial[\mathbf{y}, i], [\mathbf{y}', j]\rangle$ is the count of index $I_{HF} = 1$ finite energy holomorphic multisections in (X,J) with Lagrangian boundary $L_{\boldsymbol\alpha}\cup L_{\boldsymbol\beta}$ from $\mathbf{y}$ to $\mathbf{y}'$, whose algebraic intersection with the holomorphic strip $\mathbf{R}\times[0,1]\times\{(z')^f\}$ is $(i-j)$. We will often refer to such curves as curves *from* $[\mathbf{y}, i]$ *to* $[\mathbf{y}', j]$.

Let us write $\partial = \sum_{k=0}^{\infty}\partial_k$, where ∂_k only counts curves whose algebraic intersection with $\mathbf{R}\times[0,1]\times\{(z')^f\}$ is k.

Lemma **2.3.1.** — *The contact element* $[\mathbf{x}, 0]$ *is a cycle and its homology class does not depend on the choice of* x_i *or* x_i' *as its coordinates.*

Proof. — The proof of the first statement is the same as that for the contact element $\mathbf{x}$ in the hat version since curves from $[\mathbf{x}, 0]$ cannot intersect $\mathbf{R}\times[0,1]\times\{z^f\}$. The second statement follows from Claim I.4.9.2. □

3. The geometric U-map

3.1. *Introduction.* — In [OSz, Li], the U-map

$$U : CF^+(\Sigma, \boldsymbol\alpha, \boldsymbol\beta, z^f) \to CF^+(\Sigma, \boldsymbol\alpha, \boldsymbol\beta, z^f),$$

is defined algebraically as $U([\mathbf{y}, i]) = [\mathbf{y}, i-1]$ if $i > 0$ and $U([\mathbf{y}, 0]) = 0$. The goal of this section is to give a geometric definition of the U-map which is analogous to that of ECH.

Let z^f, $(z')^f$ be as before and let $z = (z^b, z^f) \in X = B \times \Sigma$, where $z^b \in int(B)$. Let $J^\Diamond$ be a generic C^ℓ-small perturbation of J such that $J^\Diamond = J$ away from a small neighborhood $N(z) \subset X$ of z and such that $N(z) \cap (\mathbf{R} \times [0,1] \times \{(z')^f\}) = \varnothing$. In particular, we assume that there are no $J^\Diamond$-holomorphic curves that are homologous to $\{pt\} \times \Sigma$ and pass through z.

Remark **3.1.1.** — When we refer to "C^ℓ-close" almost complex structures, etc., we assume that $\ell > 0$ is sufficiently large.

Let $\mathcal{M}_{J^\Diamond}^{I=k}([\mathbf{y}, i], [\mathbf{y}', j])$ (resp. $\mathcal{M}_{J^\Diamond}^{I=k}([\mathbf{y}, i], [\mathbf{y}', j], z)$) be the moduli space of $I_{HF} = k$ finite energy holomorphic curves in $(X, J^\Diamond)$ with Lagrangian boundary $L_\alpha \cup L_\beta$ from $[\mathbf{y}, i]$ to $[\mathbf{y}', j]$ (resp. from $[\mathbf{y}, i]$ to $[\mathbf{y}', j]$ that pass through z). There is a natural forgetful map

$$\mathcal{M}_{J^\Diamond}^{I=k}([\mathbf{y}, i], [\mathbf{y}', j], z) \to \mathcal{M}_{J^\Diamond}^{I=k}([\mathbf{y}, i], [\mathbf{y}', j]),$$

which is an injection when $I \leq 3$: If a curve u passes through z twice (or passes through z once with a singularity at z), then the nodal or singular point contributes 2 to I. Also, by our choice of $J^\Diamond$, "passing through z" is a generic codimension 2 condition, and therefore $\mathrm{ind}(u) \geq 2$. Hence, by the index inequality (I.4.5.5), $I(u) \geq 4$, a contradiction.

Also note that, by a simple count of I and the ECH index inequality for I as in Equation (I.7.5.6), an $I(u) \leq 3$ curve that passes through z cannot have a fiber component.

Definition **3.1.2** *(Geometric* U*-map). — The* geometric U-map with respect to the point z *is the map:*

$$U_z([\mathbf{y}, i]) = \sum_{[\mathbf{y}', j] \in \mathcal{S} \times \mathbf{N}} \#\mathcal{M}_{J^\Diamond}^{I=2}([\mathbf{y}, i], [\mathbf{y}', j], z) \cdot [\mathbf{y}', j].$$

Proposition **3.1.3.** — U_z *is a chain map.*

Proof. — Since we are using almost complex structures of type $J^\Diamond$, the transversality of $\mathcal{M}_{J^\Diamond}^{I=3}([\mathbf{y}, i], [\mathbf{y}', j], z)$ follows from the combination of Theorems 3.1.7 and 3.4.1 of [MS], with modifications as in Proposition I.5.8.8. The compactness follows from Lemma I.4.6.1 and the usual SFT compactness; also see [Li, Corollary 7.2]. Fiber bubbling was already eliminated. Finally, gluing is as in Propositions A.1 and A.2 of [Li, Appendix A]. □

Theorem **3.1.4.** — *There exists a chain homotopy*

$$\mathrm{H} : \mathrm{CF}^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f) \to \mathrm{CF}^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f)$$

such that

$$\mathrm{U}_z - \mathrm{U} = \mathrm{H} \circ \partial_{\mathrm{HF}} + \partial_{\mathrm{HF}} \circ \mathrm{H}. \tag{3.1.1}$$

Moreover, for all $\mathbf{y} \in \mathcal{S}$*, one has* $\mathrm{H}([\mathbf{y}, 0]) = 0$.

The rest of this section is devoted to the proof of Theorem 3.1.4.

3.2. *A model calculation.* — Let Σ be a closed surface of genus k. We consider the manifold $\mathrm{D} \times \Sigma$, where $\mathrm{D} = \{|z| \leq 1\} \subset \mathbf{C}$. Let $\pi_{\mathrm{D}} : \mathrm{D} \times \Sigma \to \mathrm{D}$ and $\pi_\Sigma : \mathrm{D} \times \Sigma \to \Sigma$ be the projections of $\mathrm{D} \times \Sigma$ onto the first and second factors. Let $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_k\}$ be the set of β-curves for Σ. Choose $z^f \in \Sigma - \boldsymbol{\beta}$ and let $z = (0, z^f) \in \mathrm{D} \times \Sigma$.

Let $\mathrm{J} = j_{\mathrm{D}} \times j_\Sigma$ be a product complex structure on $\mathrm{D} \times \Sigma$ and $\mathrm{J}^\diamondsuit$ be a generic C^ℓ-small perturbation of J such that $\mathrm{J}^\diamondsuit = \mathrm{J}$ away from a small neighborhood of z. The key feature of $\mathrm{J}^\diamondsuit$ is that all the $\mathrm{J}^\diamondsuit$-holomorphic curves that pass through z are regular.

We then define the moduli space $\mathcal{M}_{\mathrm{A}}(\mathrm{D} \times \Sigma, \mathrm{J}^*)$, $* = \varnothing$ or $\diamondsuit$, of stable maps

$$u : (\mathrm{F}, j) \to (\mathrm{D} \times \Sigma, \mathrm{J}^*)$$

in the class $\mathrm{A} = [\{pt\} \times \Sigma] + k[\mathrm{D} \times \{pt\}] \in \mathrm{H}_2(\mathrm{D} \times \Sigma, \partial \mathrm{D} \times \boldsymbol{\beta})$, such that $\partial \mathrm{F}$ has k connected components and each component of $\partial \mathrm{F}$ maps to a distinct Lagrangian $\partial \mathrm{D} \times \beta_i$, $i = 1, \dots, k$. We choose points $w_i \in \beta_i$, $i = 1, \dots, k$, and define

$$\mathbf{w} = \{(1, w_1), \dots, (1, w_k)\} \subset \mathrm{D} \times \Sigma.$$

Let $\mathcal{M}_{\mathrm{A}}(\mathrm{D} \times \Sigma, \mathrm{J}^*; z, \mathbf{w})$ be the moduli space of stable maps u as above, with the extra data of an interior puncture and k boundary punctures that map to z and $\mathbf{w}$. There is a forgetful map

$$\mathcal{M}_{\mathrm{A}}(\mathrm{D} \times \Sigma, \mathrm{J}^*; z, \mathbf{w}) \to \mathcal{M}_{\mathrm{A}}(\mathrm{D} \times \Sigma, \mathrm{J}^*),$$

which is an injection when we restrict to curves that pass through z only once and there is no singularity at z. This will be the case in our setting. The points of $\mathbf{w}$ are distinct and there is no risk of passing through the same point of $\mathbf{w}$ twice. We use the modifier "irr" to denote the subset of irreducible curves.

3.2.1. *ECH index.* — We briefly indicate the definition of the ECH index I of a homology class $\mathrm{B} \in \mathrm{H}_2(\mathrm{D} \times \Sigma, \partial \mathrm{D} \times \boldsymbol{\beta})$ which admits a representative F such that each component of $\partial \mathrm{F}$ maps to a distinct $\partial \mathrm{D} \times \beta_i$. Although we call I the "ECH index", what we are defining here is a relative version of Taubes' index from [T].

Let τ be a trivialization of $T\Sigma$ along $\boldsymbol{\beta}$, given by a nonsingular tangent vector field Y_1 along $\boldsymbol{\beta}$, and let τ' be a trivialization of TD along ∂D, given by an outward-pointing radial vector field Y_2 along ∂D. Let $Q_{(\tau,\tau')}(B)$ be the intersection number between an embedded representative u of B and its pushoff, where the boundary of u is pushed off in the direction given by $J(Y_1)$.

Definition **3.2.1.** — *The* ECH index of the homology class B *is:*

$$I(B) = c_1(T(D \times \Sigma)|_B, (\tau, \tau')) + \mu_{(\tau,\tau')}(\partial B) + Q_{(\tau,\tau')}(B).$$

The following is the relative version of the adjunction inequality:

Lemma **3.2.2** *(Index inequality). — Let* $u : (F, j) \to (D \times \Sigma, J^*)$ *be a holomorphic curve in the class* $B \in H_2(D \times \Sigma, \partial D \times \boldsymbol{\beta})$. *Then*

$$\mathrm{ind}(u) + 2\delta(u) = I(B),$$

where $\delta(u) \geq 0$ *is an integer count of the singularities.*

Proof. — Similar to the proof of Theorem I.4.5.13. □

We now calculate some ECH and Fredholm indices:

Lemma **3.2.3.** — *If* $B = [\{pt\} \times \Sigma] + k_0[D \times \{pt\}]$ *with* $k_0 \leq k$, *then*

$$I(B) = 2 - 2k + 3k_0.$$

Proof. — We compute that

$$\begin{aligned} I(B) &= I([\{pt\} \times \Sigma] + k_0[D \times \{pt\}]) \\ &= I([\{pt\} \times \Sigma]) + k_0 \cdot I([D \times \{pt\}]) + 2k_0 \cdot \langle [\{pt\} \times \Sigma], [D \times \{pt\}] \rangle \\ &= (2 - 2k) + k_0 \cdot 1 + 2k_0 = 2 - 2k + 3k_0. \end{aligned}$$

Here $\langle , \rangle$ denotes the algebraic intersection number. □

Lemma **3.2.4.** — *If* $B = [\{pt\} \times \Sigma] + k_0[D \times \{pt\}]$ *with* $k_0 \leq k$ *and* u *is an irreducible* $J^\diamondsuit$*-holomorphic curve in the class* B, *then*

$$\mathrm{ind}(u) = 2 - 2k + 3k_0 - \delta(u).$$

Proof. — Follows from Lemma 3.2.3 and the index inequality. □

3.2.2. *Main result.* — The following is the main result of this subsection:

Theorem **3.2.5.** — *If* $J^\diamondsuit$ *is generic, then the following hold:*

(1) $\mathcal{M}_A(D \times \Sigma, J^\diamondsuit; z, \mathbf{w}) = \mathcal{M}_A^{irr}(D \times \Sigma, J^\diamondsuit; z, \mathbf{w})$;
(2) $\mathcal{M}_A(D \times \Sigma, J^\diamondsuit; z, \mathbf{w})$ *is compact, regular, and* 0*-dimensional;*
(3) *the curves of* $\mathcal{M}_A(D \times \Sigma, J^\diamondsuit; z, \mathbf{w})$ *are embedded; and*
(4) $\#\mathcal{M}_A(D \times \Sigma, J^\diamondsuit; z, \mathbf{w}) \equiv 1$ *mod* 2.

Hence $\#\mathcal{M}_A(D \times \Sigma, J^\diamondsuit; z, \mathbf{w})$ is a certain relative Gromov-Witten invariant [IP] which is computed to be 1 mod 2. (What we are really computing here is a relative Gromov-Taubes invariant [T], although the two invariants coincide in this case.)

Proof. — (1) Let us write $\mathcal{M} = \mathcal{M}_A(D \times \Sigma, J^\diamondsuit; z, \mathbf{w})$. Arguing by contradiction, suppose $u \in \mathcal{M} - \mathcal{M}^{irr}$. Then u consists of an irreducible component u_0 which passes through z and $k_0 < k$ points of $\mathbf{w}$, together with $k - k_0$ copies of $D \times \{pt\}$. By Lemma 3.2.4, $\mathrm{ind}(u_0) \le 2 - 2k + 3k_0$. On the other hand, the point constraints are $(k_0 + 2)$-dimensional. Hence u_0 does not exist for generic $J^\diamondsuit$, which is a contradiction.

(2), (3) The compactness follows from the usual Gromov compactness theorem: We have already specified the homology class A and the genus bound is a consequence of Lemma 3.2.2, from which we see that the Euler characteristic term that appears in the formula for $\mathrm{ind}(u)$ is controlled by the homology class A. The regularity of $\mathcal{M}$ is immediate from the genericity of $J^\diamondsuit$ and (1). Lemma 3.2.4 implies the dimension calculation, as well as (3).

(4) We degenerate Σ along the union C of $k - 1$ separating curves into a nodal surface $\widetilde{\Sigma}$ whose irreducible components are k tori which are successively attached to one another; let $J^\diamondsuit_\tau$, $\tau \in [0, \infty)$, be the family of almost complex structures corresponding to the degeneration. We choose C so that they are disjoint from $\boldsymbol{\beta}$ and each irreducible component contains exactly one component of $\boldsymbol{\beta}$ (and hence exactly one w_i). Since the basepoint z remains in one component, the almost complex structure on $D \times \widetilde{\Sigma}$ is a product almost complex structure in all but one of the irreducible components of $D \times \widetilde{\Sigma}$. In order to attain transversality, we need to further perturb $J^\diamondsuit_\tau$ to $J^\heartsuit_\tau$ on a compact subset $K \subset \mathrm{int}(D) \times (\Sigma - C)$ such that each component of $K \cap (D \times (\Sigma - C))$ nontrivially intersects each curve of $\mathcal{M}_A^{irr}(D \times \Sigma, J^\diamondsuit_\tau; z, \mathbf{w})$. By a standard continuation argument,

$$\#\mathcal{M}_A^{irr}(D \times \Sigma, J^\diamondsuit_\tau; z, \mathbf{w}) = \#\mathcal{M}_A^{irr}(D \times \Sigma, J^\heartsuit_\tau; z, \mathbf{w});$$

from now on we will work with the latter almost complex structure.

As Σ degenerates into $\widetilde{\Sigma}$, a sequence $u^\tau \in \mathcal{M}_A^{irr}(D \times \Sigma, J^\heartsuit_\tau; z, \mathbf{w})$ of holomorphic curves with $\tau \to \infty$ (after passing to a subsequence) degenerates into a nodal holomorphic curve $u_1 \cup \cdots \cup u_k$ in $D \times \widetilde{\Sigma}$, where each u_i lies on a separate level and u_i is attached to u_{i+1} for $i = 1, \dots, k-1$. Starting with the component u_1 that passes through z, the incidence condition between u_1 and u_2 is analogous to a point constraint for u_2, and so on.

Hence it suffices to prove Theorem 3.2.5(4) for $k=1$; this is the content of Lemma 3.3.4 in Section 3.3. See Section II.2.4.4 for a similar argument. ☐

Remark **3.2.6.** — The section $\{\infty\}\times\Sigma$ is not regular, and thus neither $J_S^\diamondsuit$ nor $J_S^\heartsuit$ are generic almost complex structures. What we are computing here is a simple instance of *relative Gromov-Witten invariant* in the sense of [IP].

3.3. *Computation of* $\#\mathcal{M}_A(D\times\Sigma, J^\diamondsuit; z, \mathbf{w})$ *when* $k=1$ *and* Σ *is a torus.* — The first step is to degenerate D into $D\cup S^2$, where $0\in D$ is identified with $\infty\in S^2\cong \mathbf{C}\cup\{\infty\}$ (we will refer to the identified point by $\mathfrak{n}$) and $z=(0,z^f)\in S^2\times\Sigma$; equivalently, we are taking a 1-parameter family $J_\kappa^\diamondsuit$, $\kappa\in[0,\infty)$, and taking the limit $\kappa\to\infty$. Let $J_D^\diamondsuit\cup J_{S^2}^\diamondsuit$ denote the limit almost complex structure on $(D\times\Sigma)\cup(S^2\times\Sigma)$, which we assume to be a small perturbation of a product almost complex structure $J_D\cup J_{S^2}$ in a small neighborhood of z.

Let $v_1\cup v_2$ be a limit of a sequence $u^\kappa\in\mathcal{M}_A(D\times\Sigma, J_\kappa^\diamondsuit; z, \mathbf{w})$ of curves with $\kappa\to\infty$. Then v_1 is the trivial multisection $D\times\{w_1\}$ in $D\times\Sigma$ and

$$v_2\in\mathcal{M}_B^\diamondsuit := \mathcal{M}_B(S^2\times\Sigma, J_{S^2}^\diamondsuit; z, \mathbf{w}=\{(\infty,w_1)\}),$$

where $\mathcal{M}_B^\diamondsuit$ is the moduli space of $J_{S^2}^\diamondsuit$-holomorphic curves in $S^2\times\Sigma$ representing the homology class $B=[S^2]+[\Sigma]$ and passing through $z=(0,z^f)$ and (∞,w_1).

In order to analyze $\mathcal{M}_B^\diamondsuit$, we first describe $\mathcal{M}_B := \mathcal{M}_B(S^2\times\Sigma, J_{S^2}; z, \mathbf{w})$ for a product complex structure J_{S^2}:

Lemma **3.3.1.** — *If* $k=1$*, then:*

1. $\mathcal{M}_A(D\times\Sigma, J; z, \mathbf{w})$ *is a one-element set consisting of a degenerate curve* $(D\times\{w_1\})\cup(\{0\}\times\Sigma)$*; and*
2. $\mathcal{M}_B$ *is a two-element set consisting of degenerate curves* $v_{21}:=(S^2\times\{w_1\})\cup(\{0\}\times\Sigma)$ *and* $v_{22}:=(S^2\times\{z^f\})\cup(\{\mathfrak{n}\}\times\Sigma)$.

Proof. — (1) follows from the homological constraint

$$A=[\{pt\}\times\Sigma]+[D\times\{pt\}].$$

If $u:(F,j)\to(D\times\Sigma,J)$ is a stable map in $\mathcal{M}_A(D\times\Sigma,J;z,\mathbf{w})$, then $\pi_D\circ u$ and $\pi_\Sigma\circ u$ are degree 1 maps. This implies that F consists of two components F_1, F_2 and $\pi_D\circ u|_{F_1}$ and $\pi_\Sigma\circ u|_{F_2}$ are biholomorphisms. On the other hand, $\pi_\Sigma\circ u|_{F_1}$ maps to a point since F_1 is a disk and $\pi_D\circ u|_{F_2}$ maps to a point since otherwise the cardinality of $(\pi_D\circ u)^{-1}(pt)$ for generic pt will be larger than $\deg(\pi_D\circ u)=1$.

(2) is similar and follows from the fact that there are no degree 1 holomorphic maps from the torus Σ to S^2. ☐

By Gromov compactness and Lemma 3.3.1(2), all the curves of $\mathcal{M}_B^\diamond$ are close to the degenerate curves in $\mathcal{M}_B$ described in Lemma 3.3.1(2). Note that elements in $\mathcal{M}_B^\diamond$ can be reducible and only the irreducible component passing through z has to be regular. Simple considerations taking into account the homological and point constraints imply:

Lemma **3.3.2.** — *If* $k = 1$ *and* $J^\diamond$, $\mathbf{w}$, *and* $\boldsymbol{\beta}$ *are generic, then the only element* $v'_{22} \in \mathcal{M}_B^\diamond - \mathcal{M}_B^{\diamond,irr}$ *is close to* v_{22} *and consists of* $\{\mathfrak{n}\} \times \Sigma$ *together with one sphere in the class* $[S^2]$ *passing through* z.

We also have:

Lemma **3.3.3.** — *If* $k = 1$ *and* $J^\diamond$, $\mathbf{w}$, *and* $\boldsymbol{\beta}$ *are generic, then:*

(1) *the curves of* $\mathcal{M}_B^{\diamond,irr}$ *are embedded; and*
(2) $\mathcal{M}_B^{\diamond,irr}$ *is compact, regular, and* 0*-dimensional.*

Proof. — (1) The proof is similar to that of Lemma 3.2.5(3) and follows from the adjunction inequality [M1, M2] (compare with Lemma 3.2.2): If $v \in \mathcal{M}_B^{\diamond,irr}$, then

$$I(v) = c_1(v^*T(S^2 \times \Sigma)) + Q(v),$$

where $Q(v)$ is the self-intersection number of v, and

$$\mathrm{ind}(v) + 2\delta(v) = I(v),$$

where $\delta(v) \geq 0$ is an integer count of the singularities. Since $c_1(v^*T(S^2 \times \Sigma)) = 2$ and $Q(v) = 2$, it follows that $I(v) = 4$. On the other hand,

$$\mathrm{ind}(v) = -\chi(F) + 2c_1(v^*T(S^2 \times \Sigma)) = -0 + 2(2) = 4,$$

where F is the domain of v with $\chi(F) = 0$. Hence v is embedded by the adjunction inequality.

(2) Since v is embedded and $c_1(v^*T(S^2 \times \Sigma)) = 2$, the regularity of v without the point constraints follows from automatic transversality (cf. Hofer-Lizan-Sikorav [HLS, Theorem 1]). The regularity with point constraints is the consequence of the genericity of $J^\diamond$, $\mathbf{w}$, and $\boldsymbol{\beta}$. The rest of the assertion is immediate. □

Next we argue that $v_1 \cup v'_{22}$ cannot appear as the limit of u^κ. This can be proved by an analysis of the limit in the SFT sense (or equivalently in the relative Gromov-Witten sense): in brief, we can view the component $\{\mathfrak{n}\} \times \Sigma$ of v'_{22} as an intermediate irreducible level with image in $S^2 \times \Sigma$, is in the class $[\{pt\} \times \Sigma] + [S^2 \times \{pt\}]$, and passes through (∞, w_1) and $(0, z^f)$. Such a curve does not exist since there are no degree 1 holomorphic maps from the torus Σ to S^2. Therefore,

$$\#\mathcal{M}_A(D \times \Sigma, J^\diamond; z, \mathbf{w}) \equiv \#\mathcal{M}_B^{\diamond,irr} \bmod 2.$$

The following lemma then completes the proof of Theorem 3.1.4.

Lemma **3.3.4.** — $\#\mathcal{M}_{B}^{\Diamond,irr} \equiv 1$ *mod* 2.

Proof. — The lemma follows from [MS, Example 8.6.12], but one can also argue more explicitly by degenerating $\Sigma = T^2$ into a nodal surface $\Sigma_0 \cup \Sigma_1$, where the sphere Σ_0 contains z, the sphere Σ_1 contains w_1, and $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are the two nodes.

Consider a limit $u_0 \cup u_1$ of $u^\tau \in \mathcal{M}_{B,\tau}^{\heartsuit,irr}$ as $\tau \to \infty$, where we are using $J^\heartsuit$ instead of $J^\Diamond$ and the subscript τ indicates the dependence of $J^\heartsuit$ on $\tau \in [0,\infty)$ as we degenerate Σ. Here u_0 has image in $S^2 \times \Sigma_0$ and passes through $(0, z^f)$, and u_1 has image in $S^2 \times \Sigma_1$ and passes through (∞, w_1). Since u^τ is C^0-close to v_{21} for τ, the curve u_0 represents the homology class $[\Sigma_0]$, while the curve u_1 represents the homology class $[S^2] + [\Sigma_1]$. Moreover the images of u_0 and u_1 match at $S^2 \times \{\mathfrak{n}_1, \mathfrak{n}_2\}$. The image of u_0 is a small perturbation of the graph of a degree zero holomorphic map $\Sigma_0 \to S^2$ and the image of u_1 is a small perturbation of the graph of a degree one holomorphic map $\Sigma_1 \to S^2$. Then by elementary complex analysis there is a unique choice for u_0, while the choice for u_1 becomes unique once the intersection of its image with $S^2 \times \{\mathfrak{n}_1, \mathfrak{n}_2\}$ is fixed. Hence $\#\mathcal{M}_{B}^{\Diamond,irr} \equiv 1 \bmod 2$. □

3.4. *Family of cobordisms.* — We now describe a family of marked points $z_\tau \in X$ and a family of almost complex structures $J_\tau^\Diamond$ on X for $\tau \in [0,1)$, as well as their limits for $\tau = 1$. These families give rise to the chain homotopy H of Theorem 3.1.4.

Let $z_\tau^b \in int(B)$, $\tau \in [0,1)$, be a family of points such that $z_0^b = z^b$, $\lim_{\tau\to 1} z_\tau^b = (0,0)$, and $z_\tau^b \in \{s=0\}$ for $\tau \in [\frac{1}{2}, 1)$. Then let $z_\tau = (z_\tau^b, z^f) \in X$.

Assume that the almost complex structure J on X is a product complex structure on $\mathbf{R} \times [0,\varepsilon] \times \Sigma$ for $\varepsilon > 0$ small. We then define a family of C^ℓ-small perturbations $J_\tau^\Diamond$, $\tau \in [0,1)$, of J such that $J_\tau^\Diamond = J$ away from a small neighborhood $N(z_\tau)$ of z_τ and

$$N(z_\tau) \cap (\mathbf{R} \times [0,1] \times \{(z')^f\}) = \varnothing.$$

In the limit $\tau = 1$, the base $\widetilde{B}$ is $(B \sqcup D)/\sim$, where $D = \{|z| \leq 1\} \subset \mathbf{C}$ and $\sim$ identifies $(0,0) \in B$ with $-1 \in D$, and the total space $\widetilde{X}$ is $(X \sqcup (D \times \Sigma))/\sim$, where $((0,0),x) \sim (-1,x)$ for all $x \in \Sigma$. See Figure 2. We write w^b for the node $[(0,0)] = [-1] \in \widetilde{B}$. Let $\pi_B : X \to B$ and $\pi_D : D \times \Sigma \to D$ be the projections onto the first factors.

The limit z_1 of z_τ is in $D \times \Sigma$ and we assume that $z_1^b = 0 \in int(D)$. When $\tau = 1$, the almost complex structure $J_1^\Diamond$ restricts to the complex structure J on X and to the almost complex structure $J_D^\Diamond$, where J_D is a product complex structure on $D \times \Sigma$ and $J_D^\Diamond$ is a C^ℓ-small perturbation of J_D such that $J_D^\Diamond = J_D$ away from a small neighborhood $N(z_1)$ of z_1 and

$$N(z_1) \cap (D \times \{(z')^f\}) = \varnothing.$$

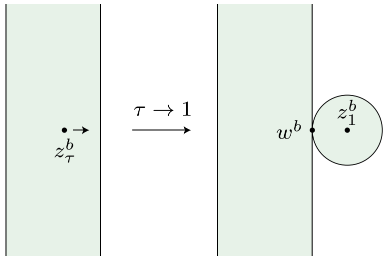


FIG. 2. — The degeneration of the base B together with the marked point z_τ^b as $\tau \to 1$. (Color figure online)

The Lagrangian boundary condition for $\tau \in [0, 1)$ is $L_{\boldsymbol\alpha} \cup L_{\boldsymbol\beta}$. In the limit $\tau = 1$, we use $L_{\boldsymbol\alpha} \cup L_{\boldsymbol\beta}$ for X and $\partial D \times \boldsymbol\beta$ for $D \times \Sigma$.

The degeneration for $\tau \to 1$ can be described in an equivalent way as a neck-stretching along a stable Hamiltonian hypersurface $\gamma \times \Sigma$, where γ is a boundary-parallel arc in the base B which separates a disk containing the z_τ^b.

3.5. *Proof of Theorem 3.1.4.* — Let u_{τ_i}, $\tau_i \to 1$, be a sequence of $I_{HF} = 2$ curves in $(X, J_{\tau_i}^\diamondsuit)$ from $[\mathbf{y}, i]$ to $[\mathbf{y}', i-k]$ that pass through z_{τ_i}. Applying SFT compactness in the neck-stretching setting and transferring the result to the nodal degeneration picture, we obtain the limit $\tilde u = u_B \cup u_D$, where $u_B \subset X$, $u_D \subset D \times \Sigma$, and u_D passes through z_1. Components of $\tilde u$ that map to the fiber $\{w^b\} \times \Sigma$ will be viewed as components of u_D.

Lemma **3.5.1.**

(1) $[u_D] = k_0[\{pt\} \times \Sigma] + 2g[D \times \{pt\}] \in H_2(D \times \Sigma)$ *for some* $0 < k_0 \le k$.
(2) $I(u_D) = 2k_0 + 2g \ge 2g + 2$.

Proof. — (1) $\deg(\pi_D \circ u_D) = 2g$, since u_{τ_i} is a degree $2g$ multisection of X for each τ_i, away from a neighborhood of $z_{\tau_i}^b$. Also, since $\langle u_{\tau_i}, B \times \{(z^f)'\}\rangle = k$ for all τ_i, it follows that $\langle u_D, D \times \{(z^f)'\}\rangle = k_0$, where $0 < k_0 \le k$. Here $k_0 > 0$ since u_D passes through z_1.

(2) is a consequence of (1) and computations as in the proof of Lemma 3.2.3. We remind the reader that the genus of Σ is $2g$. □

Lemma **3.5.2.** — $I(u_D) = 2g + 2$ *and* $I_{HF}(u_B) = 0$. *In particular,* $\mathbf{y} = \mathbf{y}'$, u_B *consists of* $2g$ *trivial strips, and* $k_0 = k = 1$.

Proof. — The gluing constraints give $I_{HF}(u_\tau) = I(u_D) + I_{HF}(u_B) - 2g = 2$. Strictly speaking, if there are (possibly multiply-covered) fiber components over $z = -1$ in D, then we should view $\tilde u$ as an SFT limit, in which case there will be intermediate levels with image in $D \times \Sigma$, where D has nodes at $z = \pm 1$, and there are no fiber components over $z = \pm 1$. We can then view u_D as the union of all the levels besides u_B, to which one can apply gluing constraints. By the regularity of J and the index inequality, we have $I_{HF}(u_B) \geq 0$. The first sentence of the lemma then follows from Lemma 3.5.1(2); the second sentence is a consequence of the first. □

The first sentence of Theorem 3.1.4 follows from the usual construction of chain homotopies in Floer theory: By Lemma 3.5.2, U_z is chain homotopic to aU, where a is the count of holomorphic curves u_D in $(D \times \Sigma, J_D^\diamond)$ that pass through z_1 and $\mathbf{w} = \{(0, y_1), \dots, (0, y_{2g})\}$, where $\mathbf{y} = \{y_1, \dots, y_{2g}\}$. Since $a = 1$ modulo 2 by Theorem 3.2.5, U_z is chain homotopic to U.

Next we prove the second sentence of Theorem 3.1.4. For all $\mathbf{y} \in \mathcal{S}$, $H([\mathbf{y}, 0])$ is obtained by counting $I_{HF} = 1$ curves that pass through z_τ for some $\tau \in (0, 1)$ and that do not cross the holomorphic strip $\mathbf{R} \times [0, 1] \times \{(z')^f\}$. There are no such curves since $\mathbf{R} \times [0, 1] \times \{z^f\}$ is holomorphic and homologous to $\mathbf{R} \times [0, 1] \times \{(z')^f\}$: if a curve passes through z_τ, its intersection with $\mathbf{R} \times [0, 1] \times \{z^f\}$ is strictly positive by the positivity of intersections, and so is its intersection with $\mathbf{R} \times [0, 1] \times \{(z')^f\}$.

4. The cobordism X_+

In this section we give the construction of the symplectic cobordism (X_+, Ω_{X_+}) from $[0, 1] \times \Sigma$ to M, together with the Lagrangian submanifold $L_{\boldsymbol{\alpha}} \subset \partial X_+$.

4.1. *Construction of* (X_+, Ω_{X_+}). — We describe the construction of X_+, leaving some key details for later:[2] First we construct fibrations $\pi_0 : X_+^0 \to B_+^0$ and $\pi_1 : X_+^1 \to D^2$ with fibers diffeomorphic to Σ and $S_{1/2}$. Here $B_+^0 = ([0, \infty) \times \mathbf{R}/2\mathbf{Z}) - B_+^c$ with coordinates (s, t) and B_+^c is the subset $[2, \infty) \times [1, 2]$ with the corners rounded. We then glue X_+^0 and X_+^1 and smooth a boundary component $\mathcal{B}$ of $X_+^0 \cup X_+^1$ to obtain $\widetilde{\mathcal{B}} \simeq M$. Finally we attach the negative end $X_+^2 = (-\infty, 0] \times \widetilde{\mathcal{B}}$ to obtain X_+.

Let $\delta > 0$ be a small irrational number and N a large positive number which depends on δ and whose dependence will be described later.

Lemma **4.1.1.** — *There exists a symplectic manifold* (X_+, Ω_{X_+}) *which depends on* $\delta > 0$ *and which satisfies the following:*

(1) *There is a symplectic surface* $S_{z^f} := \{z^f\} \times (B_+^0 \cup D^2)$, *obtained by gluing sections* $\{z^f\} \times B_+^0 \subset X_+^0$ *and* $\{z^f\} \times D^2 \subset X_+^1$.
(2) $\Omega_{X_+} = d\Theta^+$ *for some* 1*-form* Θ^+ *on* $X_+ - N(S_{z^f})$, *where* $N(S_{z^f})$ *is a small neighborhood of* S_{z^f}.
(3) Θ^+ *is exact on the Lagrangian submanifold* $L_{\boldsymbol{\alpha}} \subset \partial X_+$.
(4) *On the positive end*

$$\pi_0^{-1}([3, \infty) \times [0, 1]) = [3, \infty) \times \Sigma \times [0, 1] \subset X_+^0$$

[2] Compare with the description in Section 1.0.1, keeping in mind that the notation will be slightly different.

of X_+, Ω_{X_+} *restricts to* $\widetilde{\omega} + ds \wedge dt$, *where* $\widetilde{\omega}$ *is an area form on* Σ. *Moreover,*

$$L_{\boldsymbol{\alpha}} \cap \{s \geq 3\} = ([3,\infty) \times \{0\} \times \boldsymbol{\beta}') \cup ([3,\infty) \times \{1\} \times \boldsymbol{\alpha}),$$

where $\boldsymbol{\beta}'$ *is isotopic to* $\boldsymbol{\beta}$.

(5) *On the negative end* X_+^2 *of* X_+, Ω_{X_+} *restricts to the negative symplectization of a contact form* λ_- *on* $\widetilde{\mathcal{B}} \simeq M$ *which is adapted to the open book decomposition* $(S, \hbar)$.

(6) *The manifold* $\widetilde{\mathcal{B}} \simeq M$ *admits a decomposition into three disjoint pieces: the mapping torus* $N_{(S_0,\hbar)}$, *a closed neighborhood* $N(K)$ *of the binding* K, *and an open thickened torus* $\mathcal{N}$ *in between that we refer to as the "no man's land".*

(7) *All the orbits of the Reeb vector field* R_{λ_-} *of* λ_- *in int*$(N(K)) \cup \mathcal{N}$ *have* λ_-*-action* $\geq \frac{1}{2\delta} - \kappa$, *where* $\kappa > 0$ *is independent of* δ. *Moreover,* $T_+ = \partial N(K)$ *(resp.* $T_- = \partial N_{(S_0,\hbar)}$*) is a positive (resp. negative) Morse-Bott torus of meridian orbits.*

(8) *There is an embedding of* W_+, *defined in Section I.5.1.1, into* X_+ *such that the restriction* $\pi_1 : W_+ \cap X_+^0 \to B_+^0$ *is a fibration with fiber* S_0, $W_+ \cap X_+^1 = \varnothing$, $W_+ \cap X_+^2 = (-\infty, 0] \times N_{(S_0,\hbar)}$, *and* $W_+ \cap N(S_{z^f}) = \varnothing$.

Here X_+, Ω_{X_+}, Θ^+, $L_{\boldsymbol{\alpha}}$, *and* λ_- *depend on* $\delta > 0$.

The S^1-family $\mathcal{P}_+$ (resp. $\mathcal{P}_-$) of simple orbits of T_+ (resp. T_-) can be viewed equivalently as a pair e', h' (resp. e, h) consisting of an elliptic orbit and a hyperbolic orbit. The proof of Lemma 4.1.1 will be given in Section 4.3.

Let $A_{[-1,N]} \simeq [-1, N] \times S^1$ be a small neighborhood of $\partial S_0 = \{0\} \times S^1$ in Σ with coordinates (r_1, θ_1), such that $z^f \notin A_{[-1,N]}$, $A_{[-1,0]} \subset S_0$ and $A_{[0,N]} \subset S_{1/2}$. Here we write $A_{\mathfrak{I}} = \mathfrak{I} \times S^1$ if $\mathfrak{I}$ is a subset of $[-1, N]$. Also let $N(z^f) \subset S_{1/2} - A_{[0,N]} - \boldsymbol{\alpha} - \boldsymbol{\beta}$ be a small ball $D_\tau = \{r' \leq \tau\}$ about z^f, where we are using polar coordinates (r', θ').

The actual construction of (X_+, Ω_{X_+}) is a bit involved, and consists of several steps.

Step 1. The following lemma is a rephrasing of Lemma I.2.1.2 and its proof.

Lemma **4.1.2.** — *After possibly isotoping* $\hbar$ *relative to* ∂S_0, *there exists a factorization* $\hbar = \hbar_0 \circ \hbar_1$ *and a contact form* $\lambda = f_t(x)dt + \beta_t(x)$, $(x, t) \in S_0 \times [0, 2]$, *on* $N_{(S_0,\hbar_0)}$ *with Reeb vector field* R_λ, *such that the following hold:*

(1) $\hbar : S_0 \times \{0\} \xrightarrow{\sim} S_0 \times \{0\}$ *is the first return map of* R_λ.
(2) $\hbar$ *has no elliptic periodic point of period* $\leq 2g$ *in int*(S_0), *as required for technical reasons in II.1.0.1.*
(3) $\hbar_0 = id$ *on* $A_{[-1/2,0]}$.
(4) $\hbar_1$ *is the flow of* R_λ *from* $S_0 \times \{0\}$ *to* $S_0 \times \{2\}$.[3]
(5) R_λ *is parallel to* ∂_t *on* $(S_0 - A_{[-1,0]}) \times [0, 2]$. *In particular,* $\hbar_1 = id$ *on* $S_0 - A_{[-1,0]}$.

[3] In a departure from the stable Hamiltonian vector field $R_0 = \partial_t$ from Section I.5.1, we are not assuming R_λ to be parallel to ∂_t on all of $S_0 \times [0, 2]$.

(6) *$f_t(r_1,\theta_1) = 1+\varepsilon r_1^2/2$ and $\beta_t(r_1,\theta_1) = (C+r_1)d\theta_1$ on $A_{[-1/2,0]}$, for $\varepsilon > 0$ sufficiently small and $C > 0$. In particular, f_t and β_t are independent of t and R_λ is parallel to $\partial_t - \varepsilon r_1 \partial_\theta$ on $A_{[-1/2,0]}$.*

(7) $|d_2 f_t|_{A_{[-1/2,0]}}|_{C^0} \leq \delta$ *and* $\frac{1}{2} \leq f_t \leq 2$.

Here $\varepsilon > 0$ depends on $\delta > 0$, d_2 is the differential in the S_0-direction, and the C^0-norm is with respect to a fixed Riemannian metric on S_0.

Step 2. We then extend $h_0, h_1, h \in \mathrm{Diff}(S_0, \partial S_0)$ to $h_0^+, h_1^+, h^+ = h_0^+ \circ h_1^+ \in \mathrm{Diff}(\Sigma)$ and the contact form λ to the contact form $\lambda_+ = f_t dt + \beta_t$ to $N_{(\Sigma - N(z^f), h_0^+)}$, all of which depend on $\delta > 0$, as follows:

(3') $h_0^+ = id$ on $S_{1/2}$.

(4') $h_1^+|_{\Sigma - N(z^f)}$ is the flow of R_{λ_+} from $(\Sigma - N(z^f)) \times \{0\}$ to $(\Sigma - N(z^f)) \times \{2\}$ and $h_1^+|_{N(z^f)} = id$.

(5') f_t and β_t are independent of t on $S_{1/2} - N(z^f)$. Hence R_{λ_+} is parallel to $\partial_t + X_f$, where X_f is the Hamiltonian vector field satisfying $i_{X_f}\omega = d_2 f$ and ω is an area form on Σ which agrees with $d_2\beta_t$ on $\Sigma - N(z^f)$.

(6a') $f_t(r', \theta') = const > 0$ and $\beta_t(r',\theta') = (-C' + r')d\theta'$ near $\partial N(z^f)$, for $-C' > 0$. In particular, R_{λ_+} is parallel to ∂_t near the mapping torus of $\partial N(z^f)$.

(6b') $f_t(r_1,\theta_1) = 1 + \varepsilon r_1^2/2$ near $A_{\{0\}}$ and $\beta_t(r_1,\theta_1) = (C+r_1)d\theta_1$ on $A_{[0,N]}$.

(7') $|d_2 f_t|_{S_{1/2} - N(z^f)}|_{C^0} \leq \delta$ and $\frac{1}{2} \leq f_t|_{S_{1/2}-N(z^f)} \leq 2$.

Without loss of generality we may assume that $\boldsymbol{\alpha} \times \{1\}$ is Legendrian with respect to λ_+. This is an easy consequence of the Legendrian realization principle; see for example [H, Theorem 3.7].

Step 3 (Construction of $(X_+^0, \Omega^0_{X_+})$). Let

$$\widetilde{X}_+^0 = ([0,\infty) \times \Sigma \times [0,2])/(s,x,2) \sim (s, h_0^+(x), 0)$$

and let $\pi_0 : \widetilde{X}_+^0 \to [0,\infty) \times \mathbf{R}/2\mathbf{Z}$ be the projection $(s,x,t) \mapsto (s,t)$. We then set

$$X_+^0 := \pi_0^{-1}(B_+^0).$$

Let $g : [0, \frac{1}{2}] \to \mathbf{R}$ be a smooth function such that $g(r) = 1 + \varepsilon r^2/2$ near $r = 0$, $0 < g'(r) \leq \delta$ for $r \in (0, \frac{1}{2})$, $g'(r)$ is monotonically decreasing for $r \in (\frac{1}{4}, \frac{1}{2})$, $g'(\frac{1}{2}) = 0$, and $g(\frac{1}{2}) = 1 + \varepsilon$. In particular, this requires $2\varepsilon < \delta$. Then let

$$\lambda_{+,s} = f_{s,t} dt + \beta_t, \quad s \in [0,\infty),$$

be a 1-parameter family of contact forms[4] on $N_{(\Sigma - N(z^f), h_0^+)}$ such that the following hold:

(a) $\lambda_{+,s} = \lambda_+$ if $s \geq \frac{3}{2}$ or $(x,t) \in N_{(S_0, h_0)}$.

[4] Note that β_t does not depend on s.

(b) $\lambda_{+,s}$ is independent of s if $s \in [0, \frac{1}{2}]$.
(c) $f_{0,t}(r_1, \theta_1) = g(r_1)$ on $A_{[0,1/2]}$.
(d) $f_{0,t}|_{S_{1/2}-A_{[0,1/2]}-N(z^f)} = 1 + \varepsilon$. In particular, $d\lambda_{+,0} = d_2\beta_t$ and $R = \partial_t$ on the mapping torus of $S_{1/2} - A_{[0,1/2]} - N(z^f)$.
(e) $f_{s,t}$ is a constant $C_s > 0$ near $\partial N(z^f)$.
(f) $|d_2 f_{s,t}|_{A_{[-1/2,0]}\cup S_{1/2}-N(z^f)}|_{C^0} \le \delta$, $|\partial_s f_{s,t}|_{A_{[-1/2,0]}\cup S_{1/2}-N(z^f)}|_{C^0} \le \delta$ and $\frac{1}{2} \le f_{s,t}|_{\Sigma - N(z^f)} \le 2$ for all s, t.

We then define:

$$\Omega^0_{X_+} := \widetilde{\omega} + ds \wedge dt,$$

where

$$\widetilde{\omega} = \begin{cases} d\lambda_{+,s} & \text{on } X^0_+ - (N(z^f) \times B^0_+); \\ \omega & \text{on } N(z^f) \times B^0_+; \end{cases}$$

and ω is an area form on Σ which agrees with $d_2\beta_t$ on $\Sigma - N(z^f)$. The 2-form $\Omega^0_{X_+}$ is symplectic by an easy calculation which uses (f).

Step 4 (Construction of $(X^1_+, \Omega^1_{X_+})$ and primitives Θ^+_0, Θ^+_1). Let

$$X^1_+ := S'_{1/2} \times D^2, \quad S'_{1/2} := S_{1/2} - A_{[0,1/2]}.$$

We use polar coordinates (r_2, θ_2) on $D^2 = \{r_2 \le 1\}$. We identify neighborhoods of $\{0\} \times S'_{1/2} \times \mathbf{R}/2\mathbf{Z} \subset \partial X^0_+$ and $S'_{1/2} \times \partial D^2 \subset \partial X^1_+$ as follows:

$$\phi_{01} : [-\varepsilon', \varepsilon'] \times S'_{1/2} \times \mathbf{R}/2\mathbf{Z} \xrightarrow{\sim} S'_{1/2} \times \{(r_2, \theta_2) \mid e^{-\pi\varepsilon'} \le r_2 \le e^{\pi\varepsilon'}\},$$
$$(s, x, t) \mapsto (x, e^{\pi s}, \pi t),$$

where $\varepsilon' > 0$ is sufficiently small.

Let ω_{D^2} be an area form on D^2 satisfying:

$$\omega_{D^2} = \begin{cases} r_2 dr_2 d\theta_2 & \text{near } r_2 = 0; \\ \frac{1}{\pi^2 r_2} dr_2 d\theta_2 & \text{near } r_2 = 1. \end{cases}$$

We then define

$$\Omega^1_{X_+} := \widetilde{\omega}|_{S'_{1/2}} + \omega_{D^2}.$$

An easy calculation shows that $\omega_{D^2} = ds \wedge dt$, and hence $\Omega^1_{X_+} = \Omega^0_{X_+}$, on their overlap.

We write $\omega_{D^2} = d(\phi(r_2) d\theta_2)$, where $\phi : [0, 1] \to \mathbf{R}$ satisfies

$$\phi(r_2) = \begin{cases} r_2^2/2 & \text{near } r_2 = 0; \\ \frac{1}{\pi^2} \log r_2 + \frac{1}{10} & \text{near } r_2 = 1. \end{cases}$$

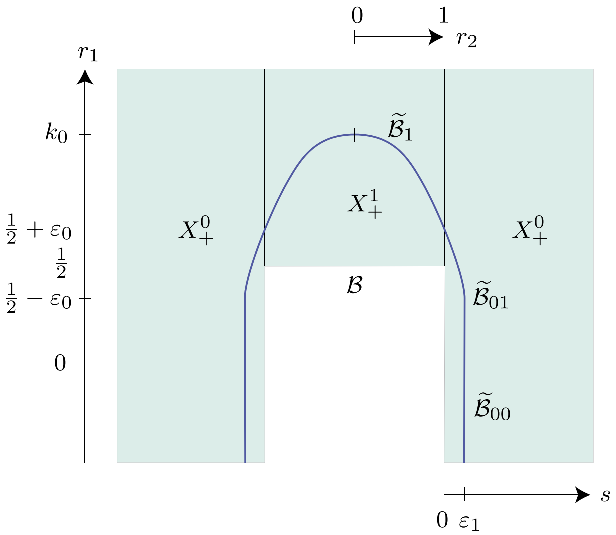


FIG. 3. — Schematic diagram for rounding the corner of $\mathcal{B}$. The diagram shows a neighborhood $N(\mathcal{B})$ of $\mathcal{B}$, where we are projecting $X^0_+ \cap N(\mathcal{B})$ to coordinates (s, r_1) and $X^1_+ \cap N(\mathcal{B})$ to coordinates (r_2, r_1). (Color figure online)

Then $\phi(r_2)d\theta_2 = (s + \frac{\pi}{10})dt$ on their overlap. The choice of the constant $\frac{\pi}{10} < 1$ will be used in the proof of Lemma 5.4.2. We then define primitives Θ^+_i of $\Omega^i_{X_+}$, $i = 0, 1$, as follows:

$$\textbf{(4.1.1)} \qquad \Theta^+_0 = \lambda_{+,s} + (s + \tfrac{\pi}{10})dt \quad \text{on } X^0_+ - (N(z^f) \times B^0_+);$$

$$\textbf{(4.1.2)} \qquad \Theta^+_1 = \lambda_{+,0} + \phi(r_2)d\theta_2 \quad \text{on } X^1_+ - (N(z^f) \times D^2).$$

We have $\Theta^+_0 = \Theta^+_1$ on their overlap.

Step 5 (Corner smoothing). We now have a 4-manifold $X^0_+ \cup X^1_+$ with a concave corner along $(\partial S'_{1/2}) \times \partial D^2$. The component $\mathcal{B}$ of $\partial(X^0_+ \cup X^1_+)$ that contains the corner is homeomorphic to M and $(\partial S'_{1/2}) \times D^2$ is a neighborhood of the binding $(\partial S'_{1/2}) \times \{0\}$.

In this step we round the corner of $\mathcal{B}$ to obtain the smoothing $\widetilde{\mathcal{B}} \subset X^0_+ \cup X^1_+$. We write $\widetilde{\mathcal{B}}_i = \widetilde{\mathcal{B}} \cap X^i_+$, $i = 0, 1$. We define the contact form λ_- on $\widetilde{\mathcal{B}}$ so that $\lambda_-|_{\widetilde{\mathcal{B}}_i} = \Theta^+_i|_{\widetilde{\mathcal{B}}_i}$, $i = 0, 1$. Here the notation $|_A$ refers to the pullback to A. See Figure 3.

Construction of $\widetilde{\mathcal{B}}_0$. There exist $\varepsilon_0, \varepsilon_1 > 0$ small with $\frac{\varepsilon_1}{2\varepsilon_0} < \delta$ and $\varepsilon_1 < \varepsilon'$ and a smooth map $\psi : [0, \frac{1}{2} + \varepsilon_0] \to \mathbf{R}$ such that:

- $\psi(r_1) = \varepsilon_1$ on $[0, \frac{1}{2} - \varepsilon_0]$;
- $\psi'(r_1)$ is monotonically decreasing and $-\delta \le \psi'(r_1) < 0$ on $(\frac{1}{2} - \varepsilon_0, \frac{1}{2} + \varepsilon_0)$; and
- $\psi(\frac{1}{2} + \varepsilon_0) = 0$ and $\psi'(\frac{1}{2} + \varepsilon_0) = -\delta$.

We then let $\widetilde{\mathcal{B}}_0 = \widetilde{\mathcal{B}}_{00} \cup \widetilde{\mathcal{B}}_{01}$, where

$$\widetilde{\mathcal{B}}_{00} = \{s = \varepsilon_1\} \times N_{(S_0, h_0)},$$

$$\widetilde{\mathcal{B}}_{01} = \{s = \psi(r_1), r_1 \in [0, \tfrac{1}{2} + \varepsilon_0]\} \times \mathbf{R}/2\mathbf{Z} \times S^1.$$

Here $\mathbf{R}/2\mathbf{Z} \times S^1$ has coordinates (t, θ_1).

Lemma **4.1.3.** — *There exists a unique* $r_1^* \in (0, \frac{1}{2} + \varepsilon_0)$ *such that each orbit in* $\widetilde{\mathcal{B}}_{01} \cap \{r_1 \neq r_1^*\}$ *is directed by some* $\partial_t + \delta'\partial_{\theta_1}$, *where* $0 < |\delta'| \leq \delta$ *and* δ' *depends on the orbit. Also* $\widetilde{\mathcal{B}}_{01} \cap \{r_1 = \frac{1}{2} + \varepsilon_0\}$ *is directed by* $\partial_t + \delta\partial_{\theta_1}$.

Proof. — The 1-form $\lambda_-|_{\widetilde{\mathcal{B}}_{00}}$ is clearly a contact form and

$$\lambda_-|_{\widetilde{\mathcal{B}}_{01}} = (\psi(r_1) + f_{0,t}(r_1, \theta_1) + \pi/10)dt + (C + r_1)d\theta_1, \tag{4.1.3}$$

with respect to coordinates (r_1, θ_1, t). The Reeb vector field R_{λ_-} is parallel to $\partial_t - \frac{\partial}{\partial r_1}(\psi + f_{0,t})\partial_{\theta_1}$. Let $r_1^* \in [0, \frac{1}{2} + \varepsilon_0]$ be the point where $\frac{\partial}{\partial r_1}(\psi + f_{0,t}) = 0$. Then $0 < -\frac{\partial}{\partial r_1}(\psi + f_{0,t}) \leq \delta$ for $r_1 \in [r_1^*, \frac{1}{2} + \varepsilon_0]$, $-\frac{\partial}{\partial r_1}(\psi + f_{0,t})(\frac{1}{2} + \varepsilon_0) = \delta$, and $0 < \frac{\partial}{\partial r_1}(\psi + f_{0,t}) \leq \delta$ for $r_1 \in (0, r_1^*)$, which imply the lemma. □

Construction of $\widetilde{\mathcal{B}}_1$. Let $\zeta : [0, 1] \to \mathbf{R}$ be a smooth map such that:

- $\zeta(r_2) = k_0 - k_1 r_2^2/2$ near $r_2 = 0$, where $k_0, k_1 \gg 0$;
- $\zeta'' < 0$ on $(0, 1]$;
- $\zeta(1) = \frac{1}{2} + \varepsilon_0$.

We then define $\widetilde{\mathcal{B}}_1 = \{r_1 = \zeta(r_2)\}$.

Lemma **4.1.4.** — *There exist* $k_0, k_1 \gg 0$, $N = N(k_0, k_1) \gg 0$, *and* ζ *such that* $R_{\lambda_-}|_{\widetilde{\mathcal{B}}_1}$ *is directed by* $\pi\partial_{\theta_2} + \delta\partial_{\theta_1}$, *which agrees with* $\partial_t + \delta\partial_{\theta_1}$ *on* $\widetilde{\mathcal{B}}_0$.

Proof. — $\lambda_-|_{\widetilde{\mathcal{B}}_1}$ is given by

$$\lambda_-|_{\widetilde{\mathcal{B}}_1} = (\phi(r_2) + (1 + \varepsilon)/\pi)d\theta_2 + (C + \zeta(r_2))d\theta_1, \tag{4.1.4}$$

with respect to coordinates $(\theta_1, r_2, \theta_2)$. The Reeb vector field R_{λ_-} is parallel to $\pi\partial_{\theta_2} - \pi\frac{\phi'}{\zeta'}\partial_{\theta_1}$. By choosing $k_0, k_1 \gg 0$, $N(k_0, k_1) \gg 0$, and ζ suitably, we may assume that $-\frac{\phi'}{\zeta'}(r_2) = \frac{\delta}{\pi}$ for all $r_2 \in (0, 1]$. □

We also define $N(K) \subset \widetilde{\mathcal{B}}$ as the closed neighborhood of the binding $K = \{r_2 = 0\}$ that is bounded by the torus $\{r_1 = r_1^*\}$. The region $\mathcal{N} = \{0 < r_1 < r_1^*\} \subset \widetilde{\mathcal{B}}$ will be called "no man's land".

Step 6 (Construction of $(X_+^2, \Omega_{X_+}^2)$). Let $X_+^{01} \subset X_+^0 \cup X_+^1$ be the closure of the component of $(X_+^0 \cup X_+^1) - \widetilde{\mathcal{B}}$ that does not contain $\mathcal{B}$. We then glue the negative cylindrical end

$$(X_+^2, \Omega_{X_+}^2) := ((-\infty, 0] \times \widetilde{\mathcal{B}}, d(e^{s'}\lambda_-))$$

to X_+^{01} along $\widetilde{\mathcal{B}}$, where s' is the coordinate for $(-\infty, 0]$. This concludes the construction of (X_+, Ω_{X_+}).

4.2. *Further definitions.*
Hamiltonian structure on $\Sigma \times [0,1]$. Let $\overline{\omega} = \widetilde{\omega}|_{s=\frac{3}{2}}$. The Hamiltonian structure on $\Sigma \times [0,1]$ at the positive end of X_+ is given by $(dt, \overline{\omega}|_{\Sigma\times[0,1]})$. Let h_2^+ be the flow of the corresponding Hamiltonian vector field from $\Sigma \times \{0\}$ to $\Sigma \times \{1\}$; *this is different from* h_1^+, *which is the flow from* $\Sigma \times \{0\}$ *to* $\Sigma \times \{2\}$. Also note that we do not necessarily have $h_2^+ = id$ by construction.
Lagrangian submanifold $L_{\boldsymbol{\alpha}}$. As in Section I.5.2.1, we define the Lagrangian submanifold $L_{\boldsymbol{\alpha}} \subset \partial X_+$ by placing a copy of $\boldsymbol{\alpha}$ on the fiber $\pi^{-1}(3,1)$ over $(3,1) \in \partial B_+^0$ and using the symplectic connection Ω_{X_+} to parallel transport $\boldsymbol{\alpha}$ along the boundary component $(\partial B_+^0) \cap \{s \geq 1\}$ of B_+^0. Observe that

$$L_{\boldsymbol{\alpha}} \cap \{s \geq 3\} = ([3,\infty) \times \{0\} \times h_0^+(\boldsymbol{\alpha})) \cup ([3,\infty) \times \{1\} \times \boldsymbol{\alpha}). \tag{4.2.1}$$

Lemma **4.2.1.** — $\boldsymbol{\beta}' := h^+ \circ (h_2^+)^{-1}(\boldsymbol{\alpha})$ *is isotopic to* $\boldsymbol{\beta}$.

Proof. — Observe that h_1^+ and h_2^+ are isotopic to the identity. Then h^+ is isotopic to h_0^+ where $h_0^+|_{S_{1/2}} = id$ and $h_0^+|_{S_0}$ is isotopic to h. The lemma then follows. □

Submanifolds S_z, C_θ, *and* $\mathcal{H}$. Given $z \in N(z^f)$, let

$$S_z = \{z\} \times (B_+^0 \cup D^2),$$

where $\{z\} \times B_+^0 \subset X_+^0$ and $\{z\} \times D^2 \subset X_+^1$. Also let

$$C_\theta = (\{\theta\} \times B_+^0) \cup (\{\theta\} \times (-\infty, 0]_{s'} \times \mathbf{R}/2\mathbf{Z}),$$

where $\theta \in \partial S_0$, and let $\mathcal{H} = \cup_{\theta \in \partial S_0} C_\theta$.
Definition of W_+. Let W_+ be the closure of the component of $X_+ - \mathcal{H}$ which is disjoint from $S_{(z')^f}$. In particular, the restriction $\pi_1 : W_+ \cap X_+^0 \to B_+^0$ is a fibration with fiber S_0, $W_+ \cap X_+^1 = \varnothing$, and $W_+ \cap X_+^2 = (-\infty, 0] \times N_{(S_0, h)}$. The cobordism W_+ is diffeomorphic to the cobordism used to define the map Φ in Section I.5.1.

4.3. *Proof of Lemma 4.1.1.* — (1), (5), (6), (8) are clear from the construction.
(2) follows by letting $\Theta^+ = \Theta_i^+$, $i = 0, 1, 2$, where defined.
(3) By construction, $L_{\boldsymbol{\alpha}}$ is Lagrangian and $d\Theta^+|_{L_{\boldsymbol{\alpha}}} = 0$. It then suffices to observe that $\Theta^+ = 0$ on $L_{\boldsymbol{\alpha}} \cap \pi^{-1}(3,1)$. This follows from the fact that $\boldsymbol{\alpha} \times \{1\}$ is a Legendrian submanifold of $(N_{(\Sigma - N(z^f), h_0^+)}, \lambda_+)$.
(4) The first sentence follows from the construction and the second sentence follows from Lemma 4.2.1.
(7) By Lemma 4.1.4, the Reeb vector field R_{λ_-} has no closed orbits in $\widetilde{\mathcal{B}}_1$ since $\delta > 0$ is irrational. By Lemma 4.1.3 and Equation (4.1.3), each orbit of R_{λ_-} in $\widetilde{\mathcal{B}}_{01} \cap \{r_1 \neq r_1^*\}$

has λ_--action $\geq \frac{1}{2\delta} - (C + \frac{1}{2})$, where $C > 0$ is independent of δ. The second sentence of (7) is immediate from the construction of λ_-.

5. The chain map Φ^+

The goal of this section is to define the chain map

$$\Phi^+ : CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f) \to ECC(M, \lambda_-),$$

which is induced by the symplectic cobordism (X_+, Ω_{X_+}) and an admissible almost complex structure J^+. We can write $\boldsymbol{\beta} = h_2^+ \circ h^+ \circ (h_2^+)^{-1}(\boldsymbol{\alpha})$, in view of Equation (4.2.1) and Lemma 4.2.1 and the fact that h_2^+ is the flow of the Hamiltonian vector field of $\widetilde{\omega}|_{s=s_0}$, $s_0 \gg 0$, from $\Sigma \times \{0\}$ to $\Sigma \times \{1\}$ before normalization.

For simplicity we identify $X_+ \cap \{s \geq s_0\} \simeq [s_0, \infty) \times [0, 1] \times \Sigma$ *with coordinates* (s, t, x) *so that* $h_2^+ = id$ *and the Hamiltonian vector field is* ∂_t.

5.1. *Almost complex structures.* — Let $\overline{\omega} = \widetilde{\omega}|_{s=3/2}$.

Lemma **5.1.1.** — *There exists a family* $(\overline{\lambda}_\tau, \overline{\omega})$, $\tau \in [0, 1]$, *of stable Hamiltonian structures on* $N_{(S_0, h_0)}$ *such that* $\overline{\lambda}_1 = \lambda$, $\overline{\lambda}_\tau$ *is a contact form for* $\tau > 0$, *and* $\overline{\lambda}_0 = dt$. *The* 1*-forms* $\overline{\lambda}_\tau = f_{t,\tau} dt + \beta_{t,\tau}$ *can be normalized so that* $\frac{1}{2} < |f_{t,\tau}| \leq 2$.

Proof. — Follows from the discussion of Section I.3.1. □

Definition **5.1.2.** — *An almost complex structure* J^+ *on* X_+ *is* (X_+, Ω_{X_+})*-admissible if the following hold:*

(1) J^+ *is tamed by* Ω_{X_+}*;*
(2) J^+ *is* s*-invariant for* $\{s \geq \frac{3}{2}\} \cap X_+^0$ *and is adapted to the stable Hamiltonian structure* $(dt, \overline{\omega}|_{\Sigma\times[0,1]})$ *at the positive end;*
(3) J^+ *is* s'*-invariant for* $\{s' \leq -\frac{1}{2}\} \cap X_+^2$ *and is adapted to the contact form* λ_- *at the negative end;*
(4) *the restriction* J_+ *of* J^+ *to* W_+ *is* C^ℓ*-close to a regular admissible almost complex structure* J_+^0 *on* W_+ *with respect to* $(\overline{\lambda}_0, \overline{\omega})$ *(cf. Definitions I.5.4.1 and I.5.8.5);*
(5) *the surfaces* $S_{(z')^f}$ *and* C_θ *are* J^+*-holomorphic for all* $\theta \in \partial S_0$.

Let J, J' *be the adapted almost complex structures that agree with* J^+ *at the positive and negative ends.*

Note that (4) imposes additional conditions on Ω_{X_+} and λ_-. In practice, the order in which we construct Ω_{X_+} and J^+ is a little convoluted: (i) choose a regular J_+^0, (ii) choose $\tau > 0$ sufficiently small and J_+ sufficiently close to J_+^0, (iii) construct Ω_{X_+} using $\overline{\lambda}_\tau$ in place of λ, and (iv) extend J^+ to the rest of X_+.

Let $\mathcal{J}_{X_+}$ be the set of all (X_+, Ω_{X_+})-admissible almost complex structures.

5.2. *The ECH index.* — Let $\mathcal{P} = \mathcal{P}_{\lambda_-}$ be the set of simple orbits of R_{λ_-} and let $\mathcal{O} = \mathcal{O}_{\lambda_-}$ be the set of orbit sets constructed from $\mathcal{P}$.

Let $J^+ \in \mathcal{J}_{X_+}$ be an admissible almost complex structure. Let $\mathcal{M}_{J^+}(\mathbf{y}, \boldsymbol{\gamma})$ be the set of holomorphic maps $u : (\dot{F}, j) \to (X_+, J^+)$ from $\mathbf{y} \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ to $\boldsymbol{\gamma} \in \mathcal{O}$, such that each component of $\partial\dot{F}$ is mapped to a distinct component of $L_{\boldsymbol{\alpha}}$ and each component of $L_{\boldsymbol{\alpha}}$ is used exactly once. Here (F, j) is a compact Riemann surface with boundary, $\dot{F} = F - \mathbf{q}_+ - \mathbf{q}_-$, $\mathbf{q}_+$ is the set of boundary punctures, and $\mathbf{q}_-$ is the set of interior punctures. Elements of $\mathcal{M}_{J^+}(\mathbf{y}, \boldsymbol{\gamma})$ will be called X_+-*curves.*

Let $\check{X}_+$ be X_+ with the ends $\{s > 3\}$ and $\{s' < -1\}$ removed and let

$$Z_{\mathbf{y},\boldsymbol{\gamma}} = (L_{\boldsymbol{\alpha}} \cap \check{X}_+) \cup (\{3\} \times [0, 1] \times \mathbf{y}) \cup (\{-1\} \times \boldsymbol{\gamma})$$

as in Section I.5.4.2. The class $[u]$ of $u \in \mathcal{M}_{J^+}(\mathbf{y}, \boldsymbol{\gamma})$ is the relative homology class of the compactification $\check{u}$ in $H_2(\check{X}_+, Z_{\mathbf{y},\boldsymbol{\gamma}})$. Given $A \in H_2(\check{X}_+, Z_{\mathbf{y},\boldsymbol{\gamma}})$, we write $\mathcal{M}_{J^+}(\mathbf{y}, \boldsymbol{\gamma}, A) \subset \mathcal{M}_{J^+}(\mathbf{y}, \boldsymbol{\gamma})$ for the subset of X_+-curves u in the class A.

Definition **5.2.1** *(Filtration $\mathcal{F}$). — Given a X_+-curve u that limits to* $\mathbf{y}$ *at the positive end and* $\boldsymbol{\gamma}$ *at the negative end, we define*

$$\mathcal{F}(u) = \langle [u], S_{(z')^f} \rangle,$$

where $\langle , \rangle$ is the algebraic intersection number. Since $S_{(z')^f}$ is a holomorphic divisor, $\mathcal{F}(u) \geq 0$. We will also refer to u as an X_+-curve from $[\mathbf{y}, \mathcal{F}(u)]$ to $\mathbf{y}$.

The definition of the ECH index given in Section I.5.6 also extends directly to our case. The ECH index of a X_+-curve from $\mathbf{y}$ to $\boldsymbol{\gamma}$ in the class A is denoted by $I_{X_+}(\boldsymbol{\gamma}, A)$.

5.3. *Homology of* X_+. — The goal of this subsection is to compute $H_2(X_+)$. Let us write $N = N_{(S_0,\hbar)}$, $N_0 = N_{(S_{1/2},\hbar^+|_{S_{1/2}})}$ and $\overline{N} = N_{(\Sigma,\hbar^+)}$.

Lemma **5.3.1.** — $H_2(N) \cong H_2(M)$ *and* $H_1(N) \cong H_1(M) \oplus \mathbf{Z}$*, where the extra* $\mathbf{Z}$ *factor is generated by a meridian of the binding.*

Proof. — The lemma follows from the exact sequence of the pair (M, N). ☐

Lemma **5.3.2.** — $H_2(X_+^0) \cong H_2(N) \oplus H_2(N_0) \oplus H_2(\Sigma)$.

Proof. — Observe that X_+^0 is homotopy equivalent to $\overline{N}$. We compute $H_2(\overline{N})$ using the Mayer-Vietoris sequence:

$$H_2(N \cap N_0) \xrightarrow{i} H_2(N) \oplus H_2(N_0) \to H_2(\overline{N}) \to H_1(N \cap N_0)$$
$$\xrightarrow{j} H_1(N) \oplus H_1(N_0).$$

Since $i=0$ and $\ker j = \mathbf{Z}\langle\partial S_0\rangle = \mathbf{Z}\langle\partial S_{1/2}\rangle$, the lemma follows. □

Lemma **5.3.3.** — $H_2(X_+) \cong H_2(M) \oplus H_2(\Sigma)$.

Proof. — X_+ is homotopy equivalent to $X_+^0 \cup X_+^1$ and $X_+^0 \cap X_+^1 \cong N_0$. Since X_+^1 is homotopy equivalent to $S_{1/2}$, the Mayer-Vietoris sequence becomes:

$$H_2(N_0) \xrightarrow{i} H_2(X_+^0) \to H_2(X_+) \to H_1(N_0) \xrightarrow{j} H_1(X_+^0) \oplus H_1(S_{1/2}).$$

The map i surjects onto the factor $H_2(N_0)$ in the decomposition of $H_2(X_+^0)$ coming from Lemma 5.3.2. The map j is injective, since $H_1(N_0) \cong H_1(S_{1/2}) \oplus H_1(S^1)$ by the Künneth formula, the restriction $j : H_1(S_{1/2}) \to H_1(S_{1/2})$ is an isomorphism, and the restriction $j : H_1(S^1) \to H_1(\overline{N}) \simeq H_1(X_+^0)$ is injective because the image of the generator of $H_1(S^1)$ is dual to the fiber Σ. The lemma then follows from Lemma 5.3.1. □

5.4. *Energy bound.*

Definition **5.4.1.** — *Let $\mathcal{C}_+$ be the set of nondecreasing functions $\phi : [0, +\infty) \to [0, 1]$ such that $\phi(s) = s + \frac{\pi}{10}$ near $s = 0$*[5] *and let $\mathcal{C}_-$ be the set of nondecreasing functions $\psi : (-\infty, 0] \to [0, 1]$ such that $\psi(s') = e^{s'}$ near $s' = 0$. Let*

$$\Omega^+_{\phi,\psi} := \begin{cases} \widetilde{\omega} + d\phi(s) \wedge dt & \textit{on } X_+^0 \cap X_+^{01}; \\ \Omega^1_{X_+} & \textit{on } X_+^1 \cap X_+^{01}; \\ d(\psi(s')\lambda_-) & \textit{on } X_+^2, \end{cases}$$

where $(\phi, \psi) \in \mathcal{C}_+ \times \mathcal{C}_-$.[6] *Then the* energy *of an X_+-curve $u : \dot{F} \to X_+$ from $[\mathbf{y}, k]$ to $\boldsymbol{\gamma}$ is given by:*

$$(\mathbf{5.4.1}) \qquad E(u) = \sup_{\phi,\psi} \int_F u^* \Omega^+_{\phi,\psi},$$

where the supremum is taken over all pairs $(\phi, \psi) \in \mathcal{C}_+ \times \mathcal{C}_-$.

The condition imposed on the intersection with $S_{(z')^f}$ gives an energy bound:

Lemma **5.4.2** *(Energy bound).* — *For all $k \in \mathbf{N}$, there exists $N_k > 0$ such that $E(u) \le N_k$ for all $\mathbf{y} \in \mathcal{S}_{\alpha,\beta}$, $\boldsymbol{\gamma} \in \mathcal{O}$, and $u \in \mathcal{M}^{\mathcal{F}=k}_{J^+}(\mathbf{y}, \boldsymbol{\gamma})$.*

Proof. — Let $u : (\dot{F}, j) \to (X_+, J^+)$ be an element of $\mathcal{M}^{\mathcal{F}=k}_{J^+}(\mathbf{y}, \boldsymbol{\gamma})$. By (2) and (3) of Lemma 4.1.1, $\Omega_{X_+} = d\Theta^+$ on $X_+^\circ := X_+ - N(S_{z^f})$ and Θ^+ is exact on the Lagrangian

[5] See the discussion in the second paragraph of the proof of Lemma 5.4.2 which justifies this definition.
[6] ϕ, ψ used here are not to be confused with ϕ, ψ which appeared in Section 4.1.

$L_{\boldsymbol{\alpha}}$. Hence $\int_{\partial\dot{F}} u^*\Theta^+$ only depends on $\mathbf{y}$. Since $\Theta^+ = (s+\frac{\pi}{10})dt + \lambda_+$ along $\operatorname{Im} u(\partial\dot{F})$ by Equation (4.1.1) and Section 4.1, Step 3, Item (a), there exists a constant $C(\mathbf{y})$ such that

$$(\mathbf{5.4.2}) \qquad \int_{\partial\dot{F}} u^*\lambda_+ < C(\mathbf{y}).$$

Let $v : \dot{F}' \to X_+^\circ$ be a representative of the homology class $[u] - k[\Sigma] \in H_2(\check{X}_+, Z_{\mathbf{y},\boldsymbol{\gamma}})$. Since the energy is obtained by integrating a closed form,

$$(\mathbf{5.4.3}) \qquad E(u) = E(v) + k\int_\Sigma \widetilde{\omega}.$$

Now $\Omega^+_{\phi,\psi} = d\Theta^+_{\phi,\psi}$ on X_+°, where

$$\Theta^+_{\phi,\psi} = \begin{cases} \lambda_{+,s} + \phi(s)dt & \text{on } X_+^0 \cap X_+^\circ \cap X_+^{01}; \\ \Theta_1^+ & \text{on } X_+^1 \cap X_+^\circ \cap X_+^{01}; \\ \psi(s')\lambda_- & \text{on } X_+^2. \end{cases}$$

By Equations (4.1.1) and (4.1.2), Θ_1^+ can be written as $\lambda_{+,s} + (s+\frac{\pi}{10})dt$ on $X_+^0 \cap X_+^1 \cap X_+^\circ \cap X_+^{01}$. Observe that, since $\frac{\pi}{10} < 1$, there exist $\phi \in \mathcal{C}_+$ such that $\phi(s) = s + \frac{\pi}{10}$ near $s=0$; the compatibility with Θ_1^+ justifies the definition of $\mathcal{C}_+$.

By Stokes' theorem,

$$(\mathbf{5.4.4}) \qquad \begin{aligned} E(v) &\le \int_{\{s\}\times[0,1]\times\mathbf{y}, s\ge 3/2} \lambda_+ + \sup_{\phi\in\mathcal{C}_+}\lim_{s\to\infty}\int_{\{s\}\times[0,1]\times\mathbf{y}} \phi dt \\ &\quad + \int_{\partial\dot{F}'} v^*\lambda_+ + \sup_{\phi\in\mathcal{C}_+}\int_{\partial\dot{F}'}\phi dt - \inf_{\psi\in\mathcal{C}_-}\int_{\boldsymbol{\gamma}}\psi\lambda_- \\ &\le 4g + \int_{[0,1]\times\mathbf{y}}\lambda_+ + \int_{\partial\dot{F}'} v^*\lambda_+. \end{aligned}$$

Recall that $\lambda_{+,s} = \lambda_+$ for $s \ge \frac{3}{2}$. In the above calculation,

$$\sup_{\phi\in\mathcal{C}_+}\lim_{s\to\infty}\int_{\{s\}\times[0,1]\times\mathbf{y}}\phi dt = 2g, \quad \sup_{\phi\in\mathcal{C}_+}\int_{\partial\dot{F}'}\phi dt = 2g, \quad \inf_{\psi\in\mathcal{C}_-}\int_{\boldsymbol{\gamma}}\psi\lambda_- = 0.$$

Combining Equations (5.4.2), (5.4.3), and (5.4.4), we obtain

$$E(u) \le 4g + C(\mathbf{y}) + \int_{[0,1]\times\mathbf{y}}\lambda_+ + k\int_\Sigma\widetilde{\omega},$$

which is the desired bound. □

5.5. *Regularity.* — Define the subset $\mathcal{M}^h_{J^+}(\mathbf{y}, \boldsymbol{\gamma}, A) \subset \mathcal{M}_{J^+}(\mathbf{y}, \boldsymbol{\gamma}, A)$ consisting of holomorphic curves without vertical fiber components. As in Lemma I.5.8.2, the set $\mathcal{J}^{reg}_{X_+}$ of *regular* $J^+ \in \mathcal{J}_{X_+}$ for which all the moduli spaces $\mathcal{M}^h_{J^+}(\mathbf{y}, \boldsymbol{\gamma}, A)$ are transversally cut out is a dense subset of $\mathcal{J}_{X_+}$. We can restrict attention to $\mathcal{M}^h_{J^+}(\mathbf{y}, \boldsymbol{\gamma}, A)$ for the following reason:

Lemma **5.5.1.** — *If* $J^+ \in \mathcal{J}^{reg}_{X_+}$ *and* $u \in \mathcal{M}_{J^+}(\mathbf{y}, \boldsymbol{\gamma}, A) - \mathcal{M}^h_{J^+}(\mathbf{y}, \boldsymbol{\gamma}, A)$, *then* $I_{X_+}(u) \geq 2 + 2g$.

Proof. — Suppose $u = u_1 \cup u_2$, where u_1 is regular and u_2 is homologous to $k \geq 1$ times a fiber. Since $\langle u_1, u_2 \rangle = k \cdot 2g$,

$$\begin{aligned} I(u) &= I(u_1) + I(u_2) + 2k(2g) \\ &\geq 0 + k(2 - 2g) + 4kg \geq k(2 + 2g). \end{aligned}$$

Here $I(u_1) \geq 0$ since $I(u_1) \geq \operatorname{ind}(u_1)$ by the index inequality and $\operatorname{ind}(u_1) \geq 0$ by the regularity of u_1. □

5.6. *Holomorphic curves in* X_+ *without positive ends.* — In this subsection and the next, we make essential use of the assumption $g(S) \geq 2$.

Let $S'' = S_{1/2} - A_{[0,N]}$ and let $\overline{S}'' = S'' \cup \{\infty\}$ be the one-point compactification of S''. We define the "projection" $\pi_{\overline{S}''} : X_+ \to \overline{S}''$ as follows:

- on X^0_+, $\pi_{\overline{S}''}(s, x, t) = x$ if $x \in S''$ and $\pi_{\overline{S}''}(s, x, t) = \infty$ if $x \notin S''$;
- on X^1_+, $\pi_{\overline{S}''}(x, r_2, \theta_2) = x$ if $x \in S''$ and $\pi_{\overline{S}''}(x, r_2, \theta_2) = \infty$ if $x \notin S''$;
- $\pi_{\overline{S}''}(X^2_+) = \{\infty\}$.

Lemma **5.6.1.** — *If* $u : \dot{F} \to (X_+, J^+)$ *is a holomorphic map without positive ends, then* $g(F) \geq 2$.

Proof. — The map $\pi_{\overline{S}''} \circ u$ can be extended to a continuous map $f : F \to \overline{S}''$. Observe that the curve u must intersect $S_{(z')^f}$ because the symplectic form is exact on $X_+ - S_{(z')^f}$. Hence $\deg f > 0$. Now we use the following fact: If $f : \Sigma_1 \to \Sigma_2$ is a positive degree map between closed oriented surfaces, then $g(\Sigma_1) \geq g(\Sigma_2)$. Since $g(S) = g(\overline{S}'') \geq 2$, it follows that $g(F) \geq 2$. □

Lemma **5.6.2.** — *There are no* $I = 0$ *closed holomorphic curves in* (X_+, J^+).

Proof. — We argue by contradiction. Let $A = [u_*(F)]$. By Lemma 5.3.3, the intersection form on $H_2(X_+)$ is trivial. Hence $A \cdot A = 0$. If $I(A) = A \cdot A + c_1(A) = 0$, then it follows that $c_1(A) = 0$.

Suppose that u is simple. Then $\chi(F) \geq 0$ by the adjunction formula. This contradicts Lemma 5.6.1. In particular $I(u) > 0$ by the regularity of u and the index inequality. If v is a degree d branched cover of u in the class A, then $I(v) = I(dA) = dI(A) \geq d$ using the formula

$$I(dA) = dI(A) + (d^2 - d)A \cdot A. \tag{5.6.1}$$

Lemma **5.6.3.** — *A multiply-covered holomorphic curve u with only negative ends has* $I(u) > 0$.

Proof. — This follows from the inequality

$$I(dC) \geq dI(C) + \frac{(d^2 - d)}{2}(2g(C) - 2 + \mathrm{ind}(C) + h) \tag{5.6.2}$$

from [Hu, Section 5.1], where C is a simple curve, ind(C) is the Fredholm index of C (which is nonnegative), and h is the number of hyperbolic ends. Here $2g(C) - 2 > 0$ by Lemma 5.6.1. □

5.7. *The map* Φ^+. — Let $J^+ \in \mathcal{J}^{reg}_{X_+}$. The chain map Φ^+ is given as follows:

$$\Phi^+ : (CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f), \partial) \to (ECC(M, \lambda_-), \partial'),$$
$$[\mathbf{y}, i] \mapsto \sum_{\boldsymbol{\gamma}, A} \#\mathcal{M}^{\mathcal{F}=i, I_{X_+}=0}_{J^+}(\mathbf{y}, \boldsymbol{\gamma}, A) \cdot \boldsymbol{\gamma},$$

where the summation is over all $\boldsymbol{\gamma} \in \mathcal{O}_{\lambda_-}$ and $A \in H_2(\check{X}_+, Z_{\mathbf{y},\boldsymbol{\gamma}})$. Here ∂' is the usual ECH differential on $ECC(M, \lambda_-)$.

By a combination of Lemma 5.4.2 and the Gromov-Taubes compactness theorem (cf. Section I.3.4), the sum in the definition of Φ^+ is finite. Hence Φ^+ is well-defined.

Theorem **5.7.1.** — *If $g(S) \geq 2$, then Φ^+ is a chain map.*

Proof. — Similar to that of Theorem I.6.2.4, with slight modifications in view of Lemmas 5.6.2 and 5.6.3. □

Remark **5.7.2.** — One can define the twisted coefficient analog of Φ^+, taking into account Lemma 5.3.3.

5.8. *Restriction to* Φ. — In this subsection δ still denotes the constant that appears in the construction of λ_-. Let $\mathcal{P}|_N$ be the subset of $\mathcal{P}$ consisting of orbits that are contained in $N = N_{(S_0, h)}$. Also let $\gamma_\theta \in \mathcal{P}_-$ be the orbit corresponding to $\theta \in \partial S_0$.

Lemma **5.8.1.** — *For $\delta > 0$ sufficiently small, if $u \in \mathcal{M}_{J^+}^{\mathcal{F}=0}(\mathbf{y}, \boldsymbol{\gamma})$, $\mathbf{y} \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $\mathbf{y} \subset S_0$, and $\boldsymbol{\gamma} \in \mathcal{O}$, then $\boldsymbol{\gamma}$ is constructed from $\mathcal{P}|_N \cup \{e', h'\}$.*

Proof. — If $\mathcal{M}_{J^+}^{\mathcal{F}=0}(\mathbf{y}, \boldsymbol{\gamma})$ is nonempty, then by considerations similar to those of Lemma 5.4.2:

$$4g + C(\mathbf{y}) + \int_{[0,1]\times\mathbf{y}} \lambda_+ \geq \mathcal{A}_{\lambda_-}(\boldsymbol{\gamma}),$$

where $\mathcal{A}_{\lambda_-}(\boldsymbol{\gamma})$ is the action of $\boldsymbol{\gamma}$ with respect to λ_-. By taking the maximum of the left-hand side over all $\mathbf{y}$, we obtain an upper bound for $\mathcal{A}_{\lambda_-}(\boldsymbol{\gamma})$ which is independent of $\mathbf{y}$ and δ. By Lemma 4.1.1(7), all the orbit sets $\boldsymbol{\gamma}$ in $int(N(K)) \cup \mathcal{N}$ satisfy $\mathcal{A}_{\lambda_-}(\boldsymbol{\gamma}) \geq \frac{1}{2\delta} - \kappa$. Hence, for $\delta > 0$ sufficiently small, no negative end of u is asymptotic to an orbit in $int(N(K)) \cup \mathcal{N}$. □

Lemma **5.8.2.** — *If $u \in \mathcal{M}_{J^+}^{\mathcal{F}=0}(\mathbf{y}, \boldsymbol{\gamma})$, where $\mathbf{y} \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $\mathbf{y} \subset S_0$, and $\boldsymbol{\gamma}$ is constructed from $\mathcal{P}|_N \cup \{e', h'\}$, then $\mathrm{Im}(u) \subset W_+$ and $\boldsymbol{\gamma} \in \mathcal{O}|_N$.*

Proof. — Let $u \in \mathcal{M}_{J^+}^{\mathcal{F}=0}(\mathbf{y}, \boldsymbol{\gamma})$ such that $u(\dot{F}) \not\subset W_+$.

Suppose that u is not a multi-level Morse-Bott building. Then $u(\dot{F}) \cap C_{\theta_0} \neq \varnothing$ for some $\theta_0 \in \partial S_0 - \boldsymbol{\alpha} - \boldsymbol{\beta}$, and moreover we may assume that γ_{θ_0} is not an asymptotic limit of u at $-\infty$. Since J^+ is admissible, all the curves C_θ are holomorphic. Hence $\langle u(\dot{F}), C_{\theta_0}\rangle > 0$ by the positivity of intersections.

Let D_θ, $\theta \in \partial S_0$, be a meridian disk of the solid torus $\mathcal{N} \cup N(K)$ that is bounded by $\{\theta\} \times \mathbf{R}/2\mathbf{Z}$ and is disjoint from e' and h', and let $D_{\theta,s'} = \{s'\} \times D_\theta \subset X_+^2$, where $s' < 0$ and $\theta \in \partial S_0$. We then define

$$C_{\theta,s'_0} := (C_\theta - \{s' < s'_0\}) \cup D_{\theta,s'_0},$$

where $s'_0 < 0$. When s'_0 is sufficiently negative, the curve $u(\dot{F})$ intersects C_{θ_0,s'_0} only in the region $C_{\theta_0} - \{s' < s'_0\}$, since $\boldsymbol{\gamma}$ is constructed from $\mathcal{P}|_N \cup \{e', h'\}$ and D_{θ_0} does not intersect e' and h'. Hence $\langle u(\dot{F}), C_{\theta_0,s'_0}\rangle > 0$. Now, since $[S_{(z')^f}] = [C_{\theta_0,s'_0}]$ in $H_2(\check{X}_+, \partial\check{X}_+ - Z_{\mathbf{y},\boldsymbol{\gamma}})$, we have

$$\mathcal{F}(u) = \langle [u], S_{(z')^f}\rangle = \langle [u], C_{\theta_0,s'_0}\rangle > 0.$$

This contradicts our assumption that $\mathcal{F}(u) = 0$.

If u is a multi-level Morse-Bott building, then we need to make the appropriate modifications (left to the reader), but the same argument goes through. For example, we need to replace C_{θ_0} by a multi-level building $C_{\theta_0} \cup (\mathbf{R} \times \gamma_{\theta_0}) \cup \cdots \cup (\mathbf{R} \times \gamma_{\theta_0})$. Note that if u is a Morse-Bott building, then it could have a component u_1 with a negative end that limits to some γ_{θ_1}, followed by a gradient trajectory from θ_1 to θ_2, and then by a component u_2 with a positive end that limits to γ_{θ_2}. □

Theorem **5.8.3.** — *For* $\delta > 0$ *sufficiently small, if* $u \in \mathcal{M}_{J^+}^{\mathcal{F}=0}(\mathbf{y}, \boldsymbol{\gamma})$, $\mathbf{y} \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $\mathbf{y} \subset S_0$, *and* $\boldsymbol{\gamma} \in \mathcal{O}$, *then* $\mathrm{Im}(u) \subset W_+$ *and* $\boldsymbol{\gamma} \in \mathcal{O}|_N$.

Proof. — Follows from Lemmas 5.8.1 and 5.8.2. □

Corollary **5.8.4.** — $\Phi^+([\mathbf{x}, 0]) = e^{2g}$, *where e is the elliptic orbit of the negative Morse-Bott family on* $T_- = \partial N_{(S_0,\hbar)}$.

Proof. — By Theorem 5.8.3, any curve $u \in \mathcal{M}_{J^+}^{\mathcal{F}=0}(\mathbf{x}, \boldsymbol{\gamma})$ must have image in W_+. Then, by Lemma I.6.2.3 and its consequence in Theorem I.6.2.4, the only curves from $\mathbf{x}$ that do not intersect $S_{(z')^f}$ are curves of type C_θ. □

The restriction Φ of Φ^+ to (W_+, J_+) is given as follows:

$$\Phi : \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f) \to ECC_{2g}(M, \lambda_-),$$

$$[\mathbf{y}, 0] \mapsto \sum_{\boldsymbol{\gamma}, A} \# \mathcal{M}_{J_+}^{I_{W_+}=0}(\mathbf{y}, \boldsymbol{\gamma}, A) \cdot \boldsymbol{\gamma},$$

where $\mathcal{M}_{J_+}^{I_{W_+}=0}(\mathbf{y}, \boldsymbol{\gamma}, A)$ is the subset of $\mathcal{M}_{J^+}(\mathbf{y}, \boldsymbol{\gamma}, A)$ consisting of curves with image in W_+.

Theorem **5.8.5.** — Φ *is a quasi-isomorphism.*

Proof. — The almost complex structure J_+ is sufficiently close to J_+^0. For J_+^0, the analogous chain map was shown to be a quasi-isomorphism (Theorem II.1.0.1). Considerations similar to those of Theorem I.3.6.1 imply that Φ is a quasi-isomorphism. □

5.9. *Commutativity with the* U*-map.* — Let z^b be a point in $\mathbf{R} \times [0, 1]$ with t-coordinate $\frac{1}{2}$ and let $z = (z^b, z^f) \in X$. Let U_z be the geometric U-map with respect to z on the HF side. On the ECH side, let $z' = (s, z^M)$ be a generic point in $\mathbf{R} \times int(N(K))$ near the binding K. We define $U' = U'_{z'}$ so that $\langle U'(\boldsymbol{\gamma}), \boldsymbol{\gamma}' \rangle$ is the count of $I_{ECH} = 2$ curves in the symplectization $(\mathbf{R} \times M, J')$ from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$ that pass through z'.

Theorem **5.9.1.** — *There exists a chain homotopy*

$$K : CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f) \to ECC(M, \lambda_-)$$

which satisfies

$$U' \circ \Phi^+ - \Phi^+ \circ U_z = \partial' \circ K + K \circ \partial.$$

Proof. — The commutativity of Φ^+ with the U-maps up to homotopy is obtained by moving the point constraint in the cobordism X_+ from $s = +\infty$ to $s = -\infty$.

The 1-parameter family of points $(z_\tau)_{\tau\in\mathbf{R}}$ is chosen as follows: For $\tau \geq 0$, let $z_\tau = (z_\tau^b, z^f)$, where z_τ^b approaches $(s,t) = (+\infty, \frac{1}{2})$ as $\tau \to +\infty$ and z_0^b is near the center of the disk $D^2 = \{r_2 \leq 1\}$. Next, for $\tau \in [-1, 0]$, let $z_\tau = (z_0^b, z_\tau^f)$ so that $(z_0^b, z_{-1}^f) \in \{0\} \times \widetilde{\mathcal{B}}$ is near the binding K. For $\tau \leq -1$, let $z_\tau = (\tau + 1, z^M) \in (-\infty, 0] \times M$, where $z^M \in M = \widetilde{\mathcal{B}}$ is a point near the binding. Finally, we consider a small perturbation of $(z_\tau)_{\tau\in\mathbf{R}}$ to make it generic (without changing its name).

We define the 1-parameter family of almost complex structures $(J_\tau^+)_{\tau\in\mathbf{R}}$ so that J_τ^+ is C^ℓ-close to J^+ and agrees with J^+ outside a small neighborhood of z_τ.

The rest of the chain homotopy argument is standard, with the exception of the obstruction theory that was carried out in [HT1, HT2]. □

Theorem **5.9.2.** — *For $\delta > 0$ sufficiently small, if $\mathbf{y} \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\mathbf{y} \subset S_0$, then* $K([\mathbf{y}, 0]) = 0$.

Proof. — The coefficient $\langle K([\mathbf{y}, 0]), \boldsymbol{\gamma}\rangle$ is given by the count of $I_{X_+} = 1$ curves from $\mathbf{y}$ to $\boldsymbol{\gamma}$ that pass through z_τ for some τ and do not intersect $S_{(z')^f}$. If such a curve u exists, then $\mathrm{Im}(u) \not\subset W_+$. This is not possible by the proof of Theorem 5.8.3. □

6. Proof of Theorem 1.0.1

In this section we prove Theorem 1.0.1. In Section 6.1 we prove an algebraic result (Theorem 6.1.5) which is sufficient to prove that Φ^+ is a quasi-isomorphism. The conditions of Theorem 6.1.5 are verified in Section 6.4.

6.1. *Some algebra.*

Definition **6.1.1.** — *Let (A, d) be a chain complex. We say that a chain map $f\colon A \to A$ is* homologically almost nilpotent *(abbreviated* han*) if for every $x \in H(A)$ there exists $n \in \mathbf{N}$ such that $(f_*)^n(x) = 0$.*

Prototypical examples of *han* maps are the U-maps in HF^+ and ECH.

Let (A, d_A) and (B, d_B) be chain complexes with *han* maps $U_A\colon A \to A$ and $U_B\colon B \to B$ and let $\Phi^+\colon A \to B$ be a chain map such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\Phi^+} & B \\ {\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle U_B} \\ A & \xrightarrow{\Phi^+} & B \end{array}$$

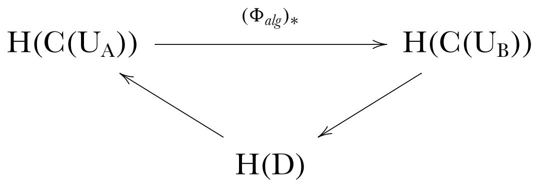
commutes up to a chain homotopy K. We form a chain complex $D = A \oplus A \oplus B \oplus B$ with differential

$$d_D = \begin{pmatrix} d_A & 0 & 0 & 0 \\ U_A & d_A & 0 & 0 \\ \Phi^+ & 0 & d_B & 0 \\ K & \Phi^+ & U_B & d_B \end{pmatrix}.$$

Given a chain map f, we denote its mapping cone by $C(f)$.

Lemma **6.1.2.** — *There is an exact triangle:*

(**6.1.1**)

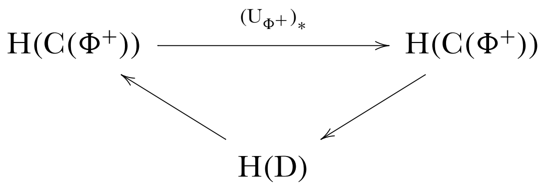


where $\Phi_{alg} = \begin{pmatrix} \Phi^+ & 0 \\ K & \Phi^+ \end{pmatrix}$.

Proof. — From the shape of d_D, it is evident that (D, d_D) is the mapping cone of $\Phi_{alg} : C(U_A) \to C(U_B)$. □

Lemma **6.1.3.** — *There is an exact triangle:*

(**6.1.2**)



where $U_{\Phi^+} = \begin{pmatrix} U_A & 0 \\ K & U_B \end{pmatrix}$.

Proof. — Let $C(\Phi^+) = A \oplus B$ be the cone of Φ^+ with differential $d_{\Phi^+} = \begin{pmatrix} d_A & 0 \\ \Phi^+ & d_B \end{pmatrix}$. Then $U_{\Phi^+} : (C(\Phi^+), d_{\Phi^+}) \to (C(\Phi^+), d_{\Phi^+})$ is a chain map. Hence the complex $(D', d_{D'})$, where $D' = A \oplus B \oplus A \oplus B$ and

$$d_{D'} = \begin{pmatrix} d_{\Phi^+} & 0 \\ U_{\Phi^+} & d_{\Phi^+} \end{pmatrix} = \begin{pmatrix} d_A & 0 & 0 & 0 \\ \Phi^+ & d_B & 0 & 0 \\ U_A & 0 & d_A & 0 \\ K & U_B & \Phi^+ & d_B \end{pmatrix},$$

is the cone of U_{Φ^+}. Moreover $f : D \to D'$ where

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is an isomorphism of complexes. □

Lemma **6.1.4.** — U_{Φ^+} *is a* han *map.*

Proof. — Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
H(B) & \xrightarrow{i_*} & H(C(\Phi^+)) & \xrightarrow{j_*} & H(A) \\
\downarrow{\scriptstyle U_B^n} & & \downarrow{\scriptstyle U_{\Phi^+}^n} & & \downarrow{\scriptstyle U_A^n} \\
H(B) & \xrightarrow{i_*} & H(C(\Phi^+)) & \xrightarrow{j_*} & H(A) \\
\downarrow{\scriptstyle U_B^m} & & \downarrow{\scriptstyle U_{\Phi^+}^m} & & \downarrow{\scriptstyle U_A^m} \\
H(B) & \xrightarrow{i_*} & H(C(\Phi^+)) & \xrightarrow{j_*} & H(A)
\end{array}$$

Given $x \in H(C(\Phi^+))$, we choose $n \in \mathbf{N}$ sufficiently large so that $U_A^n(j_*(x)) = j_*(U_{\Phi^+}^n(x)) = 0$. Then $U_{\Phi^+}^n(x) = i_*(y)$ for some $y \in H(B)$. Next choose $m \in \mathbf{N}$ sufficiently large so that $U_B^m(y) = 0$. Then $U_{\Phi^+}^{n+m}(x) = U_{\Phi^+}^m(i_*(y)) = i_*(U_B^m(y)) = 0$. □

Theorem **6.1.5.** — *If* Φ_{alg} *is a quasi-isomorphism, then* Φ^+ *is a quasi-isomorphism.*

Proof. — If Φ_{alg} is a quasi-isomorphism, then $H(D) = 0$ by Exact Triangle (6.1.1). This in turn implies that U_{Φ^+} is a quasi-isomorphism by Exact Triangle (6.1.2). However the *han* map U_{Φ^+} cannot be a quasi-isomorphism, unless $H(C(\Phi^+)) = 0$. Finally, the triangle

$$\begin{array}{ccc}
H(A) & \xrightarrow{\Phi^+_*} & H(B) \\
\nwarrow & & \swarrow \\
& H(C(\Phi^+)) &
\end{array}$$

implies that Φ^+ is a quasi-isomorphism. □

We finish this subsection with a lemma which compares the homology of $C(U)$ with that of $\ker U$.

Lemma **6.1.6.** — *Let* (C, d) *be a chain complex and let* $U : C \to C$ *be a chain map. If* U *is surjective, then the inclusion*

$$\begin{aligned} i : \ker U &\to C(U) \\ x &\mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} \end{aligned}$$

is a quasi-isomorphism.

Proof. — Let $\overline{U} : C/\ker U \to C$ be the map induced by U. We have a short exact sequence of complexes

$$0 \to \ker U \to C(U) \to C(\overline{U}) \to 0,$$

which induces the exact triangle:

$$\begin{array}{ccc} H(\ker U) & \xrightarrow{\quad i_* \quad} & H(C(U)) \\ & \nwarrow \qquad \swarrow & \\ & H(C(\overline{U})) & \end{array}$$

Since U is surjective, $\overline{U}$ is an isomorphism. Hence $H(C(\overline{U})) = 0$ and the lemma follows. □

6.2. *Heegaard Floer chain complexes.* — Recall the subcomplex $\widehat{CF'}(S_0, \mathbf{a}, h(\mathbf{a}))$ of $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f)$ from Section I.4.9.3, which is generated by $\mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$; let

$$j' : \widehat{CF'}(S_0, \mathbf{a}, h(\mathbf{a})) \to \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f)$$

be the natural inclusion map. We are viewing

$$\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f) \subset CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f)$$

as the subcomplex generated by elements of the form $[\mathbf{y}, 0]$. The chain complex $\widehat{CF}(S_0, \mathbf{a}, h(\mathbf{a}))$ is the quotient $\widehat{CF'}(S_0, \mathbf{a}, h(\mathbf{a}))/\sim$, defined in Section I.4.9.3.

Lemma **6.2.1.** — *There is an isomorphism* $j : \widehat{HF}(S_0, \mathbf{a}, h(\mathbf{a})) \to \widehat{HF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f)$ *given by* $[Z] \mapsto [Z]$.

Proof. — This follows from the discussion of Theorem I.4.9.4. Note that the natural candidate

$$\widehat{CF}(S_0, \mathbf{a}, h(\mathbf{a})) \to \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f), \quad [Z] \to Z$$

for a chain map is not a well-defined map. □

Lemma **6.2.2.** — *The inclusion* $i : \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f) \to \mathrm{C}(\mathrm{U})$ *given by* $\mathbf{y} \mapsto \begin{pmatrix} [\mathbf{y}, 0] \\ 0 \end{pmatrix}$ *is a quasi-isomorphism.*

Proof. — This follows from Lemma 6.1.6, since $\mathrm{U}([\mathbf{y}, i]) = [\mathbf{y}, i-1]$ for $i \geq 1$ and $\ker \mathrm{U} \simeq \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f)$. □

6.3. *ECH chain complexes.* — We describe several ECH chain complexes that are related to $(\mathrm{ECC}(\mathrm{M}, \lambda_-), \partial')$ and are constructed from certain subsets $\mathcal{S}$ of the set $\mathcal{P} = \mathcal{P}_{\lambda_-}$ of simple orbits of R_{λ_-}. Many of these appeared in [0, Section 9]. Let U' be the U-map of $\mathrm{ECC}(\mathrm{M}, \lambda_-)$ with respect to $(s_0, z^{\mathrm{M}}) \in \mathbf{R} \times \mathrm{M}$, where z^{M} is a generic point which is sufficiently close to the binding.

Let $\mathcal{O}_{\mathcal{S}}$ be the set of orbit sets that are constructed from $\mathcal{S}$. Then $\mathcal{S}$ is *closed* if $\boldsymbol{\gamma}' \in \mathcal{O}_{\mathcal{S}}$, whenever $\boldsymbol{\gamma} \in \mathcal{O}_{\mathcal{S}}$, $\boldsymbol{\gamma}' \in \mathcal{O}_{\mathcal{P}}$, and $\langle \partial' \boldsymbol{\gamma}, \boldsymbol{\gamma}' \rangle \neq 0$ or $\langle \mathrm{U}' \boldsymbol{\gamma}, \boldsymbol{\gamma}' \rangle \neq 0$. If $\mathcal{S}$ is closed, then let $(\mathrm{A}_{\mathcal{S}}, \partial'_{\mathcal{S}})$ be the subcomplex of $\mathrm{ECC}(\mathrm{M}, \lambda_-)$ generated by $\mathcal{O}_{\mathcal{S}}$ and let $\mathrm{U}'_{\mathcal{S}}$ be the restriction of U' to $\mathrm{A}_{\mathcal{S}}$. Let $\mathcal{P}|_{\mathrm{N}} \subset \mathcal{P}$ be the set of orbits in the mapping torus N. The subsets

$$\mathcal{S}_1 = \mathcal{P}|_{\mathrm{N}} \cup \{e', h'\},\ \mathcal{S}_2 = \mathcal{P}|_{\mathrm{N} \cup \mathcal{N}} \cup \{e', h'\},\ \mathcal{P}|_{\mathrm{N}} \cup \{h'\},\ \mathcal{P}|_{\mathrm{N} \cup \mathcal{N}} \cup \{h'\},\ \mathcal{P}|_{\mathrm{N}}$$

are closed and we write $\mathrm{A}_i = \mathrm{A}_{\mathcal{S}_i}$, $\partial'_i = \partial'_{\mathcal{S}_i}$, and $\mathrm{U}'_i = \mathrm{U}'_{\mathcal{S}_i}$ for $i = 1, 2$, as well as

$$\widehat{\mathrm{ECC}}^{\natural}(\mathrm{N}) = \mathrm{A}_{\mathcal{P}|_{\mathrm{N}} \cup \{h'\}},\quad \widehat{\mathrm{ECC}}^{\natural\natural}(\mathrm{N}) = \mathrm{A}_{\mathcal{P}|_{\mathrm{N} \cup \mathcal{N}} \cup \{h'\}},\quad \mathrm{ECC}(\mathrm{N}) = \mathrm{A}_{\mathcal{P}|_{\mathrm{N}}}.$$

Also let $\mathrm{ECC}_{2g}(\mathrm{N}) \subset \mathrm{ECC}(\mathrm{N})$ be the subcomplex generated by orbit sets $\boldsymbol{\gamma}$ satisfying $\langle \boldsymbol{\gamma}, \mathrm{S} \times \{t\} \rangle = 2g$. Let

$$q_1 : \mathrm{ECC}_{2g}(\mathrm{N}) \to \widehat{\mathrm{ECC}}^{\natural}(\mathrm{N}), \qquad q_2 : \mathrm{ECC}_{2g}(\mathrm{N}) \to \widehat{\mathrm{ECC}}^{\natural\natural}(\mathrm{N})$$

be the chain maps given by the natural inclusion. Then we have the following:

Lemma **6.3.1.** — *The chain maps* q_1 *and* q_2 *are quasi-isomorphisms.*

Proof. — The chain map q_1 is a quasi-isomorphism by Section II.5 and Section 0.9.9. By a direct limit argument similar to that of Proposition 0.7.2.1, there is a quasi-isomorphism $r : \widehat{\mathrm{ECC}}^{\natural\natural}(\mathrm{N}) \to \widehat{\mathrm{ECC}}^{\natural}(\mathrm{N})$ such that $r \circ q_2 = q_1$. This implies that q_2 is also a quasi-isomorphism. □

Lemma **6.3.2.** — *The inclusions* $p_1 : \widehat{\mathrm{ECC}}^{\natural}(\mathrm{N}) \to \mathrm{C}(\mathrm{U}'_1)$ *and* $p_2 : \widehat{\mathrm{ECC}}^{\natural\natural}(\mathrm{N}) \to \mathrm{C}(\mathrm{U}'_2)$ *given by* $\Gamma \mapsto \begin{pmatrix} \Gamma \\ 0 \end{pmatrix}$ *are quasi-isomorphisms.*

Proof. — This follows from Lemma 6.1.6. The map U'_i, $i = 1, 2$, is given by:

$$U'_i((e')^k(h')^l\Gamma) = (e')^{k-1}(h')^l\Gamma, \tag{6.3.1}$$

where $\Gamma \in \mathcal{O}|_N$ or $\mathcal{O}|_{N\cup\mathcal{N}}$; see Claim 0.9.9.3 for a similar calculation. Hence U'_i is surjective, $\ker U'_1 = \widehat{ECC}^{\natural}(N)$, and $\ker U'_2 = \widehat{ECC}^{\natural\natural}(N)$. This implies the lemma. □

Lemma **6.3.3.** — *The inclusion* $\mathfrak{i} : \widehat{ECC}^{\natural\natural}(N) \to C(U')$ *given by* $\Gamma \mapsto \begin{pmatrix}\Gamma\\0\end{pmatrix}$ *is a quasi-isomorphism.*

Proof. — This is similar to the argument in [0, Section 9].

Choose an identification $\eta : H_1(N(K); \mathbf{Z}) \xrightarrow{\sim} \mathbf{Z}$ such that the homology class of the binding is 1. Define the filtration $\mathcal{F} : ECC(M) \to \mathbf{Z}^{\geq 0}$ such that

$$\mathcal{F}\left(\sum_i \boldsymbol{\gamma}_i \otimes \Gamma_i\right) = \max_i \eta([\boldsymbol{\gamma}_i]),$$

where $\boldsymbol{\gamma}_i \in \mathcal{O}|_{N(K)}$ and $\Gamma_i \in \mathcal{O}|_{N\cup\mathcal{N}}$. Let $\mathcal{F}^{\natural\natural} : \widehat{ECC}^{\natural\natural}(N) \to \mathbf{Z}^{\geq 0}$ be its restriction to $\widehat{ECC}^{\natural\natural}(N)$. (Note that $\mathcal{F}^{\natural\natural}$ is a trivial filtration.) Next define the filtration $\widehat{\mathcal{F}} : C(U') \to \mathbf{Z}^{\geq 0}$ such that

$$\widehat{\mathcal{F}}\begin{pmatrix}\sum_i \boldsymbol{\gamma}_i \otimes \Gamma_i\\ \sum_j \boldsymbol{\gamma}'_j \otimes \Gamma'_j\end{pmatrix} = \max_{i,j}\{\eta([\boldsymbol{\gamma}_i]), \eta([\boldsymbol{\gamma}'_j])\}.$$

The map $\mathfrak{i}$ is an $(\mathcal{F}^{\natural\natural}, \widehat{\mathcal{F}})$-filtered chain map. The induced map

$$E^1(\mathfrak{i}) : E^1(\mathcal{F}^{\natural\natural}) \to E^1(\widehat{\mathcal{F}})$$

on the E^1-level agrees with the isomorphism $(p_2)_*$; the proof is similar to that of Section 0.9. If a filtered chain map between filtered chain complexes which are bounded below is an isomorphism on the E^r-level, then it is a quasi-isomorphism. This implies that $\mathfrak{i}$ is a quasi-isomorphism. □

6.4. *Completion of proof of Theorem 1.0.1.* — By Theorems 3.1.4, 5.7.1, and 5.9.1, the map

$$\Phi^+ : CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f) \to ECC(M, \lambda_-)$$

is a chain map which commutes with U and U' up to the chain homotopy $K^+ = K + \Phi^+ \circ H$, where H is given in Theorem 3.1.4 and K is given in Theorem 5.9.1. Here U is the original algebraically-defined U-map on $(CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z^f), \partial)$ and U' is the U-map on $(ECC(M, \lambda_-), \partial')$.

In view of Theorem 6.1.5, the quasi-isomorphism statement of Theorem 1.0.1 immediately follows from:

Theorem **6.4.1.** — *The algebraic map* Φ_{alg} *is a quasi-isomorphism.*

Let $\Phi' : \widehat{CF}'(S_0, \mathbf{a}, h(\mathbf{a})) \to ECC_{2g}(N)$ be the map from Definition I.6.2.1. The map Φ' descends to $\Phi : \widehat{CF}(S_0, \mathbf{a}, h(\mathbf{a})) \to ECC_{2g}(N)$, which was shown to be a quasi-isomorphism in [I, II]. Here we are using $ECC_{2g}(N)$ instead of $PFC_{2g}(N)$, but there is no substantial difference; see Theorem I.3.6.1.

Observe that there is a discrepancy between the algebra and the geometry: the map Φ_{alg} which we are using here is not the map Φ, and we need to reconcile the two.

Proof. — If $Z \in \widehat{CF}'(S_0, \mathbf{a}, h(\mathbf{a}))$, then $\Phi^+(Z) = \Phi'(Z)$ by Theorem 5.8.3. We observed in Theorem 3.1.4 that $H(Z) = 0$. Moreover, $K(Z) = 0$ by Theorem 5.9.2 and thus $K^+(Z) = 0$. Hence

$$\Phi_{alg}\begin{pmatrix} Z \\ 0 \end{pmatrix} = \begin{pmatrix} \Phi^+(Z) \\ K^+(Z) \end{pmatrix} = \begin{pmatrix} \Phi'(Z) \\ 0 \end{pmatrix},$$

and the following diagram is commutative:

$$\begin{array}{ccc} \widehat{CF}'(S_0, \mathbf{a}, h(\mathbf{a})) & \xrightarrow{\Phi'} & ECC_{2g}(N) \\ {\scriptstyle i\circ j'}\downarrow & & \downarrow{\scriptstyle \mathrm{i}\circ q_2} \\ C(U) & \xrightarrow{\Phi_{alg}} & C(U'). \end{array}$$

This gives rise to the following commutative diagram of homology groups:

$$\begin{array}{ccc} \widehat{HF}(S_0, \mathbf{a}, h(\mathbf{a})) & \xrightarrow{\Phi_*} & ECH_{2g}(N) \\ {\scriptstyle i_*\circ j}\downarrow & & \downarrow{\scriptstyle (\mathrm{i}\circ q_2)_*} \\ H(C(U)) & \xrightarrow{(\Phi_{alg})_*} & H(C(U')). \end{array}$$

Since j, i_*, Φ_*, $(q_2)_*$, and i_* are isomorphisms by Lemma 6.2.1, Lemma 6.2.2, [I, II], Lemma 6.3.1, and Lemma 6.3.3, Φ_{alg} itself is a quasi-isomorphism. □

Finally, the statement about Φ^+ mapping the contact class to the contact class follows from Corollary 5.8.4.

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Declarations:

Competing Interests

The authors declare no competing interests.

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