MARTIN-LÖF IDENTITY TYPES IN C-SYSTEMS

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ABSTRACT

This paper continues a series of papers that develop a new approach to syntax and semantics of dependent type theories. Here we study the interpretation of the rules of the identity types in the intensional Martin-Löf type theories on the C-systems that arise from universe categories. In the first part of the paper we develop constructions that produce interpretations of these rules from certain structures on universe categories while in the second we study the functoriality of these constructions with respect to functors of universe categories. The results of the first part of the paper play a crucial role in the construction of the univalent model of type theory in simplicial sets.

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1. Introduction

> *He that delivereth knowledge desireth to deliver it in such form as may be soonest believed and not as may be easiest examined.*
> "On the Impediments of Knowledge", from Valerius Terminus by Francis Bacon.

The concept of a C-system in its present form was introduced in [20]. The type of C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [3, 4], but the definition of a C-system is slightly different from the Cartmell's foundational definition.

In the past decade or more, it has been a tradition in the study of type theories to consider, as the main mathematical object associated with a type theory, not a C-system but a category with families (see [5]). As was observed recently all of the constructions of [17, 19, 21, 22] and of the present paper (but not of [20] or [16]!) can be either used directly or reformulated in a straightforward way to provide similar results for categories with families. This modification will be discussed in a separate paper or papers.

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In this introductory explanation we will distinguish between the syntactic and semantic C-systems. By a syntactic C-system we will mean a C-system that is a regular subquotient of a C-system of the form $CC(R, LM)$, where R is a monad on sets and LM is a left module over R, see [16, 20]. In particular, the C-systems of all of the various versions of dependent type theory of Martin-Löf "genus" are syntactic type systems in the sense of this definition.

By a semantic C-system we will mean a C-system whose underlying category is a full subcategory in a category of "mathematical" nature such as the category of sets or the category of sheaves of sets.

Usually one knows some good properties (e.g., consistency) of a given semantic C-system and tries to prove similar good properties of a syntactic C-system by constructing a functor from the syntactic one to the semantic one.

To construct such a functor one tries to show that the syntactic C-system is an initial one among C-systems equipped with some collection of additional operations and then to construct operations of the required form on the semantic one. A pioneering example of this approach can be found in [11].

In this paper we investigate the set of three interconnected operations on C-systems that, in the case of the syntactic C-systems, corresponds to the set of inference rules that is known as the rules for identity types in intensional Martin-Löf type theories (first published in [9]).[1] Since the key ingredient of this structure is known in type theory as the J-eliminator we call it the J-structure.

We do not use the "sequent" notation that is so widespread in the literature on type theory for general C-systems—we restrict its use only to examples where we assume the C-system to be a syntactic one.

The reason for this restriction is that translating sequent-like notation into the algebraic notation of C-systems or categories with families requires considerable mastery of various conventions connected to the use of dependently typed systems. An example of such a translation is the description of an object $\mathsf{Id}_3(T)$ corresponding to the sequent-like expression $(\Gamma, x : T, y : T, e : \mathsf{Id}\, T\, x\, y;\ )$ in Construction 2.1.4.

For a syntactic C-system $\mathbf{C}$ we are allowed to use sequent notation, for the following reason. First, since $\mathbf{C}$ in this case is a subquotient of $CC(R, LM)$ our notation needs only to provide a reference to elements of sets associated with $CC(R, LM)$ itself. There, the first T refers to an element of $LM(\{1, \dots, l\})$, where l is the length of Γ and $LM(X)$ is the set of type expressions in the raw syntax with free variables from a set X and the second T refers to an element of $LM(\{1, \dots, l + 1\})$ that is the image of the first T under the map

$$LM(\{1, \dots, l\}) \to LM(\{1, \dots, l + 1\})$$

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[1] There is also a simpler set of rules corresponding to the identity types in the extensional Martin-Löf type theory (first published in [10]). Cartmell, in his notion of a strong M-L structure [3, p. 3.36], considers the set of rules for the extensional theory.

defined by the inclusion $\{1, \dots, l\} \subset \{1, \dots, l+1\}$. In this case the map does not depend on T. We should distinguish between Id as a structure on the C-system and the corresponding syntactic construction (because they have different types). If we denote the syntactic "identity types" by $\mathsf{Id}^s\, \mathrm{T}\, t_1\, t_2$ then for the sequence

$$\Gamma, x : \mathrm{T}, y : \mathrm{T}, e : \mathsf{Id}^s\, \mathrm{T}\, x\, y;$$

to define an element of $Ob(\mathrm{CC}(\mathrm{LM}, \mathrm{R}))$, the expression $\mathsf{Id}^s\, \mathrm{T}\, x\, y$ must refer to an element of $\mathrm{LM}(\{1, \dots, l+2\})$ and its form shows that we assume that there is an operation

$$\mathsf{Id}^s : \mathrm{LM} \times \mathrm{R} \times \mathrm{R} \to \mathrm{LM}$$

(a natural transformation of functors that is a linear morphism of left R-modules) and $\mathsf{Id}^s\, \mathrm{T}\, x\, y$ is the "named variables" notation for $\mathsf{Id}^s_{1,\dots,l+2}(\mathrm{T}, l+1, l+2)$.

We do not continue this explanation of how to construct J-structures on syntactic C-systems. This will be done in a separate paper. Let us remark, however, that constructing J-structures on syntactic C-systems is relatively easy, and that the difficult questions about J-structures on such C-systems are those related to the initiality properties of the resulting objects.

While constructing J-structures on syntactic C-systems is relatively straightforward, constructing non-degenerate[2] J-structures on semantic C-systems or categories with families proves to be very difficult.

The first instance of such a construction, due to Martin Hofmann and Thomas Streicher, appeared in [7]. It was done in the language of categories with families and the underlying category there was the category of groupoids.

The construction of Hofmann and Streicher was substantially extended and generalized in the Ph.D. thesis of Michael Warren [23, 24] and his subsequent papers such as [25] and [1].

Further important advances were achieved in the work of Benno van den Berg and Richard Garner [13].

The two main results of the first part of this paper provide a new approach to the construction of J-structures on semantic C-systems, an approach that can be used to construct the J-structure on the C-system of the univalent model.

Construction 2.2.18 provides a simple way of extending a J1-structure on a universe $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ in a category $\mathcal{C}$ to a full J-structure.

Construction 2.4.16 provides a method of constructing a J-structure on the C-system $\mathrm{CC}(\mathcal{C}, p)$ from a J-structure on p.

Combined, they provide a method of constructing a J-structure on $\mathrm{CC}(\mathcal{C}, p)$ from a J1-structure on p.

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[2] See Remark 2.1.8 for the definition of *degenerate* J-structure.

We also discuss two sets of conditions on a pair of classes of morphisms TC and FB in a locally Cartesian closed category that can be used in combination with Construction 2.2.18 to construct J-structures. These conditions often hold for the classes of trivial cofibrations and fibrations in model categories (or categories with weak factorization systems) providing a way of constructing C-systems with J-structures starting from such categories.

In this paper we continue to use the diagrammatic order of writing composition of morphisms, i.e., for $f : X \to Y$ and $g : Y \to Z$ the composition of f and g is denoted by $f \circ g$.

In this paper, as in the preceding papers [17, 19, 21, 22], we often have to consider structures on categories that are not invariant under equivalences and their interaction with structures that are invariant under equivalence.

The methods used in this paper are fully constructive and the paper is written in "formalization ready" style, with all the proofs provided in enough detail to ensure that there are no hidden difficulties for the formalization of all of the results presented here.

Except for the section that discusses the use of classes TC and FB, the methods we use are also completely elementary in the sense that they rely only on the quasi-algebraic language of categories with various structures.

The key Definition 2.2.11 and its relation to the J-structures on categories $CC(\mathcal{C}, p)$ first appeared in [14].

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1.1. *A note from the academic executor*

After the death of Vladimir Voevodsky in September, 2017, Daniel Grayson was appointed as the *academic executor*, in order to help arrange for the publication of his works. This paper had been submitted for publication, comments had been received twice from the referee, and Voevodsky had started rewriting the paper. Grayson has edited the paper, following the advice of the referee, and has made additional changes. Where anything substantively mathematical is involved, a footnote from the academic executor has been inserted.[3] In order to make it possible and convenient for readers to review Grayson's changes, all existing versions of the paper have been entered into a *github* repository, and editing was done in an incremental fashion, with *commit messages* describing the work done at each step. The repository is visible at

https://github.com/DanGrayson/VV-paths-Csystems-univ.

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[3] Note from the academic executor: ... such as this one.

The paper is visible as an entry in the archival record of Voevodsky's work at

http://www.math.ias.edu/Voevodsky/voevodsky-publications_abstracts.html#1505.06446,

and a link there will allow the reader to view all existing drafts of the paper. If those links are ever broken, perhaps a search for the randomly chosen string

`864338b2cec01ffb32fdaaa6bd8fafb803c070e7`

will allow the files to be located.

The academic executor thanks Benedikt Ahrens for many useful contributions to the editing process.

2. J-structures on C-systems and on universe categories

We begin by recalling the definition of "C-system" from [20]. C-systems were introduced by John Cartmell ([3], [4, p. 237]) and studied further by Thomas Streicher (see [11, Def. 1.2, p. 47]). Both authors used the name "contextual categories" for these structures.

By a pre-category C we mean a pair of sets $Mor(C)$ and $Ob(C)$ with four maps

$$\partial_0, \partial_1 : Mor(C) \to Ob(C)$$

$$Id : Ob(C) \to Mor(C)$$

and

$$\circ : Mor(C)_{\partial_1} \times_{\partial_0} Mor(C) \to Mor(C)$$

which satisfy the well known conditions of unity and associativity (note that we write composition of morphisms in the form $f \circ g$ or fg where $f : X \to Y$ and $g : Y \to Z$). These objects would usually be called categories but we reserve the name "category" for those uses of these objects that are invariant under equivalence.

Definition **2.0.1.** — *A C0-system is a pre-category* CC *with additional structure of the form*

1. *a function* $l : Ob(CC) \to \mathbf{N}$,
2. *an object* pt,
3. *a map* $\mathsf{ft} : Ob(CC) \to Ob(CC)$,
4. *for each* $X \in Ob(CC)$ *a morphism* $\mathsf{p}_X : X \to \mathsf{ft}(X)$,
5. *for each* $X \in Ob(CC)$ *such that* $l(X) > 0$ *and each morphism* $f : Y \to \mathsf{ft}(X)$ *an object* $f^*(X)$ *and a morphism* $\mathsf{q}(f, X) : f^*X \to X$,

which satisfies the following conditions:

1. $l^{-1}(0) = \{pt\}$
2. *for* X *such that* $l(X) > 0$ *one has* $l(\mathsf{ft}(X)) = l(X) - 1$
3. $\mathsf{ft}(pt) = pt$
4. *pt is a final object,*
5. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$ *and* $f : Y \to \mathsf{ft}(X)$ *one has* $l(f^*(X)) > 0$, $\mathsf{ft}(f^*X) = Y$ *and the square*

$$\begin{tikzcd} f^*X \arrow[r, "\mathsf{q}(f,X)"] \arrow[d, "\mathsf{p}_{f^*X}"'] & X \arrow[d, "\mathsf{p}_X"] \\ Y \arrow[r, "f"] & \mathsf{ft}(X) \end{tikzcd} \tag{2.1}$$

commutes,

6. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$ *one has* $id^*_{\mathsf{ft}(X)}(X) = X$ *and* $\mathsf{q}(id_{\mathsf{ft}(X)}, X) = id_X$,
7. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$, $g : Z \to Y$ *and* $f : Y \to \mathsf{ft}(X)$ *one has* $(gf)^*(X) = g^*(f^*(X))$ *and* $\mathsf{q}(gf, X) = \mathsf{q}(g, f^*X)\mathsf{q}(f, X)$.

Remark **2.0.2.** — In this definition *pt* stands for "point" and serves as our notation for a final object of a category. The name "ft" stands for "father" which is the name given to this map in [11, Def. 1.1].

For $f : Y \to X$ in CC we let $\mathsf{ft}(f) : Y \to \mathsf{ft}(X)$ denote the composition $f \circ \mathsf{p}_X$.

Definition **2.0.3.** — *A C-system is a C0-system together with an operation* $f \mapsto \mathsf{s}_f$ *defined for all* $f : Y \to X$ *such that* $l(X) > 0$ *and such that the following properties hold.*

1. $\mathsf{s}_f : Y \to (\mathsf{ft}(f))^*(X)$,
2. $\mathsf{s}_f \circ \mathsf{p}_{(\mathsf{ft}(f))^*(X)} = 1_Y$,
3. $\mathsf{s}_f \circ \mathsf{q}(\mathsf{ft}(f), X) = f$,
4. *if* $X = g^*(U)$ *where* $g : \mathsf{ft}(X) \to \mathsf{ft}(U)$ *then* $\mathsf{s}_f = \mathsf{s}_{f \circ \mathsf{q}(g,U)}$.

From [20, Prop. 2.4] it follows that, in a C-system, the squares 2.1 are pullback squares, and the maps s_f are obtainable using the universal property of those pullback squares.

Remark **2.0.4.** — We record some identities for later reference.

In the case where $Y = \mathsf{ft}(X)$ and $\mathsf{ft}(f) = 1_X$, and thus f is a section of p_X, one has the following identity.

$$\mathsf{s}_f = f \tag{2.2}$$

The formation of the section s_f is natural in the sense that, for a map $h : \mathrm{Z} \to \mathrm{Y}$, one has the following identity.

$$h^*(\mathsf{s}_f) = \mathsf{s}_{h\circ f} \tag{2.3}$$

In the case where one takes $\mathrm{Y} = \mathrm{X}$, and $f = 1_{\mathrm{X}}$, we let $\delta(\mathrm{X})$ denote the diagonal morphism $\delta(\mathrm{X}) := \mathsf{s}_{1_\mathrm{X}} : \mathrm{X} \to \mathsf{p}_\mathrm{X}^*(\mathrm{X})$.

$$\begin{tikzcd}
\mathsf{p}_\mathrm{X}^*(\mathrm{X}) \arrow[r, "\mathsf{q}(\mathsf{p}_\mathrm{X}{,}\mathrm{X})"] \arrow[d, shift right, "\mathsf{p}_{\mathsf{p}_\mathrm{X}^*(\mathrm{X})}"'] & \mathrm{X} \arrow[d, "\mathsf{p}_\mathrm{X}"] \\
\mathrm{X} \arrow[u, shift right, "\delta(\mathrm{X})"'] \arrow[ur, "1_\mathrm{X}"'] \arrow[r, "\mathsf{p}_\mathrm{X}"'] & \mathsf{ft}(\mathrm{X})
\end{tikzcd} \tag{2.4}$$

Using (2.3), one shows, for a map $f : \mathrm{Y} \to \mathrm{X}$, the following identity.

$$f^*(\delta(\mathrm{X})) = \mathsf{s}_f \tag{2.5}$$

Now we also recall the notations $\mathrm{O}b_n(\Gamma)$ and $\mathrm{O}bwt_n(\Gamma)$ from [19, §3], as well as the notation $\partial(o)$ from [20, §3].

Let Γ be an object in $\mathbf{C}$. The set $\mathrm{O}b_n(\Gamma)$ is the set of objects Δ of $\mathbf{C}$ such that $l(\Delta) - n = l(\Gamma)$ and $\mathsf{ft}^n(\Delta) = \Gamma$.

For $n > 0$, the set $\widetilde{\mathrm{O}b}_n(\Gamma)$ is the set of morphisms of the form $o : \mathsf{ft}(\Delta) \to \Delta$ for some $\Delta \in \mathrm{O}b_n(\Gamma)$ that are sections of the corresponding canonical projection, i.e., where $o \circ p_\Delta = 1$; in that case $\partial(o)$ denotes Δ.

Similarly, for an object Δ with $l(\Delta) > 0$, we let $\widetilde{\mathrm{O}b}(\Delta)$ denote the set of sections of p_Δ.

Observe that s_f above is an element of $\widetilde{\mathrm{O}b}_1(\mathrm{Y})$, and $\delta(\mathrm{X})$ above is an element of $\widetilde{\mathrm{O}b}_1(\mathrm{X})$.

2.1. *The notion of J-structure on a C-system*

To define the notion of "J-structure" on a C-system $\mathbf{C}$ we will actually define three types of structure: a J0-structure, a J1-structure over a J0-structure, and a J2-structure over a J1-structure—then a J-structure will be the same as a triple $(\mathsf{Id}, \mathsf{refl}, \mathsf{J})$, where Id is a J0-structure, refl is a J1-structure over Id, and J is a J2-structure over refl.

Definition **2.1.1.** — *A* J0-structure *on a C-system* $\mathbf{C}$ *is a family of functions*

$$\mathsf{Id}_\Gamma : \{(o_1, o_2) \mid o_1, o_2 \in \widetilde{\mathrm{O}b}_1(\Gamma), \partial(o_1) = \partial(o_2)\} \to \mathrm{O}b_1(\Gamma),$$

given for all $\Gamma \in \mathrm{O}b$, *that is natural in* Γ, *i.e., such that for any* $f : \Gamma' \to \Gamma$, *one has the identity*

$$f^*(\mathsf{Id}_\Gamma(o_1, o_2)) = \mathsf{Id}_{\Gamma'}(f^*(o_1), f^*(o_2)). \tag{2.6}$$

When there is no risk of ambiguity, we will write $\mathsf{Id}(o_1, o_2)$ for $\mathsf{Id}_\Gamma(o_1, o_2)$.

Definition **2.1.2.** — *Let* Id *be a J0-structure on* $\mathbf{C}$*. A* J1-structure *over* Id *is a family of functions*

$$\mathsf{refl} : \widetilde{Ob}_1(\Gamma) \to \widetilde{Ob}_1(\Gamma)$$

given for all $\Gamma \in Ob$ *such that:*

1. refl *is natural in* Γ,
2. *for any* Γ *and* $o \in \widetilde{Ob}_1(\Gamma)$ *one has*

$$\partial(\mathsf{refl}(o)) = \mathsf{Id}_\Gamma(o, o) \tag{2.7}$$

To define the notion of a J2-structure over a given J1-structure we will need to describe two constructions first.

Problem **2.1.3.** — Given a J0-structure Id, to construct a family of functions

$$\mathsf{Id}_3 : Ob_1(\Gamma) \to Ob_3(\Gamma),$$

such that for $f : \Gamma' \to \Gamma$ and $T \in Ob_1(\Gamma)$ one has $f^*(\mathsf{Id}_3(T)) = \mathsf{Id}_3(f^*(T))$.

Construction **2.1.4** *(For Problem 2.1.3).* — The objects and some of the morphisms involved in this construction can be seen in the following diagram, in which the downward maps are canonical projections.

$$\begin{tikzcd}
\mathsf{p}^*_{\mathsf{p}^*_T(T)}(\mathsf{p}^*_T(T)) \arrow[r] \arrow[d] & \mathsf{p}^*_T(T) \arrow[r, "q(\mathsf{p}_T, T)"] \arrow[d, shift right=2, "\mathsf{p}_{\mathsf{p}^*_T(T)}"'] & T \arrow[d, "\mathsf{p}_T"] \\
\mathsf{p}^*_T(T) \arrow[r, "\mathsf{p}_{\mathsf{p}^*_T(T)}"] \arrow[u, shift left=3, "\mathsf{p}^*_{\mathsf{p}^*_T(T)}(\delta(T))"] \arrow[u, shift right=3, "\delta(\mathsf{p}^*_T(T))"'] & T \arrow[r, "\mathsf{p}_T"] \arrow[u, shift right=2, "\delta(T)"'] & \Gamma
\end{tikzcd} \tag{2.8}$$

Since $\mathsf{p}^*_{\mathsf{p}^*_T(T)}(\delta(T))$ and $\delta(\mathsf{p}^*_T(T))$ are sections of the same canonical projection, we may make the following definition.

$$\mathsf{Id}_3(T) := \mathsf{Id}_{p^*_T(T)}(\mathsf{p}^*_{\mathsf{p}^*_T(T)}(\delta(T)), \delta(\mathsf{p}^*_T(T))) \tag{2.9}$$

The fact that $\mathsf{Id}_3(T) \in Ob_3(\Gamma)$ follows now from the fact that $\mathsf{ft}^2(\mathsf{p}^*_T(T)) = \mathsf{ft}(T) = \Gamma$. The proof that Id_3 is natural in $f : \Gamma' \to \Gamma$ is omitted. □

Problem **2.1.5.** — Given a J0-structure Id and a J1-structure refl over it, to construct for all $\Gamma \in Ob$ and $T \in Ob_1(\Gamma)$ a morphism

$$\mathsf{rf}_T : T \to \mathsf{Id}_3(T)$$

over Γ, such that for any $f : \Gamma' \to \Gamma$ one has $f^*(\mathsf{rf}_T) = \mathsf{rf}_{f^*(T)}$.

Construction **2.1.6** *(For Problem 2.1.5).* — We have the following chain of equations.

$$\begin{aligned}
&\delta(\mathrm{T})^*(\mathsf{Id}_3(\mathrm{T}))\\
&=\delta(\mathrm{T})^*(\mathsf{Id}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})}((\mathsf{p}^*_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})}(\delta(\mathrm{T})))),\delta(\mathsf{p}^*_{\mathrm{T}}(\mathrm{T}))) && \text{(by Definition 2.9)}\\
&=\mathsf{Id}_{\mathrm{T}}(\delta(\mathrm{T})^*(\mathsf{p}^*_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})}(\delta(\mathrm{T}))),\delta(\mathrm{T})^*(\delta(\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})))) && \text{(by naturality (2.6))}\\
&=\mathsf{Id}_{\mathrm{T}}((\delta(\mathrm{T})\circ\mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})})^*(\delta(\mathrm{T})),\delta(\mathrm{T})^*(\delta(\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})))) && \text{(by def'n. of C-system)}\\
&=\mathsf{Id}_{\mathrm{T}}((1_{\mathrm{T}})^*(\delta(\mathrm{T})),\delta(\mathrm{T})^*(\delta(\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})))) && \text{(by def'n. of C-system)}\\
&=\mathsf{Id}_{\mathrm{T}}(\delta(\mathrm{T}),\delta(\mathrm{T})^*(\delta(\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})))) && \text{(by def'n. of C-system)}\\
&=\mathsf{Id}_{\mathrm{T}}(\delta(\mathrm{T}),\mathsf{s}_{\delta(\mathrm{T})}) && \text{(by (2.5))}\\
&=\mathsf{Id}_{\mathrm{T}}(\delta(\mathrm{T}),\delta(\mathrm{T})) && \text{(by (2.2))}
\end{aligned}$$

This shows that we have the canonical square in the following diagram.

(**2.10**)

$$\begin{tikzcd}
\mathsf{Id}_\Gamma(\delta(\mathrm{T}),\delta(\mathrm{T})) \arrow[r, "\mathsf{q}(\delta(\mathrm{T}),\mathsf{Id}_3(\mathrm{T}))"] \arrow[d, shift left, "\mathsf{p}_{\mathsf{Id}_\Gamma(\delta(\mathrm{T}),\delta(\mathrm{T}))}"] & \mathsf{Id}_3(\mathrm{T}) \arrow[d] \\
\mathrm{T} \arrow[u, shift left, "\mathsf{refl}(\delta(\mathrm{T}))"] \arrow[r, "\delta(\mathrm{T})"'] \arrow[ur, dashed, "\mathsf{rf}_{\mathrm{T}}"] & \mathsf{p}^*_{\mathrm{T}}(\mathrm{T})
\end{tikzcd}$$

Since $\mathsf{refl}(\delta(\mathrm{T}))$ is a morphism $\mathrm{T}\to\mathsf{Id}(\delta(\mathrm{T}),\delta(\mathrm{T}))$ and is a section of the corresponding canonical projection, we may introduce the following definition.

$$\mathsf{rf}_{\mathrm{T}}:=\mathsf{refl}(\delta(\mathrm{T}))\circ\mathsf{q}(\delta(\mathrm{T}),\mathsf{Id}_3(\mathrm{T})) \tag{2.11}$$

The proof that for any $f:\Gamma'\to\Gamma$ one has $f^*(\mathsf{rf}_{\mathrm{T}})=\mathsf{rf}_{f^*(\mathrm{T})}$ is omitted. □

Definition **2.1.7.** — *Let* Id *and* refl *be a J0-structure and a J1-structure over it. A* J2-structure *over* (Id, refl) *is data of the form: for all* $\Gamma\in\mathrm{O}b$, *for all* $\mathrm{T}\in\mathrm{O}b_1(\Gamma)$, *for all* $\mathrm{P}\in\mathrm{O}b_1(\mathsf{Id}_3(\mathrm{T}))$, *for all* $s0\in\widetilde{\mathrm{O}b}(\mathsf{rf}^*_{\mathrm{T}}(\mathrm{P}))$, *an element* $\mathsf{J}(\Gamma,\mathrm{T},\mathrm{P},s0)$ *of* $\widetilde{\mathrm{O}b}(\mathrm{P})$ *such that:*

1. J *satisfies the ι-rule. For* $\Gamma,\mathrm{T},\mathrm{P},s0$ *as above one has*

$$\mathsf{rf}^*_{\mathrm{T}}(\mathsf{J}(\Gamma,\mathrm{T},\mathrm{P},s0))=s0$$

2. J *is natural in* Γ, *i.e., for any* $f:\Gamma'\to\Gamma$ *and* $\mathrm{T},\mathrm{P},s0$ *as above one has*

$$f^*(\mathsf{J}(\Gamma,\mathrm{T},\mathrm{P},s0))=\mathsf{J}(\Gamma',f^*(\mathrm{T}),f^*(\mathrm{P}),f^*(s0)),$$

where the right hand side of the equation is well-defined because of the naturality in f *of* Id_3 *and* rf^*.

Remark **2.1.8.** — A J0-structure is called degenerate or extensional if for all $T \in Ob_{\geq 1}(\mathbf{C})$ and $o, o' \in \widetilde{Ob}(T)$ one has[4]

$$\widetilde{Ob}(\mathsf{Id}_3(o, o')) = \begin{cases} \emptyset & \text{if } o \neq o' \\ pt & \text{if } o = o' \end{cases}$$

One can easily see that any two extensional J0-structures are equal and that any extensional J0-structure has a unique extension to a full J-structure that is also called extensional.

We will not consider these extensional versions of J in the present version of the paper.

Remark **2.1.9.** — When one studies J-structures on C-systems that have no (Π, λ)-structures it is important, as emphasized, for example, in [8], [6, Remark 2.4.1], and [12], to consider a more complex structure than the one that we consider here.[5] This more complex structure can be seen as a family of structures eJ_n, where $eJ_0 = \mathsf{J}$ (as above in Definition 2.1.7), and where eJ_n over $(\mathsf{Id}, \mathsf{refl})$ is a collection of data of the form: for all $\Gamma \in Ob$, for all $T \in Ob_1(\Gamma)$, for all $\Delta \in Ob_n(\mathsf{Id}_3(T))$, for all $P \in Ob_1(\Delta)$, for all $s0 \in \widetilde{Ob}(\mathsf{rf}^*_T(P))$, an element $eJ_n(\Gamma, T, \Delta, s0)$ in $\widetilde{Ob}(P)$ such that eJ_n satisfies the obvious analogue of ι-rule and such that it is natural in Γ. See also Remark 2.2.19.

2.2. *The notion of J-structure on a universe in a category*

Let $\mathcal{C}$ be a category,[6] and let $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ be a morphism in $\mathcal{C}$. Recall [17] that a *universe structure* on p is a choice of pullback squares of the form

(**2.12**)
$$\begin{tikzcd}
(X; F) \arrow[r, "\mathsf{Q}(F)"] \arrow[d, "\mathsf{p}_{X,F}"'] & \widetilde{\mathcal{U}} \arrow[d, "p"] \\
X \arrow[r, "F"] & \mathcal{U}
\end{tikzcd}$$

for all objects X and all morphisms $F : X \to \mathcal{U}$. We refer to the pullbacks given by a universe structure as *canonical pullbacks*. A universe in $\mathcal{C}$ is a morphism with a universe structure on it, and a universe category is a category with a universe and a choice of a final object *pt*.

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[4] The condition stated is the classical way of saying that there is an equivalence between the types $\widetilde{Ob}(\mathsf{Id}_3(o, o'))$ and $(o = o')$.

[5] Note from the academic executor: So presumably, this paper will find its applications in situations where the C-systems do have (Π, λ)-structures.

[6] Note from the academic executor: Not all of the constructions appearing in the sequel will be invariant under equivalence of categories, and hence, in a formalization using Voevodsky's Univalent Foundations, such categories will not be assumed to be "univalent". Indeed, in Voevodsky's view, univalent categories are so important that non-univalent categories shouldn't even be called categories, hence the introduction of the term "C-system".

Definition **2.2.1.** — *Given a sequence of maps*

$$\mathrm{F}_{i+1} : (\dots(\mathrm{X}; \mathrm{F}_1); \dots; \mathrm{F}_i) \to \mathcal{U}$$

for each i*, we will use the notation* $(\mathrm{X}; \mathrm{F}_1, \dots, \mathrm{F}_n)$ *for* $(\dots(\mathrm{X}; \mathrm{F}_1); \dots; \mathrm{F}_n)$.

For $f : \mathrm{W} \to \mathrm{X}$ and $g : \mathrm{W} \to \widetilde{\mathcal{U}}$ satisfying $f \circ \mathrm{F} = g \circ p$ we will let $f * g : \mathrm{W} \to (\mathrm{X}; \mathrm{F})$ denote the unique morphism such that

$$(f * g) \circ \mathsf{p}_{\mathrm{X},\mathrm{F}} = f \tag{2.13}$$

$$(f * g) \circ \mathsf{Q}(\mathrm{F}) = g \tag{2.14}$$

Observe that if $h : \mathrm{W}' \to \mathrm{W}$ is a map, then

$$h \circ (f * g) = (h \circ f) * (h \circ g) \tag{2.15}$$

When we need to distinguish canonical squares arising from different universe structures we may write $(\mathrm{X}; \mathrm{F})_p$, $\mathrm{Q}_p(\mathrm{F})$, and $f *_p g$. We may also write $(\mathrm{X}; \mathrm{F})'$ for $(\mathrm{X}; \mathrm{F})_{p'}$ and $(\mathrm{X}; \mathrm{F})'_i$ for $(\mathrm{X}; \mathrm{F})_{p'_i}$, and similarly for $\mathrm{Q}(\mathrm{F})$.

Remark **2.2.2.** — Note that we made no assumption about $\mathsf{Q}(1_{\mathcal{U}})$ being equal to $1_{\widetilde{\mathcal{U}}}$. In fact, since we want the results of this paper to be constructive, we are not allowed to make such an assumption, since whether a morphism is an identity morphism need not be decidable, and therefore we can not normalize a construction of a universe structure by doing something different when a morphism is not the identity. The importance of this observation (in the context of whether a simplex is degenerate) was emphasized by [2].

Definition **2.2.3.** — *For* $\mathrm{X}' \xrightarrow{f} \mathrm{X} \xrightarrow{\mathrm{F}} \mathcal{U}$ *we let* $\mathsf{Q}(f, \mathrm{F})$ *denote the morphism*

$$(\mathsf{p}_{\mathrm{X}',f\circ\mathrm{F}} \circ f) * \mathsf{Q}(f \circ \mathrm{F}) : (\mathrm{X}'; f \circ \mathrm{F}) \to (\mathrm{X}; \mathrm{F})$$

As shown in [17, Lemma 2.3], the left hand square in the following diagram is a pullback square.

$$\begin{tikzcd}
(\mathrm{X}'; f \circ \mathrm{F}) \arrow[r, "\mathsf{Q}(f,\mathrm{F})"'] \arrow[d, "\mathsf{p}_{\mathrm{X}',f\circ\mathrm{F}}"'] \arrow[rr, bend left, "\mathsf{Q}(f\circ\mathrm{F})"] & (\mathrm{X}; \mathrm{F}) \arrow[r, "\mathsf{Q}(\mathrm{F})"'] \arrow[d, "\mathsf{p}_{\mathrm{X},\mathrm{F}}"] & \widetilde{\mathcal{U}} \arrow[d, "p"'] \\
\mathrm{X}' \arrow[r, "f"] \arrow[rr, bend right, "f\circ\mathrm{F}"'] & \mathrm{X} \arrow[r, "\mathrm{F}"] & \mathcal{U}
\end{tikzcd} \tag{2.16}$$

Following [21, 2.30] we define for any universe $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ and any $V \in \mathcal{C}$ a contravariant functor from $\mathcal{C}$ to the category of sets,

$$\mathsf{D}_p(-, V) : X \mapsto \amalg_{F:X\to\mathcal{U}} \mathrm{H}om((X; F), V).$$

For $F : X \to \mathcal{U}$ and $G : (X; F) \to V$, the corresponding element of $\mathsf{D}_p(X, V)$ will be written as the pair (F, G). The action of the functor on a morphism $f : X' \to X$ is given by

$$\mathsf{D}_p(f, V) : (F, G) \mapsto (f \circ F, \mathsf{Q}(f, F) \circ G).$$

Recall that for an object X of $\mathcal{C}$, the *slice* category $\mathcal{C}/X$ is the category whose objects are morphisms $p : Y \to X$ of $\mathcal{C}$, and whose morphisms from $p : Y \to X$ to $p' : Y' \to X$ are those morphisms $f : Y \to Y'$ that make the evident triangle commute. We may also use (Y, p) as notation for the object p, to prevent ambiguity. When $\mathcal{C}$ is locally Cartesian closed, then the slice categories are Cartesian closed, and we use $\underline{\mathrm{H}om}((Y, p), (Y', p'))$ to denote the internal Hom in the slice category.

When $\mathcal{C}$ is a locally Cartesian closed category, any morphism $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ defines a functor

$$\mathsf{I}_p : \mathcal{C} \to \mathcal{C}/\mathcal{U}$$

which sends an object V to

$$\mathsf{I}_p(V) := \underline{\mathrm{H}om}((\widetilde{\mathcal{U}}, p), (\mathcal{U} \times V, \mathsf{pr}_1)).$$

We denote by

$$\mathsf{prI}_p(V) : \mathsf{I}_p(V) \to \mathcal{U} \tag{\textbf{2.17}}$$

the arrow of $\mathsf{I}_p(V)$.

We have constructed in [21, Construction 2.6.4] (originally in [18, Construction 3.9]) a family of bijections

$$\eta^!{}_{p,X,V} : \mathrm{H}om(X, \mathsf{I}_p(V)) \to \mathsf{D}_p(X, V),$$

which are natural in X and V. We let $\eta^{!^{-1}}$ denote the inverse bijections

$$\eta^{!^{-1}}{}_{p,X,V} : \mathsf{D}_p(X, V) \to \mathrm{H}om(X, \mathsf{I}_p(V)).$$

For $F : X \to \mathcal{U}$ and $G : (X; F) \to V$, we will abbreviate $\eta^{!^{-1}}_{p,X,V}(F, G)$ to $\eta^{!^{-1}}_{p}(F, G)$; this should cause no confusion, because X and V are determined by F and G. Similarly, for $H : X \to \mathsf{I}_p(V)$, we will abbreviate $\eta^!_{p,X,V}(H)$ to $\eta^!_p(H)$, provided that X and V can be determined from the context. We may even write $\eta^!$ for $\eta^!_p$ and $\eta'^{,!}$ for $\eta^!_{p'}$.

Using the functorial structure on the mapping $\mathrm{V} \mapsto (\mathcal{U} \times \mathrm{V}, \mathsf{pr}_1)$ together with the naturality of internal Hom-objects in the second argument we get a functoriality structure on I_p,

$$(f : \mathrm{V} \to \mathrm{V}') \mapsto (\mathsf{I}_p(f) : \mathsf{I}_p(\mathrm{V}) \to \mathsf{I}_p(\mathrm{V}')).$$

Similarly, using the functoriality of $\underline{\mathrm{H}om}$ in the second argument (see, e.g., [21, §4.2]) we obtain, for any universe $p : \widetilde{\mathcal{U}} \to \mathcal{U}$, any universe $p' : \widetilde{\mathcal{U}}' \to \mathcal{U}$, any map $h : \widetilde{\mathcal{U}}' \to \widetilde{\mathcal{U}}$ over $\mathcal{U}$, and object V, a morphism

$$\mathsf{I}^h(\mathrm{V}) : \mathsf{I}_p(\mathrm{V}) \to \mathsf{I}_{p'}(\mathrm{V}).$$

Lemma **2.2.4.** — *Given a map $f : \mathrm{V} \to \mathrm{V}'$ and a map $h : \widetilde{\mathcal{U}}' \to \widetilde{\mathcal{U}}$ over $\mathcal{U}$, as in the notation introduced above, the following square is commutative.*

$$\begin{tikzcd}
\mathsf{I}_{p'}(\mathrm{V}) \arrow[r, "\mathsf{I}_{p'}(f)"] \arrow[d, "\mathsf{I}^h(\mathrm{V})"'] & \mathsf{I}_{p'}(\mathrm{V}') \arrow[d, "\mathsf{I}^h(\mathrm{V}')"] \\
\mathsf{I}_p(\mathrm{V}) \arrow[r, "\mathsf{I}_p(f)"] & \mathsf{I}_p(\mathrm{V}')
\end{tikzcd}$$

Proof. — This is a particular case of the commutative square of [21, Lemma 4.1.5]. ☐

Lemma **2.2.5.** — *Let $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ and $p' : \widetilde{\mathcal{U}}' \to \mathcal{U}$ be two morphisms with universe structures and $f : \widetilde{\mathcal{U}}' \to \widetilde{\mathcal{U}}$ be a morphism over $\mathcal{U}$. For $\mathrm{V} \in \mathcal{C}$ let $\mathsf{I}^f(\mathrm{V})$ be the corresponding morphism $\mathsf{I}_{p'}(\mathrm{V}) \to \mathsf{I}_p(\mathrm{V})$. Then for any X the square*

$$\begin{tikzcd}
\mathsf{D}_p(\mathrm{X}, \mathrm{V}) \arrow[r, "\eta^{!\,-1}_{p,\mathrm{X},\mathrm{V}}"] \arrow[d, "\mathsf{D}^f(\mathrm{X},\mathrm{V})"'] & \mathrm{H}om(\mathrm{X}, \mathsf{I}_p(\mathrm{V})) \arrow[d, "- \circ \mathsf{I}^f(\mathrm{V})"] \\
\mathsf{D}_{p'}(\mathrm{X}, \mathrm{V}) \arrow[r, "\eta^{!\,-1}_{p',\mathrm{X}.\mathrm{V}}"] & \mathrm{H}om(\mathrm{X}, \mathsf{I}_{p'}(\mathrm{V})),
\end{tikzcd}$$

where the left hand vertical arrow is defined by

$$\mathsf{D}^f(\mathrm{X}, \mathrm{V}) : (\mathrm{F}, \mathrm{F}') \mapsto (\mathrm{F}, \mathrm{F}^*(f) \circ \mathrm{F}'),$$

commutes.

Proof. — Since $\eta^{!^{-1}}$ is defined as an inverse to $\eta^!$ it is sufficient to show that for any $g \in Hom(X, \mathsf{I}_p(V))$ one has $\eta'^{,!}(g \circ \mathsf{I}^f(V)) = D^f(X, V)(\eta^!(g))$. Let

$$\mathsf{pr} = \mathsf{prI}_p(V) : \mathsf{I}_p(V) \to \mathcal{U}$$
$$\mathsf{pr}' = \mathsf{prI}_{p'}(V) : \mathsf{I}_{p'}(V) \to \mathcal{U}$$

(cf. (2.17)) be the canonical projections. Let

$$\mathsf{st} = \mathsf{st}_p(V) : (\mathsf{I}_p(V); \mathsf{pr}) \to V$$
$$\mathsf{st}' = \mathsf{st}_{p'}(V) : (\mathsf{I}_{p'}(V); \mathsf{pr}')' \to V$$

be the morphisms introduced in [21, (2.60)]. By the definition introduced in [21, (2.65)] we have

$$\eta'^{,!}(g \circ \mathsf{I}^f(V)) = (g \circ \mathsf{I}^f(V) \circ \mathsf{pr}', \mathsf{Q}'(g \circ \mathsf{I}^f(V), \mathsf{pr}') \circ \mathsf{st}')$$

and

$$\begin{aligned} D^f(X, V)(\eta^!(g)) &= D^f(X, V)(g \circ \mathsf{pr}, \mathsf{Q}(g, \mathsf{pr}) \circ \mathsf{st}) \\ &= (g \circ \mathsf{pr}, (g \circ \mathsf{pr})^*(f) \circ \mathsf{Q}(g, \mathsf{pr}) \circ \mathsf{st}). \end{aligned}$$

Therefore it is sufficient to show that

$$\mathsf{I}^f(V) \circ \mathsf{pr}' = \mathsf{pr} \tag{2.18}$$

and

$$\mathsf{Q}'(g \circ \mathsf{I}^f(V), \mathsf{pr}') \circ \mathsf{st}' = (g \circ \mathsf{pr})^*(f) \circ \mathsf{Q}(g, \mathsf{pr}) \circ \mathsf{st}. \tag{2.19}$$

The first equality asserts that $\mathsf{I}^f(V)$ is a morphism over $\mathcal{U}$, which follows from its construction.

By Lemma 3.1.1 we have

$$(g \circ \mathsf{pr})^*(f) \circ \mathsf{Q}(g, \mathsf{pr}) = \mathsf{Q}'(g, \mathsf{pr}) \circ \mathsf{pr}^*(f).$$

Next we have the following equations.

$$\begin{aligned} &\mathsf{Q}'(g \circ \mathsf{I}^f(V), \mathsf{pr}') \\ &= \mathsf{Q}'(g, \mathsf{I}^f(V) \circ \mathsf{pr}') \circ \mathsf{Q}'(\mathsf{I}^f(V), \mathsf{pr}') && \text{(by [17, Lemma 2.5])} \\ &= \mathsf{Q}'(g, \mathsf{pr}) \circ \mathsf{Q}'(\mathsf{I}^f(V), \mathsf{pr}') && \text{(by 2.18)} \end{aligned}$$

It remains to check that

$$\mathsf{Q}'(\mathsf{I}^f(V), \mathsf{pr}') \circ \mathsf{st}' = \mathsf{pr}^*(f) \circ \mathsf{st}.$$

This requires opening up the definitions [21, (2.60)] of st and st', which gives us

$$\mathsf{Q}'(\mathsf{I}^f(\mathrm{V}), \mathsf{pr}') \circ \iota' \circ ev' \circ \mathsf{pr}_2 = \mathsf{pr}^*(f) \circ \iota \circ ev \circ \mathsf{pr}_2.$$

It will suffice to prove the following equation:

$$\mathsf{Q}'(\mathsf{I}^f(\mathrm{V}), \mathsf{pr}') \circ \iota' \circ ev' = \mathsf{pr}^*(f) \circ \iota \circ ev. \tag{2.20}$$

We will obtain Equation (2.20) as a consequence of commutativity of the three squares in the following diagram.

$$\begin{tikzcd}
 & (\mathsf{I}_p(\mathrm{V}); \mathsf{pr})' \arrow[dl, "\mathsf{pr}^*(f)"'] \arrow[dd, "\iota'"'] \arrow[dr, "\mathsf{Q}'(\mathsf{I}^f(\mathrm{V}),\mathsf{pr})"] & \\
(\mathsf{I}_p(\mathrm{V}); \mathsf{pr}) \arrow[dd, "\iota"] & & (\mathsf{I}_{p'}(\mathrm{V}); \mathsf{pr}')' \arrow[dd, "\iota'"] \\
 & (\mathsf{I}_p(\mathrm{V}), \mathsf{pr}) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}', p') \arrow[dl, "1\times f"] \arrow[dr, "\mathsf{I}^f(\mathrm{V})\times 1"'] & \\
(\mathsf{I}_p(\mathrm{V}), \mathsf{pr}) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p) \arrow[dr, "ev"] & & (\mathsf{I}_{p'}(\mathrm{V}), \mathsf{pr}') \times_{\mathcal{U}} (\widetilde{\mathcal{U}}', p') \arrow[dl, "ev'"'] \\
 & \mathcal{U} \times \mathrm{V} &
\end{tikzcd} \tag{2.21}$$

The top two squares are particular cases of [21, Lemma 4.1.3] applied to the category of objects over $\mathcal{U}$; the maps ι and ι' are defined in the statement of the lemma. To obtain the upper right square one sets $b = 1_{\widetilde{\mathcal{U}}'}$ and $a = \mathsf{I}^f(\mathrm{V})$. To obtain the upper left square one sets $b = f$ and $a = 1_{\mathsf{I}_p(\mathrm{V})}$. The lower square is a particular case of [21, Lemma 4.1.6]. ☐

Definition **2.2.6.** — *A* J0-structure *on a universe p in a category $\mathcal{C}$ is a morphism* $\mathrm{E}q : (\widetilde{\mathcal{U}}; p) \to \mathcal{U}$.

Let $\mathrm{E}q$ be a J0-structure on p. Consider the object[7]

$$\mathrm{E}\widetilde{\mathcal{U}} := (\widetilde{\mathcal{U}}; p, \mathrm{E}q)$$

of $\mathcal{C}$ together with the composite

$$\mathrm{E}\widetilde{\mathcal{U}} \xrightarrow{\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q}} (\widetilde{\mathcal{U}}; p) \xrightarrow{\mathsf{p}_{\widetilde{\mathcal{U}},p}} \widetilde{\mathcal{U}} \xrightarrow{p} \mathcal{U}$$

of projections as an object over $\mathcal{U}$; let $p\mathrm{E}\widetilde{\mathcal{U}}$ denote that composite map.

---

[7] Note from the academic executor: As introduced in 2.2.1, $(\widetilde{\mathcal{U}}; p, \mathrm{E}q)$ is notation for $((\widetilde{\mathcal{U}}; p); \mathrm{E}q)$.

Problem **2.2.7.** — To construct a universe structure on $p\mathrm{E}\widetilde{\mathcal{U}}$.

Construction **2.2.8** *(For Problem 2.2.7).* — The three squares in the following diagram are pullback squares.

(**2.22**)
$$\begin{tikzcd}
(X; F, \mathsf{Q}(F) \circ p, \mathsf{Q}(\mathsf{Q}(F), p) \circ \mathrm{E}q) \arrow[r, "\mathsf{Q}(\mathsf{Q}(\mathsf{Q}(F), p), \mathrm{E}q)"] \arrow[d, "\mathsf{p}_{(X;F,\mathsf{Q}(F)\circ p),\mathsf{Q}(\mathsf{Q}(F),p)\circ \mathrm{E}q}"] & (\widetilde{\mathcal{U}}; p, \mathrm{E}q) \arrow[d, "\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q}"'] \arrow[ddd, bend left, "p\mathrm{E}\widetilde{\mathcal{U}}"] \\
(X; F, \mathsf{Q}(F) \circ p) \arrow[r, "\mathsf{Q}(\mathsf{Q}(F), p)"] \arrow[d, "\mathsf{p}_{(X;F,\mathsf{Q}(F)),p}"] & (\widetilde{\mathcal{U}}; p) \arrow[d, "\mathsf{p}_{\widetilde{\mathcal{U}},p}"'] \\
(X; F) \arrow[r, "\mathsf{Q}(F)"] \arrow[d, "\mathsf{p}_{X,F}"'] & \widetilde{\mathcal{U}} \arrow[d, "p"'] \\
X \arrow[r, "F"] & \mathcal{U}
\end{tikzcd}$$

Remarking that the composite of the right-hand vertical maps is $p\mathrm{E}\widetilde{\mathcal{U}}$, we define the canonical square for F relative to $p\mathrm{E}\widetilde{\mathcal{U}}$ to be the square obtained by concatenating these three squares. □

Let us denote the components of the canonical squares for $p\mathrm{E}\widetilde{\mathcal{U}}$ as follows:

(**2.23**)
$$\begin{tikzcd}
(X; F)_{\mathrm{E}} \arrow[r, "\mathsf{Q}(F)_{\mathrm{E}}"] \arrow[d, "\mathsf{p}^{\mathrm{E}}_{X,F}"'] & \mathrm{E}\widetilde{\mathcal{U}} \arrow[d, "p\mathrm{E}\widetilde{\mathcal{U}}"] \\
X \arrow[r, "F"] & \mathcal{U}
\end{tikzcd}$$

Explicitly we have

(**2.24**) $$(X; F)_{\mathrm{E}} = (X; F, \mathsf{Q}(F) \circ p, \mathsf{Q}(\mathsf{Q}(F), p) \circ \mathrm{E}q)$$

(**2.25**) $$\mathsf{Q}(F)_{\mathrm{E}} = \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(F), p), \mathrm{E}q)$$

(**2.26**) $$\mathsf{p}^{\mathrm{E}}_{X,F} = \mathsf{p}_{(X;F,\mathsf{Q}(F)\circ p),\mathsf{Q}(\mathsf{Q}(F),p)\circ\mathrm{E}q} \circ \mathsf{p}_{(X;F),\mathsf{Q}(F)\circ p} \circ \mathsf{p}_{X,F}$$

Definition **2.2.9.** — *For any map $f : X' \to X$, we will write $\mathsf{Q}(f, F)_{\mathrm{E}}$ for the canonical morphism from $(X; f \circ F)_{\mathrm{E}}$ to $(X; F)_{\mathrm{E}}$, defined analogously to Definition 2.2.3. It fits into the following* *diagram.*

(**2.27**)
$$\begin{tikzcd}
(\mathrm{X}';f\circ \mathrm{F})_{\mathrm{E}} \arrow[r, "\mathsf{Q}(f{,}\mathrm{F})_{\mathrm{E}}"'] \arrow[rr, bend left, "\mathsf{Q}(f\circ\mathrm{F})_{\mathrm{E}}"] \arrow[d, "\mathsf{p}^{\mathrm{E}}_{\mathrm{X}',f\circ\mathrm{F}}"'] & (\mathrm{X};\mathrm{F})_{\mathrm{E}} \arrow[r, "\mathsf{Q}(\mathrm{F})_{\mathrm{E}}"'] \arrow[d, "\mathsf{p}^{\mathrm{E}}_{\mathrm{X},\mathrm{F}}"] & \mathrm{E}\widetilde{\mathcal{U}} \arrow[d, "p\mathrm{E}\widetilde{\mathcal{U}}"'] \\
\mathrm{X}' \arrow[r, "f"] \arrow[rr, bend right, "f\circ\mathrm{F}"'] & \mathrm{X} \arrow[r, "\mathrm{F}"] & \mathcal{U}
\end{tikzcd}$$

Definition **2.2.10.** — *Let $p:\widetilde{\mathcal{U}}\to\mathcal{U}$ be a universe in $\mathcal{C}$ and* Eq *be a $\mathcal{J}0$-structure on p. A* J1-structure *on p over* Eq *is a morphism $\Omega:\widetilde{\mathcal{U}}\to\widetilde{\mathcal{U}}$ such that the square*

(**2.28**)
$$\begin{tikzcd}
\widetilde{\mathcal{U}} \arrow[r, "\Omega"] \arrow[d, "\Delta"'] & \widetilde{\mathcal{U}} \arrow[d, "p"] \\
(\widetilde{\mathcal{U}};p) \arrow[r, "\mathrm{E}q"] & \mathcal{U},
\end{tikzcd}$$

where $\Delta:=(1_{\widetilde{\mathcal{U}}})*(1_{\widetilde{\mathcal{U}}})$ is the diagonal of $\widetilde{\mathcal{U}}$, commutes.

The square (2.28) defines a morphism $\widetilde{\mathcal{U}}\to\mathrm{E}\widetilde{\mathcal{U}}$, which will be denoted by ω, as in the following diagram.

(**2.29**)
$$\begin{tikzcd}
\widetilde{\mathcal{U}} \arrow[dr, "\omega"] \arrow[drr, bend left, "\Omega"] \arrow[ddr, bend right, "\Delta"'] & & \\
& \mathrm{E}\widetilde{\mathcal{U}} \arrow[r, "\mathsf{Q}(\mathrm{E}q)"] \arrow[d, "\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q}"] & \widetilde{\mathcal{U}} \arrow[d, "p"] \\
& (\widetilde{\mathcal{U}};p) \arrow[r, "\mathrm{E}q"'] & \mathcal{U}
\end{tikzcd}$$

To define a J2-structure on a universe we will need to assume that $\mathcal{C}$ is a locally Cartesian closed category. Recall that a locally Cartesian closed category is a category with the choice of fiber squares based on all pairs of morphisms with a common codomain as well as the choice of relative internal Hom-objects and co-evaluation morphisms for all such pairs. For our notation related to the locally Cartesian closed categories as well as for some other notations used below see [19, 21, 22].

When a universe is considered in a locally Cartesian closed category we make no assumption about the compatibility of choices of the pullback squares of the universe structure on p and pullback squares of the locally Cartesian closed structure.

Consider the functors I_p and $\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}$. We have the following commutative square:

(**2.30**)
$$\begin{tikzcd}
\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}}) \arrow[r, "\mathsf{I}^{\omega}(\widetilde{\mathcal{U}})"] \arrow[d, "\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(p)"'] & \mathsf{I}_p(\widetilde{\mathcal{U}}) \arrow[d, "\mathsf{I}_p(p)"] \\
\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}) \arrow[r, "\mathsf{I}^{\omega}(\mathcal{U})"] & \mathsf{I}_p(\mathcal{U})
\end{tikzcd}$$

and therefore a morphism

$$co\mathrm{J} : \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}}) \longrightarrow (\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^{\omega}(\mathcal{U})) \times_{\mathsf{I}_p(\mathcal{U})} (\mathsf{I}_p(\widetilde{\mathcal{U}}), \mathsf{I}_p(p))$$

Definition **2.2.11.** — *A* J2-structure *on* $p : \widetilde{\mathcal{U}} \to \mathcal{U}$*, relative to a* $\mathcal{J}$*0-structure* Eq *and a* $\mathcal{J}$*1-structure* Ω *over* Eq*, is a morphism*

$$\mathrm{J}p : (\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^{\omega}(\mathcal{U})) \times_{\mathsf{I}_p(\mathcal{U})} (\mathsf{I}_p(\widetilde{\mathcal{U}}), \mathsf{I}_p(p)) \to \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}})$$

such that $\mathrm{J}p \circ co\mathrm{J} = 1$.

Definition **2.2.12.** — *A* $\mathcal{J}$*-structure on* $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ *is a triple* $(\mathrm{E}q, \Omega, \mathrm{J}p)$*, where* E$q$ *is a* $\mathcal{J}$*0-structure,* Ω *is a* $\mathcal{J}$*1-structure relative to* Eq*, and* Jp *is a* $\mathcal{J}$*2-structure relative to* Eq *and* Ω.

Definition **2.2.13.** — *Let* $\mathsf{Fp} = \mathsf{Fp}_{\mathrm{E}q,\Omega}$ *denote the fiber product*

$$(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^{\omega}(\mathcal{U})) \times_{\mathsf{I}_p(\mathcal{U})} (\mathsf{I}_p(\widetilde{\mathcal{U}}), \mathsf{I}_p(p))$$

and let $\mathsf{pFp}_{\mathrm{E}q,\Omega}$ *be the projection* $\mathsf{Fp}_{\mathrm{E}q,\Omega} \to \mathcal{U}$*. Let further* pr_1 *be the projection from* Fp *to* $\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U})$*, and let* pr_2 *be the projection from* Fp *to* $\mathsf{I}_p(\widetilde{\mathcal{U}})$*.*

Note that we have the following two equations.

(**2.31**)
$$\mathrm{J}p \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(p) = \mathrm{J}p \circ co\mathrm{J} \circ \mathsf{pr}_1 = \mathsf{pr}_1$$

(**2.32**)
$$\mathrm{J}p \circ \mathsf{I}^{\omega}(\widetilde{\mathcal{U}}) = \mathrm{J}p \circ co\mathrm{J} \circ \mathsf{pr}_2 = \mathsf{pr}_2$$

Definition **2.2.14.** — *Let* $\mathcal{C}$ *be a Cartesian closed category. For any objects* X, Y, Z *of* $\mathcal{C}$*, we let*

$$\mathsf{adj} : \mathrm{H}om(\mathrm{X}, \underline{\mathrm{H}om}(\mathrm{Y}, \mathrm{Z})) \xrightarrow{\cong} \mathrm{H}om(\mathrm{X} \times \mathrm{Y}, \mathrm{Z})$$

denote the corresponding adjunction bijection.

Remark **2.2.15.** — We will use the adjunction adj as follows. Consider the map $\mathsf{pr}_1 : \mathsf{Fp} \to \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}) = \underline{Hom}_{\mathrm{U}}(\mathrm{E}\widetilde{\mathcal{U}}, \mathcal{U} \times \mathcal{U})$ over $\mathcal{U}$. Applying adj yields a map $\mathsf{adj}(\mathsf{pr}_1) : \mathsf{Fp} \times_{\mathcal{U}} \mathrm{E}\widetilde{\mathcal{U}} \to \mathcal{U} \times \mathcal{U}$ over $\mathcal{U}$. Composing with $\mathsf{pr}_2 : \mathrm{E}\widetilde{\mathcal{U}} \times \mathcal{U} \to \mathcal{U}$ yields a map $\mathsf{adj}(\mathsf{pr}_1) \circ \mathsf{pr}_2 : \mathsf{Fp} \times_{\mathcal{U}} \widetilde{\mathcal{U}} \to \mathcal{U}$ (not over $\mathcal{U}$).

The same reasoning yields the map $\mathsf{adj}(\mathsf{pr}_2) \circ \mathsf{pr}_2 : \mathsf{Fp} \times_{\mathcal{U}} \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}$.

These maps will be used below.

Our solution to the following problem is the key to the construction of J-structures over a given J1-structure in categories with weak factorization systems, particularly in Quillen model categories.

Problem **2.2.16.** — Let $\mathcal{C}$ be a category with a locally Cartesian closed structure and $\mathrm{E}q, \Omega$ be a J1-structure on $(\mathcal{C}, p)$. To construct a bijection between the set of J-structures on p over $(\mathrm{E}q, \Omega)$ and the set of morphisms $(\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\mathrm{E}\widetilde{\mathcal{U}}, p\mathrm{E}\widetilde{\mathcal{U}}) \to \widetilde{\mathcal{U}}$ that split the following commutative square into two commutative triangles:

$$(\mathbf{2.33})\qquad \begin{tikzcd}
(\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p) \arrow[r, "\mathsf{adj}(\mathsf{pr}_2)\circ\mathsf{pr}_2"] \arrow[d, "1_{\mathsf{Fp}}\times\omega"'] & \widetilde{\mathcal{U}} \arrow[d, "p"] \\
(\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\mathrm{E}\widetilde{\mathcal{U}}, p\mathrm{E}\widetilde{\mathcal{U}}) \arrow[r, "\mathsf{adj}(\mathsf{pr}_1)\circ\mathsf{pr}_2"] \arrow[ur, dotted] & \mathcal{U}
\end{tikzcd}$$

(Commutativity of the square follows from the naturality of the adjunction adj and the equation $\mathsf{pr}_1 \circ \mathsf{I}^{\omega}(\mathcal{U}) = \mathsf{pr}_2 \circ \mathsf{I}_p(p)$.)

Remark **2.2.17.** — If we omit, as is customarily done, the explicit functions to the base in the notation of the fiber products, then the square (2.33) takes the following form.

$$(\mathbf{2.34})\qquad \begin{tikzcd}
(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}) \times_{\mathsf{I}_p(\mathcal{U})} \mathsf{I}_p(\widetilde{\mathcal{U}})) \times_{\mathcal{U}} \widetilde{\mathcal{U}} \arrow[r, "\mathsf{adj}(\mathsf{pr}_2)\circ\mathsf{pr}_2"] \arrow[d, "1_{\mathsf{Fp}}\times\omega"'] & \widetilde{\mathcal{U}} \arrow[d, "p"] \\
(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}) \times_{\mathsf{I}_p(\mathcal{U})} \mathsf{I}_p(\widetilde{\mathcal{U}})) \times_{\mathcal{U}} \mathrm{E}\widetilde{\mathcal{U}} \arrow[r, "\mathsf{adj}(\mathsf{pr}_1)\circ\mathsf{pr}_2"] \arrow[ur, dotted] & \mathcal{U}
\end{tikzcd}$$

Construction **2.2.18** *(For Problem 2.2.16).* — Observe first that there is a bijection between the set of morphisms

$$(\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\mathrm{E}\widetilde{\mathcal{U}}, p\mathrm{E}\widetilde{\mathcal{U}}) \to \widetilde{\mathcal{U}}$$

that split the square (2.33) into two commutative triangles and the set of morphisms

$$(\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\mathrm{E}\widetilde{\mathcal{U}}, p\mathrm{E}\widetilde{\mathcal{U}}) \to \mathcal{U} \times \widetilde{\mathcal{U}}$$

that split into two commutative triangles the commutative square:

$$\begin{tikzcd}
(\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p) \arrow[r, "\mathsf{adj}(\mathsf{pr}_2)"] \arrow[d, "1_{\mathsf{Fp}} \times \omega"'] & \mathcal{U} \times \widetilde{\mathcal{U}} \arrow[d, "1_{\mathcal{U}} \times p"] \\
(\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\mathrm{E}\widetilde{\mathcal{U}}, p\mathrm{E}\widetilde{\mathcal{U}}) \arrow[r, "\mathsf{adj}(\mathsf{pr}_1)"] \arrow[ur, dotted] & \mathcal{U} \times \mathcal{U}
\end{tikzcd}$$

The rule $f \mapsto \mathsf{adj}(f)$ gives us a bijection of the form

$$\mathrm{H}om_{\mathcal{U}}((\mathsf{Fp}, \mathsf{pFp}), (\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}}), \_))$$
$$\to \mathrm{H}om_{\mathcal{U}}((\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\mathrm{E}\widetilde{\mathcal{U}}, p\mathrm{E}\widetilde{\mathcal{U}}), (\mathcal{U} \times \widetilde{\mathcal{U}}, \mathsf{pr}_2))$$

All sections of $co\mathrm{J}$ are automatically morphisms over $\mathcal{U}$. Therefore it remains to show that this bijection defines a bijection of the subset of morphisms that are sections of $co\mathrm{J}$ and morphisms that make the two triangles commutative.

One verifies first that a morphism $f : \mathsf{Fp} \to \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}})$ is a section of $co\mathrm{J}$ if and only if $f \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(p) = \mathsf{pr}_1$ and $f \circ \mathsf{I}^{\omega}(\widetilde{\mathcal{U}}) = \mathsf{pr}_2$. This is omitted.

Next we have

$$\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(p) = \underline{\mathrm{H}om}_{\mathcal{U}}((\mathrm{E}\widetilde{\mathcal{U}}, p\mathrm{E}\widetilde{\mathcal{U}}), 1_{\mathcal{U}} \times p)$$
$$\mathsf{I}^{\omega}(\widetilde{\mathcal{U}}) = \underline{\mathrm{H}om}_{\mathcal{U}}(\omega, (\mathcal{U} \times \widetilde{\mathcal{U}}, \mathsf{pr}_2))$$

Therefore by [21, Lemma 4.1.7] one has

$$\mathsf{adj}(f \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(p)) = \mathsf{adj}(f) \circ (1_{\mathcal{U}} \times p) \quad \text{and}$$
$$\mathsf{adj}(f \circ \mathsf{I}^{\omega}(\widetilde{\mathcal{U}})) = (1_{\mathsf{Fp}} \times_{\mathcal{U}} \omega) \circ \mathsf{adj}(f),$$

and we conclude that f is a section of $co\mathrm{J}$ if and only if

$$\mathsf{adj}(f) \circ (1_{\mathcal{U}} \times p) = \mathsf{adj}(\mathsf{pr}_1) \quad \text{and}$$
$$(1_{\mathsf{Fp}} \times_{\mathcal{U}} \omega) \circ \mathsf{adj}(f) = \mathsf{adj}(\mathsf{pr}_2).$$

These two equations are the ones that assert the two triangles involving $\mathsf{adj}(f)$ commute. This completes the construction. □

Remark **2.2.19.** — It is likely to be relatively easy to generalize the constructions of this paper to the extended J-structures $e\mathrm{J}_n$ (see Remark 2.1.9). The structures $e\mathrm{J}p_n$ can be defined in the same way as $\mathrm{J}p$ but with the square (2.30) replaced by the square

$$\begin{tikzcd}
\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathsf{I}_p^n(\widetilde{\mathcal{U}})) \arrow[r, "\mathsf{I}^{\omega}(\mathsf{I}_p^n(\widetilde{\mathcal{U}}))"] \arrow[d, "\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathsf{I}_p^n(p))"'] & \mathsf{I}_p(\mathsf{I}_p^n(\widetilde{\mathcal{U}})) \arrow[d, "\mathsf{I}_p(\mathsf{I}_p^n(p))"] \\
\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathsf{I}_p^n(\mathcal{U})) \arrow[r, "\mathsf{I}^{\omega}(\mathsf{I}_p^n(\mathcal{U}))"] & \mathsf{I}_p(\mathsf{I}_p^n(\mathcal{U}))
\end{tikzcd} \tag{2.35}$$

2.3. *J-structures on universes in categories with two classes of morphisms*

Recall that a collection of morphisms R is said to have the right lifting property for the collection of morphisms L if for any commutative square of the form

$$\begin{tikzcd}
\mathrm{Z} \arrow[r, "f_{\mathrm{Z}}"] \arrow[d, "i"'] & \mathrm{E} \arrow[d, "p"] \\
\mathrm{W} \arrow[r, "f_{\mathrm{W}}"] & \mathrm{B}
\end{tikzcd}$$

such that $i \in \mathrm{L}$ and $p \in \mathrm{R}$ there exists a morphism $g : \mathrm{W} \to \mathrm{E}$ that makes the two triangles into which it splits the square commute, i.e., a morphism g such that $i \circ g = f_{\mathrm{Z}}$ and $g \circ p = f_{\mathrm{W}}$.

We are going to consider two sets of conditions (Conditions 2.3.2 and 2.3.4) on a pair of classes of morphisms FB and TC in a category with fiber products and then show in Theorems 2.3.3 and 2.3.9 how pairs satisfying conditions of each of these two sets can be used to construct J-structures on elements of FB.

Remark **2.3.1.** — This is the only part of the paper where we depart from constructions that are conservatively algebraic over the theory of categories, i.e., from constructions that can be expressed in terms of adding new quasi-algebraic operations to the theory of categories without adding new sorts to this theory.

Considering classes of morphisms in categories can be expressed in the quasi-algebraic way, but this requires adding new sorts to the theory.

This is also the only context where we use the concept "there exists" in this paper. In all the previous cases the objects that we considered were given (specified). To make the lemmas proved below into constructions and to avoid the use of "there exists" one would have to define the collection FB as a collection of pairs of a morphism p together with, for all $i \in \mathrm{TC}$, f_{W} and f_{Z} such that $f_{\mathrm{Z}} \circ p = i \circ f_{\mathrm{W}}$, a morphism g such that $i \circ g = f_{\mathrm{Z}}$ and $g \circ p = f_{\mathrm{W}}$.

Our first set of conditions is as follows:

Conditions **2.3.2.**
1. *A morphism is in* FB *if and only if it has the right lifting property for* TC*,*
2. *consider morphisms* $f : \mathrm{B}' \to \mathrm{B}$*,* $p_1 : \mathrm{E}_1 \to \mathrm{B}$*,* $p_2 : \mathrm{E}_2 \to \mathrm{B}$ *and* $i : \mathrm{E}_1 \to \mathrm{E}_2$ *such that* $p_1, p_2 \in \mathrm{FB}$*,* $i \circ p_2 = p_1$*, and* $i \in \mathrm{TC}$*. Then the morphism*

$$1_{\mathrm{B}'} \times i : (\mathrm{B}', f) \times_{\mathrm{B}} (\mathrm{E}_1, p_1) \to (\mathrm{B}', f) \times_{\mathrm{B}} (\mathrm{E}_2, p_2)$$

is in TC.

Theorem **2.3.3.** — *Let* FB *and* TC *be two classes of morphisms in a locally Cartesian closed category* $\mathcal{C}$ *that satisfy Conditions 2.3.2. Let* $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ *be a universe in* $\mathcal{C}$*, let* Eq *be a* $\mathcal{J}0$*-structure on* p*, and let* Ω *be a* $\mathcal{J}1$*-structure over* Eq*. Assume further that:*

1. p *is in* FB*,*
2. $\omega : \widetilde{\mathcal{U}} \to \mathrm{E}\widetilde{\mathcal{U}}$ *is in* TC *(see Definition 2.2.10 for the definition of* ω*).*

Then there exists an extension of $(\mathrm{E}q, \Omega)$ *to a full* $\mathcal{J}$*-structure on* p*.*

Proof. — Let us apply Construction 2.2.18 to $(\mathrm{E}q, \Omega)$. To construct the required morphism it is sufficient to establish that $1_{\mathsf{Fp}} \times \omega$ is in TC. It follows from the first of our conditions that FB is closed under pullbacks and compositions. Therefore, $p\mathrm{E}\widetilde{\mathcal{U}}$ is in FB. It remains to apply the second of our conditions. ☐

Our second set of conditions is more involved. Conditions of this set can be satisfied in the situations arising when one attempts to localize Quillen model structures and when the resulting sets of morphisms do not form a model structure. The difference is mainly concerned with the fact that the good behavior is required only for fibrations over fibrant objects. One particular example of such a situation is considered in [15, Section 3.3].

Conditions **2.3.4.**
1. 1_{pt} *is in* FB*,*
2. *let* B *be such that the morphism* $\mathrm{B} \to pt$ *is in* FB*, then a morphism* $p : \mathrm{E} \to \mathrm{B}$ *is in* FB *if and only if it has the right lifting property for* TC*,*
3. *if* $p : \mathrm{E} \to \mathrm{B}$ *and* $\mathrm{B} \to pt$ *are in* FB*,* $i : \mathrm{Z} \to \mathrm{W}$ *is in* TC *and* $f : \mathrm{W} \to \mathrm{B}$ *is an arbitrary morphism, then*

$$(i \times_{\mathcal{U}} 1_{\mathrm{E}}) : (\mathrm{Z}, i \circ f) \times_{\mathrm{B}} (\mathrm{E}, p) \to (\mathrm{W}, f) \times_{\mathrm{B}} (\mathrm{E}, p)$$

is in TC.

We will say that B is fibrant if the morphism $\mathrm{B} \to pt$ is in FB.

Lemma **2.3.5.** — *Assume Conditions 2.3.4 are satisfied. Let* $p : \mathrm{E} \to \mathrm{B}$ *be in* FB *and* $f : \mathrm{B}' \to \mathrm{B}$ *be a morphism. Assume in addition that* B *and* B′ *are fibrant, then for any pullback square of the form*

$$\begin{tikzcd}
\mathrm{E}' \arrow[r] \arrow[d, "p'"'] & \mathrm{E} \arrow[d, "p"] \\
\mathrm{B}' \arrow[r, "f"] & \mathrm{B}
\end{tikzcd}$$

the morphism p' *is in* FB.

Proof. — Since B′ is fibrant it is sufficient to verify that p' has the right lifting property for TC. This can be shown in the standard way to be a consequence of p having the right lifting property for TC. That p has this property we know because p is in FB and B is fibrant. ☐

Lemma **2.3.6.** — *Assume Conditions 2.3.4 are satisfied. Let* B *be fibrant and* $p_2 : \mathrm{E}_2 \to \mathrm{E}_1$, $p_1 : \mathrm{E}_1 \to \mathrm{B}$ *be in* FB. *Then* $p_2 \circ p_1$ *is in* FB.

Proof. — Let us show first that E_1 is fibrant, i.e., that $\pi_{\mathrm{E}_1} : \mathrm{E}_1 \to pt$ is in FB. Since pt is fibrant it is sufficient to show that π_{E_1} has the right lifting property for TC. It is shown in the standard way from the fact that both p_1 and $\pi_{\mathrm{B}} : \mathrm{B} \to pt$ have the right lifting property for TC and $\pi_{\mathrm{E}_1} = p_1 \circ \pi_{\mathrm{B}}$.

Since E_1 is fibrant we know that p_2 has the right lifting property for TC, and since B is fibrant we know that p_1 has the right lifting property for FB. From this we conclude in the standard way that $p_2 \circ p_1$ have the right lifting property for TC, and since B is fibrant this implies that $p_2 \circ p_1$ is in FB. ☐

Lemma **2.3.7.** — *Assume Conditions 2.3.4 are satisfied. Assume that* $\mathcal{U}, \mathrm{V}$ *are fibrant and that* $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ *is in* FB. *Then the morphism* $\mathsf{prl}_p(\mathrm{V}) : \mathsf{I}_p(\mathrm{V}) \to \mathcal{U}$ *is in* FB.

Proof. — Since $\mathcal{U}$ is fibrant it is sufficient to check that $\mathsf{pr} = \mathsf{prl}_p(\mathrm{V})$ has the right lifting property for TC. Consider a commutative square of the form

$$\begin{tikzcd}
\mathrm{Z} \arrow[r, "f_{\mathrm{Z}}"] \arrow[d, "i"'] & \underline{Hom}_{\mathcal{U}}((\widetilde{\mathcal{U}}, p), (\mathcal{U} \times \mathrm{V}, \mathsf{pr}_1)) \arrow[d, "\mathsf{pr}"] \\
\mathrm{W} \arrow[r, "f_{\mathrm{W}}"] & \mathcal{U}
\end{tikzcd}$$

We need to construct a morphism $f : \mathrm{W} \to \underline{Hom}_{\mathcal{U}}((\widetilde{\mathcal{U}}, p), (\mathcal{U} \times \mathrm{V}, \mathsf{pr}_1))$ that would make the two triangles commutative. The commutativity of the lower triangle means that f is a morphism over $\mathcal{U}$ which is equivalent to the assumption that $f = \mathsf{adj}^{-1}(g)$ for some $g : (W, f_W) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p) \to \mathcal{U} \times V$ over $\mathcal{U}$.

Consider the following square.

$$\begin{tikzcd}
(Z, i \circ f_W) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p) \arrow[r, "\mathsf{adj}(f_Z)"] \arrow[d, "i \times 1_{\widetilde{\mathcal{U}}}"'] & \mathcal{U} \times V \arrow[d, "\mathsf{pr}_1"] \\
(W, f_W) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p) \arrow[r] & \mathcal{U}
\end{tikzcd}$$

Here the bottom horizontal arrow is the projection to $\mathcal{U}$. By Lemma 2.3.5 we know that pr_1 belongs to FB. By our assumptions on TC and FB we know that $i \times 1_{\widetilde{\mathcal{U}}}$ is in TC. Therefore there exists a morphism $g : (W, f_W) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p) \to \mathcal{U} \times V$ that makes the two triangles commute. The commutativity of the lower triangle means that this is a morphism over $\mathcal{U}$. Therefore $\mathsf{adj}^{-1}(g)$ is defined. Set $f = \mathsf{adj}^{-1}(g)$. It remains to check that $i \circ f = f_Z$. This is equivalent to $\mathsf{adj}(i \circ f) = \mathsf{adj}(f_Z)$. Since $\mathsf{adj}(i \circ f) = (i \times 1_{\widetilde{\mathcal{U}}}) \circ \mathsf{adj}(f)$ by [18, Lemma 8.7(3)], this is equivalent to $(i \times 1_{\widetilde{\mathcal{U}}}) \circ g = \mathsf{adj}(f_Z)$ which is the commutativity of the upper triangle. □

Lemma **2.3.8.** — *Assume Conditions 2.3.4 are satisfied. Assume that* $\mathcal{U}$ *and* V *are fibrant and that* $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ *and* $r : V' \to V$ *are in* FB. *Then* $\mathsf{I}_p(r) : \mathsf{I}_p(V') \to \mathsf{I}_p(V)$ *is in* FB.

Proof. — By Lemmas 2.3.7 and 2.3.6 we know that $\mathsf{I}_p(V)$ is fibrant. Therefore it is sufficient to show that $\mathsf{I}_p(r)$ has the right lifting property for TC. Consider a commutative square of the form

$$\begin{tikzcd}
Z \arrow[r, "f_Z"] \arrow[d, "i"'] & \underline{\mathit{Hom}}_{\mathcal{U}}((\widetilde{\mathcal{U}}, p), (\mathcal{U} \times V', \mathsf{pr}_1)) \arrow[d, "\underline{\mathit{Hom}}_{\mathcal{U}}((\widetilde{\mathcal{U}}, p), 1_{\mathcal{U}} \times r)"] \\
W \arrow[r, "f_W"] & \underline{\mathit{Hom}}_{\mathcal{U}}((\widetilde{\mathcal{U}}, p), (\mathcal{U} \times V, \mathsf{pr}_1))
\end{tikzcd} \tag{2.36}$$

The lower right corner is an object over $\mathcal{U}$ through the morphism $p \triangle \mathsf{pr}_1$. Let $\mathsf{p}_W = f_W \circ (p \triangle \mathsf{pr}_{\mathcal{U}}^{\mathcal{U}, V})$ and

$$\mathsf{p}_Z = i \circ \mathsf{p}_W = f_Z \circ (p \triangle \mathsf{pr}_{\mathcal{U}}^{\mathcal{U}, V'}).$$

Consider the square

$$\begin{tikzcd}
(Z, \mathsf{p}_Z) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p) \arrow[r, "\mathsf{adj}(f_Z)"] \arrow[d, "i \times 1_{\widetilde{\mathcal{U}}}"'] & \mathcal{U} \times V' \arrow[d, "1_{\mathcal{U}} \times r"] \\
(W, \mathsf{p}_W) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p) \arrow[r, "\mathsf{adj}(f_W)"] & \mathcal{U} \times V
\end{tikzcd} \tag{2.37}$$

This square commutes. Indeed,

$$\begin{aligned}\mathsf{adj}(f_{\mathrm{Z}})\circ(1_{\mathcal{U}}\times r) &= \mathsf{adj}(f_{\mathrm{Z}}\circ\underline{Hom}_{\mathcal{U}}((\widetilde{\mathcal{U}},p),\,1_{\mathcal{U}}\times r))\\ &= \mathsf{adj}(i\circ f_{\mathrm{W}})\\ &= (i\times 1_{\widetilde{\mathcal{U}}})\circ\mathsf{adj}(f_{\mathrm{W}}),\end{aligned}$$

where the first equality is by [18, Lemma 8.7(1)] and the third by [18, Lemma 8.7(3)]. By Lemmas 2.3.5 and 2.3.6 we know that $1_{\mathcal{U}}\times r$ is in FB. By our assumption (3) on FB and TC we know that $i\times 1_{\widetilde{\mathcal{U}}}$ is in TC. Therefore, there exists a morphism $g:(\mathrm{W},\mathsf{p}_{\mathrm{W}})\times_{\mathcal{U}}(\widetilde{\mathcal{U}},p)\to\mathcal{U}\times\mathrm{V}'$ that splits this square into two commutative triangles. Since the lower triangle commutes, g is a morphism over $\mathcal{U}$ and, in particular, $g=\mathsf{adj}(f)$ for some $f:\mathrm{W}\to\underline{Hom}_{\mathcal{U}}((\widetilde{\mathcal{U}},p),(\mathcal{U}\times\mathrm{V}',\mathsf{pr}_1))$. Let us show that f splits the square (2.36) into two commutative triangles, i.e., that we have $i\circ f=f_{\mathrm{Z}}$ and $f\circ\underline{Hom}_{\mathcal{U}}((\widetilde{\mathcal{U}},p),\,1_{\mathcal{U}}\times r)=f_{\mathrm{W}}$.

The first equality is equivalent to $\mathsf{adj}(i\circ f)=\mathsf{adj}(f_{\mathrm{Z}})$ which is equivalent, by [18, Lemma 8.7(3)] to $(i\times 1_{\widetilde{\mathcal{U}}})\circ g=\mathsf{adj}(f_{\mathrm{Z}})$, which is the commutativity of the upper of the two triangles into which g splits (2.37).

The second equality is equivalent to $\mathsf{adj}(f\circ\underline{Hom}_{\mathcal{U}}((\widetilde{\mathcal{U}},p),\,1_{\mathcal{U}}\times r))=\mathsf{adj}(f_{\mathrm{W}})$, which is equivalent by [18, Lemma 8.7(1)] to $g\circ(1_{\mathcal{U}}\times r)=\mathsf{adj}(f_{\mathrm{W}})$, which is the commutativity of the lower of the two triangles into which g splits (2.37). ☐

We can now prove the second main theorem of this section.

Theorem **2.3.9.** — *Let $(\mathcal{C},p,pt)$ be a universe category, let $\mathcal{C}$ be given a locally Cartesian closed structure and let* TC *and* FB *be two collections of morphisms in $\mathcal{C}$ that satisfy Conditions 2.3.4. Let further* $\mathrm{Eq}:(\widetilde{\mathcal{U}};p)\to\mathcal{U}$ *be a $\mathcal{J}0$-structure on p, and let $\Omega:\widetilde{\mathcal{U}}\to\widetilde{\mathcal{U}}$ be a $\mathcal{J}1$-structure over* Eq. *Assume that the following conditions hold:*

1. *$\mathcal{U}$ is fibrant,*
2. *$p:\widetilde{\mathcal{U}}\to\mathcal{U}$ is in* FB,
3. *the map $\omega:\widetilde{\mathcal{U}}\to\mathrm{E}\widetilde{\mathcal{U}}$ constructed in (2.29) is in* TC.

Then there exists a $\mathcal{J}2$-structure $\mathrm{J}p$ relative to Eq *and* Ω.

Proof. — Let us use the notation of Problem 2.2.16. We need to show that under the assumptions of the current theorem there exists a morphism that splits the square of Problem 2.2.16 into two commutative triangles. Observe first that constructing such a splitting is equivalent to constructing the splitting of the square

$$\begin{tikzcd}
(\widetilde{\mathcal{U}},p)\times_{\mathcal{U}}(\mathsf{Fp},\mathsf{pFp}) \arrow[r, "\sigma\circ\mathsf{adj}(\mathsf{pr}_2)"] \arrow[d, "\omega\times 1_{\mathsf{Fp}}"'] & \mathcal{U}\times\widetilde{\mathcal{U}} \arrow[d, "1_{\mathcal{U}}\times p"] \\
(\mathrm{E}\widetilde{\mathcal{U}},p\mathrm{E}\widetilde{\mathcal{U}})\times_{\mathcal{U}}(\mathsf{Fp},\mathsf{pFp}) \arrow[r, "\sigma'\circ\mathsf{adj}(\mathsf{pr}_1)"] & \mathcal{U}\times\mathcal{U},
\end{tikzcd}$$

where

$$\sigma : (\widetilde{\mathcal{U}}, p) \times_{\mathcal{U}} (\mathsf{Fp}, \mathsf{pFp}) \to (\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\widetilde{\mathcal{U}}, p)$$
$$\sigma' : (\mathrm{E}\widetilde{\mathcal{U}}, p\mathrm{E}\widetilde{\mathcal{U}}) \times_{\mathcal{U}} (\mathsf{Fp}, \mathsf{pFp}) \to (\mathsf{Fp}, \mathsf{pFp}) \times_{\mathcal{U}} (\mathrm{E}\widetilde{\mathcal{U}}, p\mathrm{E}\widetilde{\mathcal{U}})$$

are permutations of the factors.

It is easy to show that $\mathcal{U} \times \mathcal{U}$ is fibrant. Therefore it is sufficient to show that $1_{\mathcal{U}} \times p$ is in FB and $\omega \times_{\mathcal{U}} 1_{\mathsf{Fp}}$ is in TC. The first fact follows from the assumption that p is in FB and that $\mathcal{U}$ is fibrant. To obtain the second fact let us apply condition (3) on the classes FB and TC to $\mathrm{B} = \mathcal{U}, f = p\mathrm{E}\widetilde{\mathcal{U}}$, $i = \omega$ and $p = \mathsf{pFp}$. It remains to show that pFp is in FB. We can represent pFp as the composition

$$\mathsf{Fp} \xrightarrow{\mathsf{pr}_1} \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}) \xrightarrow{\mathsf{prI}_{p\mathrm{E}\widetilde{\mathcal{U}}}} \mathcal{U}$$

The morphism $p\mathrm{E}\widetilde{\mathcal{U}}$ is in FB as a composition of pullbacks of p with respect to morphisms with fibrant domains through repeated application of Lemmas 2.3.5 and 2.3.6. Therefore, the morphism $\mathsf{prI}_{p\mathrm{E}\widetilde{\mathcal{U}}}$ is in FB by Lemma 2.3.7 and as a corollary we know that $\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U})$ is fibrant. Similarly $\mathsf{I}_p(\mathcal{U})$ is fibrant and $\mathsf{I}_p(p)$ is in FB and applying again Lemma 2.3.5 we see that pr_1 is in FB. And again by Lemma 2.3.6 we see that pFp is in FB which finishes the proof of the theorem. ☐

Corollary **2.3.10.** — *Let $\mathcal{C}$ be a locally Cartesian closed category with a Quillen model structure, p a universe in $\mathcal{C}$ and* $(\mathrm{E}q, \Omega)$ *a $\mathcal{J}1$-structure on p. Assume further that p is a fibration and $\omega : \widetilde{\mathcal{U}} \to \mathrm{E}\widetilde{\mathcal{U}}$ is a trivial cofibration and that in addition one of the following two conditions holds:*

1. *consider morphisms $f : \mathrm{B}' \to \mathrm{B}$, $p_1 : \mathrm{E}_1 \to \mathrm{B}$, $p_2 : \mathrm{E}_2 \to \mathrm{B}$ and $i : \mathrm{E}_1 \to \mathrm{E}_2$ such that p_1, p_2 are fibrations, $i \circ p_2 = p_1$, and i is a trivial cofibration. Then the morphism*

   $$1_{\mathrm{B}'} \times i : (\mathrm{B}', f) \times_{\mathrm{B}} (\mathrm{E}_1, p_1) \to (\mathrm{B}', f) \times_{\mathrm{B}} (\mathrm{E}_2, p_2)$$

   is a trivial cofibration; or
2. *$\mathcal{U}$ is fibrant and the pullback of a trivial cofibration along a fibration is a trivial cofibration.*

Then $(\mathrm{E}q, \Omega)$ *can be extended to a full $\mathcal{J}$-structure on p.*

The following result can be used to produce many examples of (non-univalent) universes with J-structures.

Let $\mathcal{C}$ be a locally Cartesian closed category with coproducts $\amalg_{n \in \mathbf{N}} \mathrm{X}_n$ of sequences of objects. We let $in_n : \mathrm{X}_n \to \amalg_n \mathrm{X}_n$ the canonical inclusion. Given a sequence of maps $f_n : \mathrm{X}_n \to \mathrm{Y}$, we let $\langle f_n \rangle_n : \amalg_n \mathrm{X}_n \to \mathrm{Y}$ denote the associated morphism. Given two sequences of objects, X_n and Y_n, along with a sequence of maps $f_n : \mathrm{X}_n \to \mathrm{Y}_n$, we let $\amalg f_n : \amalg_n \mathrm{X}_n \to \amalg_n \mathrm{Y}_n$ denote the morphism $\langle f_n \circ in_n \rangle_n$.

Assume that these coproducts satisfy the following two conditions:

1. for a sequence of morphisms $f_n : \mathrm{E}_n \to \mathrm{B}_n$ the square

$$\begin{tikzcd}
\amalg_n(\mathrm{E}_n, f_n) \times_{\mathrm{B}_n} (\mathrm{E}_n, f_n) \arrow[r, "\amalg_n \mathsf{pr}_2"] \arrow[d, "\amalg_n \mathsf{pr}_1"'] & \amalg_n \mathrm{E}_n \arrow[d, "\amalg_n f_n"] \\
\amalg_n \mathrm{E}_n \arrow[r, "\amalg_n f_n"] & \amalg_n \mathrm{B}_n
\end{tikzcd}$$

   is a pullback square,
2. for a sequence of morphisms $f_n : \mathrm{E}_n \to \mathrm{B}_n$ the square

$$\begin{tikzcd}
\amalg_n \mathrm{E}_{n+1} \arrow[r, "\langle in_{n+1} \rangle_n"] \arrow[d, "\amalg_n f_{n+1}"'] & \amalg_n \mathrm{E}_n \arrow[d, "\amalg_n f_n"] \\
\amalg_n \mathrm{B}_{n+1} \arrow[r, "\langle in_{n+1} \rangle_n"] & \amalg_n \mathrm{B}_n
\end{tikzcd}$$

   is a pullback square.

Problem **2.3.11.** — Let $\mathcal{C}$ be as above FB and TC two classes of morphisms satisfying one of the sets of Conditions 2.3.4 or 2.3.2. Assume in addition the following:

1. the coproduct of a sequence of morphisms from TC is in TC and the coproduct of a sequence of morphisms from FB is in FB,
2. the composition of a morphism from TC with an isomorphism is in TC,
3. for any morphism $f : \mathrm{X} \to \mathrm{Y}$ there is given an object $\mathrm{P}(f)$ and morphisms $i_f : \mathrm{X} \to \mathrm{P}(f)$, $q_f : \mathrm{P}(f) \to \mathrm{Y}$ such that $i_f \in \mathrm{TC}$, $q_f \in \mathrm{FB}$ and $f = i_f \circ q_f$.

To construct, for any universe $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ such that $p \in \mathrm{FB}$ a sequence of morphisms $\mathsf{p}_n : \widetilde{\mathcal{U}}_n \to \mathcal{U}_n$ such that $\mathsf{p}_0 = p$, $\mathsf{p}_n \in \mathrm{FB}$ and $\amalg_n p$, with the universe structure defined by the fiber squares of $\mathcal{C}$, has a J-structure with $\omega \in \mathrm{TC}$.

Construction **2.3.12** *(For Problem 2.3.11)*. — Define $\mathsf{p}_n : \widetilde{\mathcal{U}}_n \to \mathcal{U}_n$ inductively as follows. For $n = 0$ we take $\mathsf{p}_0 = p$. To define p_{n+1} consider the diagonal $\Delta_n : \widetilde{\mathcal{U}}_n \to (\widetilde{\mathcal{U}}_n, \mathsf{p}_n) \times_{\mathcal{U}_n} (\widetilde{\mathcal{U}}_n, \mathsf{p}_n)$ and let

$$\mathsf{p}_{n+1} = q_{\Delta_n} : \mathrm{P}(\Delta_n) \to (\widetilde{\mathcal{U}}_n, \mathsf{p}_n) \times_{\mathcal{U}_n} (\widetilde{\mathcal{U}}_n, \mathsf{p}_n)$$

so that, in particular, $\mathcal{U}_{n+1} = (\widetilde{\mathcal{U}}_n, \mathsf{p}_n) \times_{\mathcal{U}_n} (\widetilde{\mathcal{U}}_n, \mathsf{p}_n)$.

Let $\mathcal{U}_* = \amalg_n \mathcal{U}_n$, $\widetilde{\mathcal{U}}_* = \amalg_n \widetilde{\mathcal{U}}_n$ and $\mathsf{p}_* = \amalg_n \mathsf{p}_n$. According to the first of the two properties that we required from the coproducts the canonical morphism

$$\iota : \amalg_n (\widetilde{\mathcal{U}}_n, \mathsf{p}_n) \times_{\mathcal{U}_n} (\widetilde{\mathcal{U}}_n, \mathsf{p}_n) \to (\widetilde{\mathcal{U}}_*, \mathsf{p}_*) \times_{\mathcal{U}_*} (\widetilde{\mathcal{U}}_*, \mathsf{p}_*)$$

is an isomorphism. Together with the second property applied to the right-most square this gives us a diagram with pullback squares of the form:

$$\begin{tikzcd}
\amalg_n \widetilde{\mathcal{U}}_{n+1} \arrow[r, "="] \arrow[d, "r \circ \iota"'] & \amalg_n \widetilde{\mathcal{U}}_{n+1} \arrow[r, "="] \arrow[d, "r"] & \amalg_n \widetilde{\mathcal{U}}_{n+1} \arrow[r, "\langle in_{n+1} \rangle_n"] \arrow[d, "r"] & \widetilde{\mathcal{U}}_* \arrow[d, "\mathsf{p}_*"] \\
(\widetilde{\mathcal{U}}_*, \mathsf{p}_*) \times_{\mathcal{U}_*} (\widetilde{\mathcal{U}}_*, \mathsf{p}_*) \arrow[r, "\iota^{-1}"] & \amalg_n (\widetilde{\mathcal{U}}_n, \mathsf{p}_n) \times_{\mathcal{U}_n} (\widetilde{\mathcal{U}}_n, \mathsf{p}_n) \arrow[r, "="] & \amalg_n \mathcal{U}_{n+1} \arrow[r, "\langle in_{n+1} \rangle_n"] & \mathcal{U}_*,
\end{tikzcd}$$

where $r = \amalg_n \mathsf{p}_{n+1}$. Define $\mathrm{E}q$ as the composition of the lower horizontal arrows of this diagram (up to an isomorphism this is just $\langle in_{n+1} \rangle_n$). Since the squares of the diagram are pullback squares, the natural morphism

$$\iota' : \amalg_n \widetilde{\mathcal{U}}_{n+1} \to ((\widetilde{\mathcal{U}}_*, \mathsf{p}_*) \times_{\mathcal{U}_*} (\widetilde{\mathcal{U}}_*, \mathsf{p}_*), \mathrm{E}q)_{\mathcal{U}_*} (\widetilde{\mathcal{U}}_*, \mathsf{p}_*)$$

is an isomorphism. Define

$$\Omega = (\amalg_n i_{\Delta_n}) \circ \iota' \circ \langle in_{n+1} \rangle_n$$

such that then

$$\omega = (\amalg_n i_{\Delta_n}) \circ \iota'.$$

By our assumptions $\omega \in \mathrm{TC}$ and then by Theorem 2.3.3 if FB and TC satisfied Conditions 2.3.2 or by Theorem 2.3.9 if FB and TC satisfied Conditions 2.3.4 we conclude that $(\mathrm{E}q, \Omega)$ can be extended to a full J-structure on p_*. $\square$

2.4. *Constructing a J-structure on* CC$(\mathcal{C}, p)$ *from a J-structure on* p

The construction of a C-system CC$(\mathcal{C}, p)$ from a category with a universe p and a final object pt was presented in [17] and summarized in [18]. Let us recall some facts and notation. The underlying category of CC$(\mathcal{C}, p)$ is equipped with a functor *int* to $\mathcal{C}$. Note that while *int* is the identity function on Hom-sets by the construction of CC$(\mathcal{C}, p)$, the notations for an element of $\mathrm{H}om(\Gamma', \Gamma) = \mathrm{H}om(int(\Gamma'), int(\Gamma))$ may differ. In particular, for $f : \Gamma' \to \Gamma$ in CC$(\mathcal{C}, p)$ and $\mathrm{F} : int(\Gamma) \to \mathcal{U}$ in $\mathcal{C}$, we have the following equation.[8]

$$\mathsf{q}(f, (\Gamma, \mathrm{F})) = \mathsf{Q}(f, \mathrm{F}). \tag{2.38}$$

---

[8] Note from the academic executor: The notation (Γ, F) refers to the inductive construction of objects of CC$(\mathcal{C}, p)$ presented in [17]. Here Γ is an object of CC$(\mathcal{C}, p)$, so (Γ, F) is a new object of CC$(\mathcal{C}, p)$, with $\mathsf{ft}(\Gamma, \mathrm{F}) = \Gamma$ and $int(\Gamma, \mathrm{F}) = (\Gamma; \mathrm{F})$. In the equation displayed, the left side, $\mathsf{q}(f, (\Gamma, \mathrm{F}))$, refers to the C-system structure on CC$(\mathcal{C}, p)$, while the right side, $\mathsf{Q}(f, \mathrm{F})$, refers to the universe structure on $\mathcal{C}$. The equation follows from the definition [17, 2.6 (4)].

For each $\Gamma \in Ob(CC(\mathcal{C}, p))$ we have natural bijections

$$u_1 = u_{1,\Gamma} : Ob_1(\Gamma) \overset{\cong}{\to} Hom(int(\Gamma), \mathcal{U}) \tag{2.39}$$

$$\widetilde{u}_1 = \widetilde{u}_{1,\Gamma} : \widetilde{Ob}_1(\Gamma) \overset{\cong}{\to} Hom(int(\Gamma), \widetilde{\mathcal{U}}), \tag{2.40}$$

where $u_1^{-1}(\mathrm{F}) = (\Gamma, \mathrm{F})$ and where

$$\widetilde{u}_1(s) = s \circ \mathsf{Q}(u_1(\partial(s))). \tag{2.41}$$

In particular,

$$\widetilde{u}_1(s) \circ p = s \circ \mathsf{Q}(u_1(\partial(s))) \circ p = s \circ \mathsf{p}_{\partial(s)} \circ u_1(\partial(s)) = u_1(\partial(s)),$$

i.e., with respect to these bijections the function $\partial : \widetilde{Ob}_1(\Gamma) \to Ob_1(\Gamma)$ is given by composition with $p : \widetilde{\mathcal{U}} \to \mathcal{U}$.

Problem **2.4.1.** — Let $\mathrm{E}q : (\widetilde{\mathcal{U}}; p) \to \mathcal{U}$ be a J0-structure on a universe p in a category $\mathcal{C}$. To construct a J0-structure on $CC(\mathcal{C}, p)$.

Construction **2.4.2** *(For Problem 2.4.1).* — Since the canonical squares given by the universe structure on p are pullback squares, the bijections u_1 and $\widetilde{u}_1$ give us a bijection

$$\widetilde{uu} : \{o, o' \in \widetilde{Ob}_1(\Gamma) \mid \partial(o) = \partial(o')\} \overset{\cong}{\to} Hom(int(\Gamma), (\widetilde{\mathcal{U}}; p)),$$

where $\widetilde{uu}(o, o') = \widetilde{u}_1(o) * \widetilde{u}_1(o')$. We set:

$$\mathsf{Id}(o, o') := u_1^{-1}(\widetilde{uu}(o, o') \circ \mathrm{E}q). \qquad \square$$

We let $\mathsf{Id}_{\mathrm{E}q}$ denote the J0-structure on $CC(\mathcal{C}, p)$ constructed from $\mathrm{E}q$ in Construction 2.4.2. Note that

$$int(\mathsf{Id}(o, o')) = (int(\Gamma); (\widetilde{u}_1(o) * \widetilde{u}_1(o')) \circ \mathrm{E}q) \tag{2.42}$$

and

$$u_1(\mathsf{Id}(o, o')) = (\widetilde{u}_1(o) * \widetilde{u}_1(o')) \circ \mathrm{E}q \tag{2.43}$$

Recall that in [20] we let $\mathsf{p}_{\Gamma,n} : \Gamma \to \mathsf{ft}^n(\Gamma)$ denote the composition of n canonical projections $\mathsf{p}_\Gamma \circ \cdots \circ \mathsf{p}_{\mathsf{ft}^{n-1}(\Gamma)}$.

Lemma **2.4.3.** — *Let* $\mathrm{E}q$ *be a* $J0$*-structure on* p*. Let* $\Gamma \in Ob$ *and* $\mathrm{F} : int(\Gamma) \to \mathcal{U}$*. Then one has:*

$$\mathsf{Id}_3(\Gamma, \mathrm{F}) = (((\mathrm{X}, \mathrm{F}), \mathsf{Q}(\mathrm{F}) \circ p), \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q) \tag{2.44}$$

$$int(\mathsf{Id}_3(\Gamma, \mathrm{F})) = (int(\Gamma); \mathrm{F})_{\mathrm{E}} \tag{2.45}$$

$$int(\mathsf{p}_{\mathsf{Id}_3((\Gamma,\mathrm{F})),3}) = \mathsf{p}^{\mathrm{E}}_{\Gamma,\mathrm{F}} \tag{2.46}$$

$$\mathsf{Q}(\mathrm{F})_{\mathrm{E}} \circ \mathsf{Q}(\mathrm{E}q) = \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(\mathrm{F}) \circ p) \circ \mathrm{E}q) \tag{2.47}$$

Proof. — It will be helpful for the reader to refer to diagram (2.22). Let $\mathrm{T} = (\Gamma, \mathrm{F})$, and set

$$o := \mathsf{p}^*_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})}(\delta(\mathrm{T})) \in \widetilde{Ob}_1(\Gamma) \qquad \text{and}$$
$$o' := \delta(\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})) \in \widetilde{Ob}_1(\Gamma).$$

We then have

$$\mathsf{Id}_3(\mathrm{T}) = \mathsf{Id}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})}(o, o') = (\mathsf{p}^*_{\mathrm{T}}(\mathrm{T}), (\widetilde{u}_1(o) * \widetilde{u}_1(o')) \circ \mathrm{E}q).$$

Furthermore, we have

$$\mathsf{p}^*_{\mathrm{T}}(\mathrm{T}) = ((\Gamma, \mathrm{F}), \mathsf{Q}(\mathrm{F}) \circ p)$$

and

$$\widetilde{u}_1(\mathsf{p}^*_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})}(\delta(\mathrm{T}))) = \mathsf{p}_{(int(\Gamma,\mathrm{F},\mathsf{Q}(\mathrm{F})\circ p)} \circ \mathsf{Q}(\mathrm{F})$$
$$\widetilde{u}_1(\delta(\mathsf{p}^*_{\mathrm{T}}(\mathrm{T}))) = \mathsf{Q}(\mathsf{Q}(\mathrm{F}) \circ p)$$

which shows that $\widetilde{u}_1(o) * \widetilde{u}_1(o') = \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p)$ and completes the proof of the first three equations.

The last equality is a corollary of the equality $\mathsf{Q}(\mathrm{F})_{\mathrm{E}} = \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(\mathrm{F}), p), \mathrm{E}q)$ and the equality $\mathsf{Q}(f, \mathrm{F}) \circ \mathsf{Q}(\mathrm{F}) = \mathsf{Q}(f \circ \mathrm{F})$. □

Problem **2.4.4.** — Let $\mathrm{E}q : (\widetilde{\mathcal{U}}; p) \to \mathcal{U}$ be a J0-structure on a universe p in a category $\mathcal{C}$, and let $\Omega : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}$ be a J1-structure over $\mathrm{E}q$. To construct a J1-structure refl over $\mathsf{Id}_{\mathrm{E}q}$ on $\mathrm{CC}(\mathcal{C}, p)$.

Construction **2.4.5** *(For Problem 2.4.4).* — Due to the natural bijections (2.40) the morphism Ω defines maps

$$\mathsf{refl} : \widetilde{Ob}_1(\Gamma) \to \widetilde{Ob}_1(\Gamma)$$

by the formula

$$\mathsf{refl}(s) = \widetilde{u}_1^{-1}(\widetilde{u}_1(s) \circ \Omega), \tag{2.48}$$

which are natural in Γ. Equation (2.7) follows from the commutativity of the square (2.28).

We let refl_Ω denote the J1-structure constructed from Ω in Construction 2.4.5. The following technical lemma is needed only in the proof of Lemma 2.4.7.

Lemma **2.4.6.** — *For* $s \in \widetilde{Ob}_1(\Gamma)$ *one has:*

$$\mathsf{refl}_\Omega(s) \circ \mathsf{Q}(s \circ \mathsf{Q}(F) \circ \Omega \circ p) = s \circ \mathsf{Q}(F) \circ \Omega,$$

where $F = u_1(\partial(s))$.

Proof. — We have

$$u_1(\partial(\mathsf{refl}_\Omega(s))) = \widetilde{u}_1(\mathsf{refl}(s)) \circ p = \widetilde{u}_1(s) \circ \Omega \circ p = s \circ \mathsf{Q}(F) \circ \Omega \circ p$$

therefore

$$\begin{aligned}\widetilde{u}_1(\mathsf{refl}_\Omega(s)) &= \mathsf{refl}_\Omega(s) \circ \mathsf{Q}(u_1(\partial(\mathsf{refl}_\Omega(s)))) \\ &= \mathsf{refl}_\Omega(s) \circ \mathsf{Q}(s \circ \mathsf{Q}(F) \circ \Omega \circ p).\end{aligned}$$

On the other hand, by definition of refl_Ω,

$$\widetilde{u}_1(\mathsf{refl}_\Omega(s)) = \widetilde{u}_1(s) \circ \Omega = s \circ \mathsf{Q}(F) \circ \Omega.$$ □

Lemma **2.4.7.** — *Given* $\mathrm{E}q$ *and* Ω *consider the corresponding* Id *and* refl. *For* $\Gamma \in Ob(CC(\mathcal{C}, p))$ *and for* $T \in Ob_1(\Gamma)$ *let*

$$\mathsf{rf}_T : int(T) \to int(\mathsf{Id}_3(T))$$

be the morphism constructed in Construction 2.1.6. On the other hand let

$$F^*(\omega) : (int(\Gamma); F) \to (int(\Gamma); F)_E$$

be the pullback of $\omega : \widetilde{\mathcal{U}} \to E\widetilde{\mathcal{U}}$ *(defined in Definition 2.2.10) with respect to* $F := u_1(T)$, *i.e., the unique morphism*

$$(int(\Gamma); F) \to (int(\Gamma); F)_E$$

such that

$$F^*(\omega) \circ p^{E}_{int(\Gamma),F} = \mathsf{p}_{int(T)} \tag{2.49}$$

$$F^*(\omega) \circ \mathsf{Q}(F)_E = \mathsf{Q}(F) \circ \omega \tag{2.50}$$

Then

$$\mathsf{rf}_T = F^*(\omega).$$

Proof. — In view of Lemma 2.4.3, both rf_{T} and $\mathrm{F}^*(\omega)$ are morphisms from $(int(\Gamma); \mathrm{F})$ to $(int(\Gamma); \mathrm{F})_{\mathrm{E}}$. Let us denote $int(\Gamma)$ by X and $(int(\Gamma); \mathrm{F}, \mathsf{Q}(\mathrm{F}) \circ p)$ by Y. We have

$$(\mathrm{X}; \mathrm{F})_{\mathrm{E}} = (\mathrm{Y}; \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q)$$

and we can see this object as a part of the diagram with two pullback squares:

(**2.51**)
$$\begin{tikzcd}
(\mathrm{X}; \mathrm{F})_{\mathrm{E}} \arrow[r, "h_1"] \arrow[d, "\mathsf{p}_{\mathrm{Y}, \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q}"'] & \mathrm{E}\widetilde{\mathcal{U}} \arrow[r, "h_2"] \arrow[d, "\mathsf{p}_{(\widetilde{\mathcal{U}}; p), \mathrm{E}q}"] & \widetilde{\mathcal{U}} \arrow[d, "p"] \\
\mathrm{Y} \arrow[r, "\mathsf{Q}(\mathsf{Q}(\mathrm{F}), p)"] & (\widetilde{\mathcal{U}}, p) \arrow[r, "\mathrm{E}q"] & \mathcal{U}
\end{tikzcd},$$

where $h_1 := \mathsf{Q}(\mathrm{F})_{\mathrm{E}}$ and $h_2 := \mathsf{Q}(\mathrm{E}q)$.

Let $\Delta := (1_{\widetilde{\mathcal{U}}}) * (1_{\widetilde{\mathcal{U}}})$ be the diagonal of $\widetilde{\mathcal{U}}$ over $\mathcal{U}$.

The following commutative diagram of canonical pullback squares clarifies some of the forthcoming computations.

(**2.52**)
$$\begin{tikzcd}
 & & \widetilde{\mathcal{U}} \arrow[d, "\Delta"] & & \\
 & (\mathrm{X}; \mathrm{F}) \arrow[ur, "\mathsf{Q}(\mathrm{F})"] \arrow[d, "\delta(\mathrm{T})"'] & (\widetilde{\mathcal{U}}; p) \arrow[rr, "\mathsf{Q}(p)"] \arrow[dd, "\mathsf{p}_{\widetilde{\mathcal{U}}, p}"] & & \widetilde{\mathcal{U}} \arrow[dd, "p"] \\
int(\mathsf{p}_{\mathrm{T}}^*(\mathrm{T})) \arrow[r, "="] & (\mathrm{X}; \mathrm{F}, \mathsf{Q}(\mathrm{F}) \circ p) \arrow[ur, "\mathsf{Q}(\mathsf{Q}(\mathrm{F}), p)"] \arrow[rr] \arrow[dd, "\mathsf{p}_{\mathsf{p}_{\mathrm{T}}^*(\mathrm{T})}"] & & (\mathrm{X}; \mathrm{F}) \arrow[ur, "\mathsf{Q}(\mathrm{F})"] \arrow[dd, "\mathsf{p}_{(\mathrm{X}; \mathrm{F})}"] & \\
 & & \widetilde{\mathcal{U}} \arrow[rr, "p"] & & \mathcal{U} \\
int(\mathrm{T}) \arrow[r, "="] & (\mathrm{X}; \mathrm{F}) \arrow[ur, "\mathsf{Q}(\mathrm{F})"] \arrow[rr, "\mathsf{p}_{(\mathrm{X}; \mathrm{F})}"] & & \mathrm{X} \arrow[ur, "\mathrm{F}"'] &
\end{tikzcd}$$

We have the following two projections.

(**2.53**)
$$h := h_1 \circ h_2 = \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q) : (\mathrm{X}; \mathrm{F})_{\mathrm{E}} \to \widetilde{\mathcal{U}}$$

(**2.54**)
$$v := \mathsf{p}_{\mathrm{Y}, \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q} : (\mathrm{X}; \mathrm{F})_{\mathrm{E}} \to \mathrm{Y}$$

It suffices to check the following equations.

$$\mathsf{rf}_{\mathrm{T}} \circ h = \mathrm{F}^{*}(\omega) \circ h$$
$$\mathsf{rf}_{\mathrm{T}} \circ v = \mathrm{F}^{*}(\omega) \circ v$$

We have

$$\begin{aligned}
&\mathsf{rf}_{\mathrm{T}} \circ h \\
&= \mathsf{refl}_{\Omega}(\delta(\mathrm{T})) \circ \mathsf{q}(\delta(\mathrm{T}), \mathsf{Id}_3(\mathrm{T})) \circ h && \text{(by def'n (2.11))} \\
&= \mathsf{refl}_{\Omega}(\delta(\mathrm{T})) \circ \mathsf{Q}(\delta(\mathrm{T}), \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q) \circ h && \text{(by (2.38) and (2.44))} \\
&= \mathsf{refl}_{\Omega}(\delta(\mathrm{T})) \circ \mathsf{Q}(\delta(\mathrm{T}), \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q) \\
&\quad \circ \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q) && \text{(by (2.53))} \\
&= \mathsf{refl}_{\Omega}(\delta(\mathrm{T})) \circ \mathsf{Q}(\delta(\mathrm{T}) \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q) && \text{(by (2.16))} \\
&= \mathsf{refl}_{\Omega}(\delta(\mathrm{T})) \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}) \circ \Delta \circ \mathrm{E}q) && \text{(by (2.52))} \\
&= \mathsf{refl}_{\Omega}(\delta(\mathrm{T})) \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}) \circ \Omega \circ p), && \text{(by (2.28))}
\end{aligned}$$

and thus we deduce that

$$\mathsf{rf}_{\mathrm{T}} \circ h = \mathsf{refl}_{\Omega}(\delta(\mathrm{T})) \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}) \circ \Omega \circ p). \tag{2.55}$$

We have

$$\begin{aligned}
&\delta(\mathrm{T}) \circ \mathsf{Q}(u_1(\partial(\delta(\mathrm{T})))) \\
&= \delta(\mathrm{T}) \circ \mathsf{Q}(u_1((\mathrm{X}; \mathrm{F}, \mathsf{Q}(\mathrm{F}) \circ p))) && \text{(by def'n of } \partial) \\
&= \delta(\mathrm{T}) \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}) \circ p) && \text{(by def'n of } u_1) \\
&= \delta(\mathrm{T}) \circ \mathsf{Q}(\mathsf{p}_{\mathrm{X},\mathrm{F}} \circ \mathrm{F}) && \text{(by (2.12))} \\
&= \delta(\mathrm{T}) \circ \mathsf{Q}(\mathsf{p}_{\mathrm{X},\mathrm{F}}, \mathrm{F}) \circ \mathsf{Q}(\mathrm{F}) && \text{(by (2.16))} \\
&= 1_{int(\mathrm{T})} \circ \mathsf{Q}(\mathrm{F}) && \text{(by def'n of } \delta(\mathrm{T})) \\
&= \mathsf{Q}(\mathrm{F}).
\end{aligned}$$

In particular, we have the following equation, because $\mathrm{T} = (\mathrm{X}, \mathrm{F})$.

$$\delta(\mathrm{T}) \circ \mathsf{Q}(\mathsf{p}_{\mathrm{T}} \circ \mathrm{F}) = \mathsf{Q}(\mathrm{F})$$

Using that and applying Lemma 2.4.6 with s as $\delta(\mathrm{T})$ and F as $p_{\mathrm{T}} \circ \mathrm{F}$, we derive the following equation.

$$\mathsf{refl}_{\Omega}(\delta(\mathrm{T})) \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}) \circ \Omega \circ p) = \mathsf{Q}(\mathrm{F}) \circ \Omega$$

Combining that with (2.55), we deduce that

$$\mathsf{rf}_{\mathrm{T}} \circ h = \mathsf{Q}(\mathrm{F}) \circ \Omega.$$

On the other hand, we have the following sequence of equations.

$$\begin{aligned}
\mathrm{F}^*(\omega) \circ h &= \mathrm{F}^*(\omega) \circ h_1 \circ h_2 && \text{(by def'n of } h\text{)}\\
&= \mathrm{F}^*(\omega) \circ \mathsf{Q}(\mathrm{F})_{\mathrm{E}} \circ h_2 && \text{(by def'n of } h_1\text{)}\\
&= \mathsf{Q}(\mathrm{F}) \circ \omega \circ h_2 && \text{(by (2.50))}\\
&= \mathsf{Q}(\mathrm{F}) \circ \omega \circ \mathsf{Q}(\mathrm{E}q) && \text{(by def'n of } h_2\text{)}\\
&= \mathsf{Q}(\mathrm{F}) \circ \Omega && \text{(by (2.29))}
\end{aligned}$$

This proves that $\mathsf{rf}_{\mathrm{T}} \circ h = \mathrm{F}^*(\omega) \circ h$.

Both $\mathsf{rf}_{\mathrm{T}} \circ v$ and $\mathrm{F}^*(\omega) \circ v$ are morphisms $int(\mathrm{T}) \to int(\mathsf{p}^*_{\mathrm{T}}(\mathrm{T}))$. Since $int(\mathsf{p}^*_{\mathrm{T}}(\mathrm{T}))$ is a part of a pullback square with the projections being $\mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})}$ and $\mathsf{Q}(\mathsf{Q}(\mathrm{F}), p)$, to prove $\mathsf{rf}_{\mathrm{T}} \circ v = \mathrm{F}^*(\omega) \circ v$ it suffices to verify the following equations.

$$\mathsf{rf}_{\mathrm{T}} \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} = \mathrm{F}^*(\omega) \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \tag{2.56}$$

$$\mathsf{rf}_{\mathrm{T}} \circ v \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) = \mathrm{F}^*(\omega) \circ v \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \tag{2.57}$$

Similarly, because the common target in the first equation above is (X; F), which is a pullback whose projections are p_{T} and $\mathsf{Q}(\mathrm{F})$, to verify (2.56) it suffices to verify the following equations.

$$\mathsf{rf}_{\mathrm{T}} \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{p}_{\mathrm{T}} = \mathrm{F}^*(\omega) \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{p}_{\mathrm{T}} \tag{2.58}$$

$$\mathsf{rf}_{\mathrm{T}} \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{Q}(\mathrm{F}) = \mathrm{F}^*(\omega) \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{Q}(\mathrm{F}) \tag{2.59}$$

We verify (2.58) as follows.

$$\begin{aligned}
\mathsf{rf}_{\mathrm{T}} \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{p}_{\mathrm{T}} &= \delta(\mathrm{T}) \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{p}_{\mathrm{T}} && \text{(by (2.10))}\\
&= 1_{int(\mathrm{T})} \circ \mathsf{p}_{\mathrm{T}} && \text{(by def'n of } \delta(\mathrm{T})\text{)}\\
&= p_{\mathrm{T}}\\
&= \mathrm{F}^*(\omega) \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{p}_{\mathrm{T}} && \text{(by def'n of } \mathrm{F}^*(\omega)\text{)}
\end{aligned}$$

We verify (2.59) as follows.

$$\begin{aligned}
&\mathsf{rf}_{\mathrm{T}} \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{Q}(\mathrm{F})\\
&= \delta(\mathrm{T}) \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{Q}(\mathrm{F}) && \text{(by (2.10))}\\
&= 1_{int(\mathrm{T})} \circ \mathsf{Q}(\mathrm{F}) && \text{(by def'n of } \delta(\mathrm{T})\text{)}
\end{aligned}$$

$$\begin{aligned}
&= \mathsf{Q}(\mathrm{F}) && \text{(by def'n of } \delta(\mathrm{T})\text{)}\\
&= \mathsf{Q}(\mathrm{F}) \circ \Delta \circ \mathsf{p}_{\widetilde{\mathcal{U}},p} && \text{(by def'n of } \Delta\text{)}\\
&= \mathsf{Q}(\mathrm{F}) \circ \omega \circ \mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q} \circ \mathsf{p}_{\widetilde{\mathcal{U}},p} && \text{(by (2.29))}\\
&= \mathrm{F}^*(\omega) \circ \mathsf{Q}(\mathrm{F})_{\mathrm{E}} \circ \mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q} \circ \mathsf{p}_{\widetilde{\mathcal{U}},p} && \text{(by (2.50))}\\
&= \mathrm{F}^*(\omega) \circ h_1 \circ \mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q} \circ \mathsf{p}_{\widetilde{\mathcal{U}},p} && \text{(by def'n of } h_1\text{)}\\
&= \mathrm{F}^*(\omega) \circ \mathsf{p}_{\mathrm{Y},\mathsf{Q}(\mathsf{Q}(\mathrm{F}),p)\circ\mathrm{E}q} \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}),p) \circ \mathsf{p}_{\widetilde{\mathcal{U}},p} && \text{(by (2.51))}\\
&= \mathrm{F}^*(\omega) \circ v \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}),p) \circ \mathsf{p}_{\widetilde{\mathcal{U}},p} && \text{(by (2.54))}\\
&= \mathrm{F}^*(\omega) \circ v \circ \mathsf{p}_{\mathsf{p}^*_{\mathrm{T}}(\mathrm{T})} \circ \mathsf{Q}(\mathrm{F}) && \text{(by (2.52))}
\end{aligned}$$

We verify (2.57) as follows.

$$\begin{aligned}
\mathsf{rf}_{\mathrm{T}} \circ v \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}),p) &= \delta(\mathrm{T}) \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}),p) && \text{(by (2.10))}\\
&= \mathsf{Q}(\mathrm{F}) \circ \Delta && \text{(by (2.52))}\\
&= \mathsf{Q}(\mathrm{F}) \circ \omega \circ \mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q} && \text{(by (2.29))}\\
&= \mathrm{F}^*(\omega) \circ \mathsf{Q}(\mathrm{F})_{\mathrm{E}} \circ \mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q} && \text{(by (2.50))}\\
&= \mathrm{F}^*(\omega) \circ \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(\mathrm{F}),p),\mathrm{E}q) \circ \mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q} && \text{(by (2.25))}\\
&= \mathrm{F}^*(\omega) \circ \mathsf{p}_{\mathrm{Y},\mathsf{Q}(\mathsf{Q}(\mathrm{F}),p)\circ\mathrm{E}q} \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}),p) && \text{(by (2.22))}\\
&= \mathrm{F}^*(\omega) \circ v \circ \mathsf{Q}(\mathsf{Q}(\mathrm{F}),p) && \text{(by (2.54))}
\end{aligned}$$

This completes the proof of Lemma 2.4.7. □

Problem **2.4.8.** — Let $(\mathrm{E}q, \Omega, \mathrm{J}p)$ be a J-structure on a universe p. To construct for all $\Gamma \in \mathrm{O}b = \mathrm{O}b(\mathrm{CC}(\mathcal{C},p))$, for all $\mathrm{T} \in \mathrm{O}b_1(\Gamma)$, for all $\mathrm{P} \in \mathrm{O}b_1(\mathsf{Id}_3(\mathrm{T}))$, for all $s0 \in \widetilde{\mathrm{O}b}(\mathsf{rf}^*_{\mathrm{T}}(\mathrm{P}))$, an element $\mathsf{J}(\Gamma,\mathrm{T},\mathrm{P},s0)$ of $\widetilde{\mathrm{O}b}(\mathrm{P})$. (This is the type of element required by Definition 2.1.7 of a J2-structure on the C-system $\mathrm{CC}(\mathcal{C},p)$.)

Construction **2.4.9** *(For Problem 2.4.8).* — Let $\mathrm{X} := int(\Gamma)$, $\mathrm{F} := u_1(\mathrm{T}) : \mathrm{X} \to \mathcal{U}$, so $\mathrm{T} = (\Gamma, \mathrm{F})$. By Lemma 2.4.3 we have $int(\mathsf{Id}_3(\mathrm{T})) = int(\mathsf{Id}_3(\Gamma,\mathrm{F})) = (\mathrm{X};\mathrm{F})_{\mathrm{E}}$. Therefore we further have $\mathrm{G} := u_1(\mathrm{P}) : (\mathrm{X};\mathrm{F})_{\mathrm{E}} \to \mathcal{U}$ and $\widetilde{\mathrm{H}} := \widetilde{u}_1(s0) : (\mathrm{X};\mathrm{F}) \to \widetilde{\mathcal{U}}$.

Let us show first that

$$\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{F},\mathrm{G}) \circ \mathsf{l}^{\omega}(\mathcal{U}) = \eta^{!-1}_{p}(\mathrm{F},\widetilde{\mathrm{H}}) \circ \mathsf{l}_p(p)$$

We show this as follows.

$$\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{F},\mathrm{G}) \circ \mathsf{l}^{\omega}(\mathcal{U})$$

$$\begin{aligned}
&= \eta^{!-1}_{p}(F, G \circ F^{*}(\omega)) && \text{(by Lemma 2.2.5)}\\
&= \eta^{!-1}_{p}(F, \mathsf{rf}_{T} \circ G) && \text{(by Lemma 2.4.7)}\\
&= \eta^{!-1}_{p}(F, s0 \circ \mathsf{Q}(\mathsf{rf}_{T} \circ G) \circ p) && \text{(by commutativity of}\\
& && \text{the canonical square)}\\
&= \eta^{!-1}_{p}(F, \widetilde{H} \circ p) && \text{(by (2.41))}\\
&= \eta^{!-1}_{p}(F, \widetilde{H}) \circ \mathsf{I}_{p}(p) && \text{(by naturality of } \eta^{!-1}_{p,X,V}\text{).}
\end{aligned}$$

Therefore the pair $(\eta^{!-1}_{pE\widetilde{\mathcal{U}}}(F, G), \eta^{!-1}_{p}(F, \widetilde{H}))$ gives us a morphism

$$\phi(\Gamma, T, P, s0) : X \to (\mathsf{I}_{pE\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^{\omega}(\mathcal{U})) \times_{\mathsf{I}_{p}(\mathcal{U})} (\mathsf{I}_{p}(\widetilde{\mathcal{U}}), \mathsf{I}_{p}(p))$$

and composing it with Jp (cf. Definition 2.2.11) we obtain a morphism

$$\phi(\Gamma, T, P, s0) \circ Jp : X \to \mathsf{I}_{pE\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}})$$

Consider the element

$$(F_{1}, F_{2}) := \eta^{!}_{pE\widetilde{\mathcal{U}}}(\phi(\Gamma, T, P, s0) \circ Jp) \in \mathsf{D}_{pE\widetilde{\mathcal{U}}}(X, \widetilde{\mathcal{U}})$$

We have the following computation.

$$\begin{aligned}
&(F_{1}, F_{2} \circ p) && \\
&= \mathsf{D}_{pE\widetilde{\mathcal{U}}}(X, p)(F_{1}, F_{2}) && \text{(by def'n of } \mathsf{D}_{pE\widetilde{\mathcal{U}}}\text{)}\\
&= \mathsf{D}_{pE\widetilde{\mathcal{U}}}(X, p)(\eta^{!}_{pE\widetilde{\mathcal{U}}}(\phi(\Gamma, T, P, s0) \circ Jp)) && \text{(by def'n of } (F_{1}, F_{2})\text{)}\\
&= \eta^{!}_{pE\widetilde{\mathcal{U}}}(\phi(\Gamma, T, P, s0) \circ Jp \circ \mathsf{I}_{pE\widetilde{\mathcal{U}}}(p)) && \text{(by naturality of } \eta^{!}_{pE\widetilde{\mathcal{U}}}\text{)}\\
&= \eta^{!}_{pE\widetilde{\mathcal{U}}}(\phi(\Gamma, T, P, s0) \circ Jp \circ co\mathrm{J} \circ \mathsf{pr}_{1}) && \text{(by def'n of } co\mathrm{J}\text{)}\\
&= \eta^{!}_{pE\widetilde{\mathcal{U}}}(\phi(\Gamma, T, P, s0) \circ \mathsf{pr}_{1}) && \text{(by (2.2.11))}\\
&= \eta^{!}_{pE\widetilde{\mathcal{U}}}(\eta^{!-1}_{pE\widetilde{\mathcal{U}}}(F, G)) && \text{(by def'n of } \phi(\Gamma, T, P, s0)\text{)}\\
&= (F, G) &&
\end{aligned}$$

(For naturality of $\eta^{!}_{pE\widetilde{\mathcal{U}}}$, refer to [18, Problem 3.8(1)].)

Therefore, $F_{1} = F$ and $F_{2} \circ p = G$. Thus F_{2} is of type $(X; F)_{E} \to \widetilde{\mathcal{U}}$, and thus it is of the form $F_{2} = \widetilde{u}_{1}(\mathsf{J}(\Gamma, T, P, s0)))$ for some unique $\mathsf{J}(\Gamma, T, P, s0)$ such that $\partial(\mathsf{J}(\Gamma, T, P, s0)) = u_{1}^{-1}(F_{2} \circ p) = u_{1}^{-1}(G) = P$. ☐

Remark **2.4.10.** — When more than one J2-structure $\mathrm{J}p$ is under consideration, we may write $\mathsf{J}_{\mathrm{J}p} = \mathsf{J}_{\mathrm{J}p}(\Gamma, \mathrm{T}, \mathrm{P}, s0)$ instead of $\mathsf{J} = \mathsf{J}(\Gamma, \mathrm{T}, \mathrm{P}, s0)$ to indicate which J2-structure is involved in the construction of J.

Remark **2.4.11.** — Note that the defining property of $\mathsf{J} := \mathsf{J}(\Gamma, \mathrm{T}, \mathrm{P}, s0)$ is that it is the unique element of $\widetilde{\mathrm{O}b}(\mathrm{CC}(\mathcal{C}, p))$ that satisfies the equation

$$\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(u_{1,\Gamma}(\mathrm{T}), \widetilde{u}_{1,\mathsf{Id}_3(\mathrm{T})}(\mathsf{J})) = \phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathrm{J}p,$$

where

$$\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) : int(\Gamma) \to (\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^{\omega}(\mathcal{U})) \times_{\mathsf{I}_p(\mathcal{U})} (\mathsf{I}_p(\widetilde{\mathcal{U}}), \mathsf{I}_p(p))$$

is given by the pair of morphisms $(\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(u_{1,\Gamma}(\mathrm{T}), u_{1,\mathsf{Id}_3(\mathrm{T})}(\mathrm{P})), \eta^{!-1}_{p}(u_{1,\Gamma}(\mathrm{T}), \widetilde{u}_{1,\Gamma}(s0)))$.

Lemma **2.4.12.** — *Let* $\mathrm{E}q$ *be a* $\mathcal{J}0$*-structure on a universe* p, $f : \Gamma' \to \Gamma$ *a morphism in* $\mathrm{CC}(\mathcal{C}, p)$ *and* $\mathrm{F} : int(\Gamma) \to \mathcal{U}$ *a morphism in* $\mathcal{C}$. *Let* $q3 : int(\mathsf{Id}_3(\Gamma', f \circ \mathrm{F})) \to int(\mathsf{Id}_3(\Gamma, \mathrm{F}))$ *be the morphism* $\mathsf{q}(f, \mathsf{Id}_3(\Gamma, \mathrm{F}), 3)$ *defined*[9] *by* Γ, *using* $\mathsf{ft}^3(\mathsf{Id}_3(\Gamma, \mathrm{F})) = \Gamma$. *Then* $q3 = \mathsf{Q}(f, \mathrm{F})_{\mathrm{E}}$.

Proof. — Let $\mathrm{X} := int(\Gamma)$ and $\mathrm{X}' := int(\Gamma')$. By definition, $\mathsf{Q}(f, \mathrm{F})_{\mathrm{E}}$ is the unique morphism such that

$$\mathsf{Q}(f, \mathrm{F})_{\mathrm{E}} \circ \mathsf{Q}(\mathrm{F})_{\mathrm{E}} = \mathsf{Q}(f \circ \mathrm{F})_{\mathrm{E}} \qquad \text{and}$$
$$\mathsf{Q}(f, \mathrm{F})_{\mathrm{E}} \circ p^{\mathrm{E}}_{\mathrm{X},\mathrm{F}} = p^{\mathrm{E}}_{\mathrm{X}',f\circ\mathrm{F}} \circ f.$$

We will be building the proof using the following diagram.

$$\begin{tikzcd}
(\mathrm{X}', f \circ \mathrm{F})_{\mathrm{E}} \arrow[r, "{\mathsf{Q}(\mathsf{Q}(\mathsf{Q}(f,\mathrm{F}),\mathsf{Q}(\mathrm{F})\circ p),\mathsf{Q}(\mathsf{Q}(\mathrm{F}),p)\circ \mathrm{E}q)}"] \arrow[d, "p_3"'] & (\mathrm{X}; \mathrm{F})_{\mathrm{E}} \arrow[r, "{\mathsf{Q}(\mathsf{Q}(\mathsf{Q}(\mathrm{F}),p),\mathrm{E}q)}"] \arrow[d] & \mathrm{E}\widetilde{\mathcal{U}} \arrow[r, "{\mathsf{Q}(\mathrm{E}q)}"] \arrow[d, "p_1"] & \widetilde{\mathcal{U}} \arrow[d, "p"] \\
\bullet \arrow[r, "{\mathsf{Q}(\mathsf{Q}(f,\mathrm{F}),\mathsf{Q}(\mathrm{F})\circ p)}"] \arrow[d, "="] & \bullet \arrow[r, "{\mathsf{Q}(\mathsf{Q}(\mathrm{F}),p)}"] \arrow[d, "="] & \bullet \arrow[r, "\mathrm{E}q"] \arrow[d, "="] & \mathcal{U} \\
\bullet \arrow[r, "{\mathsf{Q}(\mathsf{Q}(f,\mathrm{F}),\mathsf{Q}(\mathrm{F})\circ p)}"] \arrow[d] & \bullet \arrow[r, "{\mathsf{Q}(\mathsf{Q}(\mathrm{F}),p)}"] \arrow[d] & \bullet \arrow[r, "{\mathsf{Q}(p)}"] \arrow[d, "p_2"] & \widetilde{\mathcal{U}} \arrow[d, "p"] \\
\bullet \arrow[r, "{\mathsf{Q}(f,\mathrm{F})}"] \arrow[d, "="] & \bullet \arrow[r, "{\mathsf{Q}(\mathrm{F})}"] \arrow[d, "="] & \widetilde{\mathcal{U}} \arrow[r, "p"] \arrow[d, "="] & \mathcal{U} \\
\bullet \arrow[r, "{\mathsf{Q}(f,\mathrm{F})}"] \arrow[d] & \bullet \arrow[r, "{\mathsf{Q}(\mathrm{F})}"] \arrow[d] & \bullet \arrow[d, "p"] & \\
\mathrm{X}' \arrow[r, "f"] & \mathrm{X} \arrow[r, "\mathrm{F}"] & \mathcal{U} &
\end{tikzcd}$$

---

[9] The notation $\mathsf{q}(f, \mathrm{X}, i)$ is defined in Section 3 of [20], by induction on the third parameter, which is a natural number.

By construction that is seen on this diagram we have:

$$\begin{aligned} q3 &= \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(f, F), \mathsf{Q}(F) \circ p), \mathsf{Q}(\mathsf{Q}(F), p) \circ Eq) \\ \mathsf{Q}(X, F)_E &= \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(F), p), Eq) \end{aligned}$$

and

$$\mathsf{Q}(X', f \circ F)_E = \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(f \circ F), p), Eq)$$

Therefore, the first equation that we need to verify is

$$\begin{aligned} &\mathsf{Q}(\mathsf{Q}(\mathsf{Q}(f, F), \mathsf{Q}(F) \circ p), \mathsf{Q}(\mathsf{Q}(F), p) \circ Eq) \circ \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(F), p), Eq) \\ &\quad = \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(f \circ F), p), Eq) \end{aligned}$$

By [18, Lemma 3.2] we have, together with the defining rule $\mathsf{Q}(a, A) \circ \mathsf{Q}(A) = \mathsf{Q}(a \circ A)$, also the rule:

$$\mathsf{Q}(a_1, a_2 \circ A) \circ \mathsf{Q}(a_2, A) = \mathsf{Q}(a_1 \circ a_2, A)$$

Applying it twice and then the defining rule we get:

$$\begin{aligned} \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(f, F),\mathsf{Q}(F) \circ p), \mathsf{Q}(\mathsf{Q}(F), p) \circ Eq) \circ \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(F), p), Eq) \\ = \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(f, F), \mathsf{Q}(F) \circ p) \circ \mathsf{Q}(\mathsf{Q}(F), p), Eq) \\ = \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(f, F) \circ \mathsf{Q}(F), p), Eq) \\ = \mathsf{Q}(\mathsf{Q}(\mathsf{Q}(f \circ F), p), Eq), \end{aligned}$$

which gives us the first equation. The second equation is immediate from the commutativity of the three squares that define $q3$. □

Lemma **2.4.13.** — *Let* (Eq, Ω, Jp) *be a* $\mathcal{J}$*-structure on a universe* p*. Then the morphisms of Construction 2.4.9 are natural in* Γ*, i.e., for any* $f : \Gamma' \to \Gamma$ *one has*

$$f^*(\mathsf{J}_{Jp}(\Gamma, T, P, s0)) = \mathsf{J}_{Jp}(\Gamma', f^*(T), f^*(P), f^*(s0)). \tag{2.60}$$

(This is the second condition of Definition 2.1.7 of a $\mathcal{J}$*2-structure on the C-system* $CC(\mathcal{C}, p)$*.)*

Proof. — Let us write J for $\mathsf{J}_{Jp}(\Gamma, T, P, s0)$ and J' for $\mathsf{J}_{Jp}(\Gamma', f^*(T), f^*(P), f^*(s0))$ and use the notation of Construction 2.4.9. Recall that for $f : \Gamma' \to \Gamma$ the operation f^* is defined only on $Ob_1(\Gamma)$. In all other uses it is an abbreviation for the operation $X \mapsto f^*(X, i)$ or the operation $s \mapsto f^*(s, i)$, for various values of i; these operations are defined in [20, §3] by induction on i. In particular, (2.60) is an abbreviation for

$$f^*(\mathsf{J}(\Gamma, T, P, s0), 4) = \mathsf{J}(\Gamma', f^*(T), f^*(P, 4), f^*(s0, 2))$$

which in its turn translates into the equation in $\widetilde{\mathrm{O}b}_1(\mathsf{Id}_3(f^*(\mathrm{T})))$ of the form

$$\mathsf{q}(f, \mathsf{Id}_3(\mathrm{T}), 3)^*(\mathsf{J}, 1) = \mathsf{J}'$$

We have:

$$\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{F}, \widetilde{u}_1(\mathsf{J})) = \phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathrm{J}p$$

$$\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(f \circ \mathrm{F}, \widetilde{u}_1(\mathsf{J}')) = \phi(\Gamma', f^*(\mathrm{T}), f^*(\mathrm{P}), f^*(s0)) \circ \mathrm{J}p$$

By naturality of $\eta^{!-1}$ with respect to the first argument we have

$$f \circ \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{F}, \widetilde{u}_1(\mathsf{J})) = \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(f \circ \mathrm{F}, \mathsf{Q}(f, \mathrm{F})_{\mathrm{E}} \circ \widetilde{u}_1(\mathsf{J}))$$

Therefore, by Lemma 2.4.12 we have

$$\begin{aligned} f \circ \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{F}, \widetilde{u}_1(\mathsf{J})) &= \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(f \circ \mathrm{F}, \widetilde{u}_1(\mathsf{Q}(f, \mathrm{F})^*_{\mathrm{E}}(\mathsf{J}, 1))) \\ &= \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(f \circ \mathrm{F}, \widetilde{u}_1(\mathsf{q}(f, \mathsf{Id}_3(\mathrm{T}), 3)^*(\mathsf{J}, 1))) \end{aligned}$$

Since both $\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}$ and $\widetilde{u}_1$ are bijections and thus injections, it is sufficient to show that

$$f \circ \phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathrm{J}p = \phi(\Gamma', f^*(\mathrm{T}), f^*(\mathrm{P}), f^*(s0)) \circ \mathrm{J}p$$

or that

$$f \circ \phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) = \phi(\Gamma', f^*(\mathrm{T}), f^*(\mathrm{P}), f^*(s0))$$

Since both ϕ expressions are morphism into a product this amounts to two equations that, taking into account the definition of ϕ in Construction 2.4.9 are:

$$f \circ \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{F}, \mathrm{G}) = \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(f \circ \mathrm{F}, u_1(f^*(\mathrm{P})))$$

and

$$f \circ \eta^{!-1}_{p}(\mathrm{F}, \widetilde{\mathrm{H}}) = \eta^{!-1}_{p}(f \circ \mathrm{F}, \widetilde{u}_1(f^*(s0)))$$

The first equality follows from naturality of $\eta^{!-1}$ and Lemma 2.4.12. The second equality follows from naturality of $\eta^{!-1}$. This finishes the proof of Lemma 2.4.13. ☐

Lemma **2.4.14.** — *Let* $(\mathrm{E}q, \Omega, \mathrm{J}p)$ *be a* $\mathcal{J}$*-structure on a universe* p*. Then the morphism of Construction 2.4.9 satisfies the first condition of Definition 2.1.7 of a* $\mathcal{J}2$*-structure on the C-system* $\mathrm{CC}(\mathcal{C}, p)$*, i.e., for all* Γ, T, P *and* $s0$ *as above one has*

$$\mathsf{rf}^*_{\mathrm{T}}(\mathsf{J}_{\mathrm{J}p}(\Gamma, \mathrm{T}, \mathrm{P}, s0)) = s0$$

Proof. — Let $\mathsf{J} = \mathsf{J}_{Jp}(\Gamma, T, P, s0)$. Then, using the notation of Construction 2.4.9 we have

$$\eta^{!-1}_{E\widetilde{\mathcal{U}}}(F, \widetilde{u}_1(\mathsf{J})) = \eta^{!-1}_{E\widetilde{\mathcal{U}}}(F_1, F_2) = \phi(\Gamma, T, P, s0) \circ Jp$$

Observe that

$$\eta^{!-1}_{E\widetilde{\mathcal{U}}}(F, \widetilde{u}_1(\mathsf{J})) \circ \mathsf{I}^{\omega}(\widetilde{\mathcal{U}}) = \eta^{!-1}_{p}(F, F^*(\omega) \circ \widetilde{u}_1(\mathsf{J})),$$

by naturality of $\eta^!$ with respect to change of universe, established in Lemma 2.2.5. By Lemma 2.4.7 we have $F^*(\omega) = \mathsf{rf}_T$. Therefore,

$$\eta^{!-1}_{E\widetilde{\mathcal{U}}}(F, \widetilde{u}_1(\mathsf{J})) \circ \mathsf{I}^{\omega}(\widetilde{\mathcal{U}}) = \eta^{!-1}_{p}(F, \mathsf{rf}_T \circ \widetilde{u}_1(\mathsf{J})) = \eta^{!-1}_{p}(F, \widetilde{u}_1(\mathsf{rf}^*_T(\mathsf{J})))$$

On the other hand, by (2.32),

$$\phi(\Gamma, T, P, s0) \circ Jp \circ \mathsf{I}^{\omega}(\widetilde{\mathcal{U}}) = \phi(\Gamma, T, P, s0) \circ \mathsf{pr}_2$$

which equals, by construction, $\eta^{!-1}_{p}(F, \widetilde{u}_1(s0))$. Therefore,

$$\eta^{!-1}_{p}(F, \widetilde{u}_1(\mathsf{rf}^*_T(\mathsf{J}))) = \eta^{!-1}_{p}(F, \widetilde{u}_1(s0))$$

and using again that both $\eta^{!-1}$ and $\widetilde{u}_1$ are injective we conclude that $\mathsf{rf}^*_T(\mathsf{J}) = s0$. □

Problem **2.4.15.** — Let (Eq, Ω, Jp) be a J-structure on a universe p. To construct a J-structure on $CC(\mathcal{C}, p)$ relative to Id_{Eq} and refl_{Ω}.

Construction **2.4.16** *(For Problem 2.4.15).* — One has to combine Construction 2.4.9 with Lemmas 2.4.13 and 2.4.14. □

3. Functoriality of J-structures

3.1. *A theorem about functors between categories with two universes*

Before we can formulate the definition of what it means for a universe category functor to be compatible with J-structures we need some general results about functors between categories with two universes, which we will apply in Section 3.2 to the universes $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ and $pE\widetilde{\mathcal{U}} : E\widetilde{\mathcal{U}} \to \mathcal{U}$ in a locally Cartesian closed category $\mathcal{C}$.

Given two universes $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ and $p' : \widetilde{\mathcal{U}}' \to \mathcal{U}$, with canonical squares of the form

$$\begin{tikzcd}
(X; F) \arrow[r, "\mathsf{Q}(F)"] \arrow[d, "\mathsf{p}_{X,F}"'] & \widetilde{\mathcal{U}} \arrow[d, "p"] & (X; F)' \arrow[r, "\mathsf{Q}'(F)"] \arrow[d, "p'_{X,F}"'] & \widetilde{\mathcal{U}}' \arrow[d, "p'"] \\
X \arrow[r, "F"] & \mathcal{U} & X \arrow[r, "F"] & \mathcal{U}
\end{tikzcd}$$

and given $f : \widetilde{\mathcal{U}}' \to \widetilde{\mathcal{U}}$ over $\mathcal{U}$, we let $\mathrm{F}^*(f)$ denote the unique morphism $(\mathrm{X};\mathrm{F})' \to (\mathrm{X};\mathrm{F})$ such that

$$\mathrm{F}^*(f) \circ \mathsf{Q}(\mathrm{F}) = \mathsf{Q}'(\mathrm{F}) \circ f \tag{\textbf{3.1}}$$

$$\mathrm{F}^*(f) \circ \mathsf{p}_{\mathrm{X},\mathrm{F}} = \mathsf{p}'_{\mathrm{X},\mathrm{F}} \tag{\textbf{3.2}}$$

Note that $\mathrm{F}^*(f)$ depends on the universe structures on p and p'. Even when two universe structures give the same choices of the objects $(\mathrm{X};\mathrm{F})$ and $(\mathrm{X};\mathrm{F})'$, the difference in the choice of some of the morphisms, e.g., $\mathsf{Q}(\mathrm{F})$ will affect morphisms $\mathrm{F}^*(f)$. We will need the following lemma about these morphisms.

As in Definition 2.2.3, for $\mathrm{X}' \xrightarrow{f} \mathrm{X} \xrightarrow{\mathrm{F}} \mathcal{U}$ we let $\mathsf{Q}(f,\mathrm{F})$ denote the morphism

$$(\mathsf{p}_{\mathrm{X}',f\circ\mathrm{F}} \circ f) * \mathsf{Q}(f \circ \mathrm{F}) : (\mathrm{X}'; f \circ \mathrm{F}) \to (\mathrm{X};\mathrm{F})$$

We let $\mathsf{Q}'(-)$ and $\mathsf{Q}'(-,-)$ denote the morphisms $\mathsf{Q}(-)$ and $\mathsf{Q}(-,-)$ relative to the universe p'.

Lemma **3.1.1.** — *Let* $\mathrm{X}' \xrightarrow{g} \mathrm{X} \xrightarrow{\mathrm{F}} \mathcal{U}$ *be two morphisms. Then the square*

$$\begin{tikzcd}
(\mathrm{X}'; g \circ \mathrm{F})' \arrow[r, "\mathsf{Q}'(g,\mathrm{F})"] \arrow[d, "(g\circ\mathrm{F})^*(f)"'] & (\mathrm{X};\mathrm{F})' \arrow[d, "\mathrm{F}^*(f)"] \\
(\mathrm{X}'; g \circ \mathrm{F}) \arrow[r, "\mathsf{Q}(g,\mathrm{F})"] & (\mathrm{X}'; g \circ \mathrm{F})
\end{tikzcd}$$

commutes.

Proof. — Since $(\mathrm{X};\mathrm{F})$ is a fiber product relative to the projections $\mathsf{p}_{\mathrm{X},\mathrm{F}}$ and $\mathsf{Q}(\mathrm{F})$ it is sufficient to verify that

$$\mathsf{Q}'(g,\mathrm{F}) \circ \mathrm{F}^*(f) \circ \mathsf{Q}(\mathrm{F}) = (g \circ \mathrm{F})^*(f) \circ \mathsf{Q}(g,\mathrm{F}) \circ \mathsf{Q}(\mathrm{F})$$

and

$$\mathsf{Q}'(g,\mathrm{F}) \circ \mathrm{F}^*(f) \circ \mathsf{p}_{\mathrm{X},\mathrm{F}} = (g \circ \mathrm{F})^*(f) \circ \mathsf{Q}(g,\mathrm{F}) \circ \mathsf{p}_{\mathrm{X},\mathrm{F}}$$

which easily follows from the defining equations for $\mathsf{Q}(-,-)$ and $(-)^*$. ☐

Let $(\mathcal{C}, p, pt)$, $(\mathcal{C}', p', pt')$ be two universe categories such that $\mathcal{C}$ and $\mathcal{C}'$ are equipped with locally Cartesian closed structures. Consider now a *functor of universe categories*

$$\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi}) : (\mathcal{C}, p, pt) \to (\mathcal{C}', p', pt').$$

The notion was defined in [17, 4.1] and in [18, §5]. It means that $\Phi : \mathcal{C} \to \mathcal{C}'$ is a functor, $\phi : \Phi(\mathcal{U}) \to \mathcal{U}'$ and $\widetilde{\phi} : \Phi(\widetilde{\mathcal{U}}) \to \widetilde{\mathcal{U}}'$ are morphisms, that Φ sends the chosen final object to a final object, that Φ takes canonical squares of p to pullback squares, and that the square

$$\begin{tikzcd}
\Phi(\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}"] \arrow[d, "\Phi(p)"'] & \widetilde{\mathcal{U}}' \arrow[d, "p'"] \\
\Phi(\mathcal{U}) \arrow[r, "\phi"] & \mathcal{U}'
\end{tikzcd}$$

is a pullback square. In [18, Construction 5.2] we have constructed, for any map $\mathrm{F} : \mathrm{X} \to \mathrm{V}$ in $\mathcal{C}$, a canonical isomorphism

$$\iota = \iota_{\Phi,\mathrm{X},\mathrm{F}} : (\Phi(\mathrm{X}); \Phi(\mathrm{F}) \circ \phi)' \xrightarrow{\cong} \Phi((\mathrm{X}; \mathrm{F})),$$

which results from the two objects involved being pullbacks of the same diagram, as illustrated here:

(**3.3**)
$$\begin{tikzcd}
(\Phi(\mathrm{X}); \Phi(\mathrm{F}) \circ \phi)' \arrow[dr, "\iota", "\cong"'] \arrow[drrr, bend left, "\mathsf{Q}'(\Phi(\mathrm{F})\circ\phi)"] \arrow[ddr, bend right, "\mathsf{p}'_{\Phi(\mathrm{X}),\Phi(\mathrm{F})\circ\phi}"'] & & & \\
& \Phi((\mathrm{X}; \mathrm{F})) \arrow[r, "\Phi(\mathsf{Q}(\mathrm{F}))"'] \arrow[d, "\Phi(\mathsf{p}_{\mathrm{X},\mathrm{F}})"] & \Phi(\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}"'] \arrow[d, "\Phi(p)"] & \widetilde{\mathcal{U}}' \arrow[d, "p'"] \\
& \Phi(\mathrm{X}) \arrow[r, "\Phi(\mathrm{F})"'] & \Phi(\mathrm{U}) \arrow[r, "\phi"'] & \mathcal{U}'
\end{tikzcd}$$

In [18, Construction 5.2] we have defined, for any $\mathrm{X}, \mathrm{V} \in \mathcal{C}$, a map

$$\mathbf{\Phi}^2 : \mathsf{D}_p(\mathrm{X}, \mathrm{V}) \to \mathsf{D}_p(\Phi(\mathrm{X}), \Phi(\mathrm{V})),$$

by setting

(**3.4**)
$$\mathbf{\Phi}^2(\mathrm{F}, \mathrm{G}) := (\Phi(\mathrm{F}) \circ \phi, \iota_{\mathbf{\Phi},\mathrm{X},\mathrm{F}} \circ \Phi(\mathrm{G})).$$

In [18, Construction 5.6] we have also defined a morphism

$$\chi_{\mathbf{\Phi}}(\mathrm{V}) : \Phi(\mathsf{I}_p(\mathrm{V})) \to \mathsf{I}_{p'}(\Phi(\mathrm{V})),$$

by setting

$$\chi_{\mathbf{\Phi}}(\mathrm{V}) := \eta^{!-1}_{p',\Phi(\mathsf{I}_p(\mathrm{V})),\Phi(\mathrm{V})}\big(\mathbf{\Phi}^2\big(\eta^{!}_{p,\mathsf{I}_p(\mathrm{V}),\mathrm{V}}(1_{\mathsf{I}_p(\mathrm{V})})\big)\big).$$

These constructions will be used later.

We now need to consider the situation where we have the following collection of data:

1. two universes p_1, p_2 in $\mathcal{C}$ with the common codomain $\mathcal{U}$ and a morphism $g : \widetilde{\mathcal{U}}_1 \to \widetilde{\mathcal{U}}_2$ over $\mathcal{U}$,
2. two universes p'_1, p'_2 in $\mathcal{C}'$ with the common codomain $\mathcal{U}'$ and a morphism $g' : \widetilde{\mathcal{U}}'_1 \to \widetilde{\mathcal{U}}'_2$ over $\mathcal{U}'$,
3. a functor $\Phi : \mathcal{C} \to \mathcal{C}'$,
4. a morphism $\phi : \Phi(\mathcal{U}) \to \mathcal{U}'$,
5. two morphisms $\widetilde{\phi}_i : \Phi(\widetilde{\mathcal{U}}_i) \to \widetilde{\mathcal{U}}'_i$, $i = 1, 2$, and
6. final objects pt of $\mathcal{C}$ and pt' of $\mathcal{C}'$,

and this data is such that:

1. the square

(**3.5**)
$$\begin{tikzcd}
\Phi(\widetilde{\mathcal{U}}_1) \arrow[r, "\widetilde{\phi}_1"] \arrow[d, "\Phi(g)"'] & \widetilde{\mathcal{U}}'_1 \arrow[d, "g'"] \\
\Phi(\widetilde{\mathcal{U}}_2) \arrow[r, "\widetilde{\phi}_2"] & \widetilde{\mathcal{U}}'_2
\end{tikzcd}$$

   commutes, and
2. the triples $\mathbf{\Phi}_i := (\Phi, \phi, \widetilde{\phi}_i)$, for $i = 1, 2$, are universe category functors from $(\mathcal{C}, p_i, ptk)$ to $(\mathcal{C}', p'_i, pt')$.

We will use the notations $(X; F)_1$, $(X; F)_2$, $(X; F)'_1$, and $(X; F)'_2$ for the pullback objects that are part of the four universe structures under consideration.

Let us denote the morphisms

$$\chi_{\mathbf{\Phi}_i}(V) : \Phi(\mathsf{I}_{p_i}(V)) \to \mathsf{I}_{p'_i}(\Phi(V))$$

by $\chi_i(V)$. The maps $\mathbf{\Phi}_i^2$ in the following lemma were introduced above.

Lemma **3.1.2.** — *Under the previous assumptions and notation the squares*

$$\begin{tikzcd}
\mathsf{D}_{p_2}(X, V) \arrow[r, "\Phi_2^2"] \arrow[d, "D^g(X,V)"'] & \mathsf{D}_{p_2}(\Phi(X), \Phi(V)) \arrow[d, "D^{g'}(\Phi(X),\Phi(V))"] \\
\mathsf{D}_{p_1}(X, V) \arrow[r, "\Phi_1^2"] & \mathsf{D}_{p'_1}(\Phi(X), \Phi(V))
\end{tikzcd}$$

commute.

Proof. — We will use ι_i as an abbreviation for an isomorphism ι derived from $\mathbf{\Phi}_i$. Given $(F_1, F_2) \in \mathsf{D}_{p_2}(X, V)$, we see that:

$$\begin{aligned}
&D^{g'}(\Phi(X), \Phi(V))(\mathbf{\Phi}_2^2(F_1, F_2)) \\
&= D^{g'}(\Phi(X), \Phi(V))(\Phi(F_1) \circ \phi, \iota_2 \circ \Phi(F_2)) && \text{(by def'n of } \mathbf{\Phi}_2^2\text{)} \\
&= (\Phi(F_1) \circ \phi, (\Phi(F_1) \circ \phi)^*(g') \circ \iota_2 \circ \Phi(F_2)) && \text{(by def'n of } D^{g'} \\
& && \text{in Lemma 2.2.5)}
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\mathbf{\Phi}_1^2(D^g(X, V)(F_1, F_2)) \\
&= \mathbf{\Phi}_1^2(F_1, F_1^*(g) \circ F_2) && \text{(by def'n of } D^g \text{ in Lemma 2.2.5)} \\
&= (\Phi(F_1) \circ \phi, \iota_1 \circ \Phi(F_1^*(g) \circ F_2)) && \text{(by def'n of } \mathbf{\Phi}_1^2\text{)}
\end{aligned}$$

Thus it remains to check that

$$(\Phi(F_1) \circ \phi)^*(g') \circ \iota_2 \circ \Phi(F_2) = \iota_1 \circ \Phi(F_1^*(g) \circ F_2).$$

For that it is sufficient to check that

$$(\Phi(F_1) \circ \phi)^*(g') \circ \iota_2 = \iota_1 \circ \Phi(F_1^*(g)).$$

The codomain of both morphisms is $\Phi((X; F_1)_2)$, and since Φ takes canonical squares based on p_2 to pullback squares it is sufficient to check that

$$(\Phi(F_1) \circ \phi)^*(g') \circ \iota_2 \circ \Phi(\mathsf{Q}_2(F_1)) \circ \widetilde{\phi}_2 = \iota_1 \circ \Phi(F_1^*(g)) \circ \Phi(\mathsf{Q}_2(F_1)) \circ \widetilde{\phi}_2$$

and

$$(\Phi(F_1) \circ \phi)^*(g') \circ \iota_2 \circ \Phi(\mathsf{p}_{2,X,F_1}) = \iota_1 \circ \Phi(F_1^*(g)) \circ \Phi(\mathsf{p}_{2,X,F_1})$$

We prove the first equation as follows.

$$\begin{aligned}
&(\Phi(F_1) \circ \phi)^*(g') \circ \iota_2 \circ \Phi(\mathsf{Q}_2(F_1)) \circ \widetilde{\phi}_2 \\
&= (\Phi(F_1) \circ \phi)^*(g') \circ \mathsf{Q}_2'(\Phi(F_1) \circ \phi) && \text{(by commutativity of (3.3))} \\
&= \mathsf{Q}_1'(\Phi(F_1) \circ \phi) \circ g' && \text{(by (3.1))} \\
&= \iota_1 \circ \Phi(\mathsf{Q}_1(F_1)) \circ \widetilde{\phi}_1 \circ g' && \text{(by commutativity of (3.3))} \\
&= \iota_1 \circ \Phi(\mathsf{Q}_1(F_1)) \circ \Phi(g) \circ \widetilde{\phi}_2 && \text{(by (3.5))} \\
&= \iota_1 \circ \Phi(\mathsf{Q}_1(F_1) \circ g) \circ \widetilde{\phi}_2 && \text{(by functoriality of } \Phi\text{)}
\end{aligned}$$

$$\begin{aligned}
&= \iota_1 \circ \Phi(\mathrm{F}_1^*(g) \circ \mathsf{Q}_2(\mathrm{F}_1)) \circ \widetilde{\phi}_2 && \text{(by (3.1))}\\
&= \iota_1 \circ \Phi(\mathrm{F}_1^*(g)) \circ \Phi(\mathsf{Q}_2(\mathrm{F}_1)) \circ \widetilde{\phi}_2 && \text{(by functoriality of } \Phi\text{)}
\end{aligned}$$

The second equation is proved as follows.

$$\begin{aligned}
&(\Phi(\mathrm{F}_1) \circ \phi)^*(g') \circ \iota_2 \circ \Phi(\mathsf{p}_{2,\mathrm{X},\mathrm{F}_1})\\
&= (\Phi(\mathrm{F}_1) \circ \phi)^*(g') \circ \mathsf{p}'_{2,\Phi(\mathrm{X}),\Phi(\mathrm{F}_1)\circ\phi} && \text{(by commutativity of (3.3))}\\
&= \mathsf{p}'_{1,\Phi(\mathrm{X}),\Phi(\mathrm{F}_1)\circ\phi} && \text{(by (3.2))}\\
&= \iota_1 \circ \Phi(\mathsf{p}_{1,\mathrm{X},\mathrm{F}_1}) && \text{(by commutativity of (3.3))}\\
&= \iota_1 \circ \Phi(\mathrm{F}_1^*(g) \circ \mathsf{p}_{2,\mathrm{X},\mathrm{F}_1}) && \text{(by (3.2))}\\
&= \iota_1 \circ \Phi(\mathrm{F}_1^*(g)) \circ \Phi(\mathsf{p}_{2,\mathrm{X},\mathrm{F}_1}) && \text{(by functoriality of } \Phi\text{)}
\end{aligned}$$

This finishes the proof of Lemma 3.1.2. ☐

Lemma **3.1.3.** — *Under the previous assumptions and notation the squares*

$$\begin{tikzcd}
\Phi(\mathsf{I}_{p_2}(\mathrm{V})) \arrow[r, "\chi_2(\mathrm{V})"] \arrow[d, "\Phi(\mathsf{I}^g(\mathrm{V}))"'] & \mathsf{I}_{p'_2}(\Phi(\mathrm{V})) \arrow[d, "\mathsf{I}^{g'}(\Phi(\mathrm{V}))"] \\
\Phi(\mathsf{I}_{p_1}(\mathrm{V})) \arrow[r, "\chi_1(\mathrm{V})"] & \mathsf{I}_{p'_1}(\Phi(\mathrm{V}))
\end{tikzcd}$$

commute.

Proof. — Let $\mathrm{X} = \mathsf{I}_{p_2}(\mathrm{V})$. Then we have:

$$\begin{aligned}
&\chi_2(\mathrm{V}) \circ \mathsf{I}^{g'}(\Phi(\mathrm{V}))\\
&= \eta^{!^{-1}}_{p'_2,\Phi(\mathrm{X}),\Phi(\mathrm{V})}(\mathbf{\Phi}_2^2(\eta^!_{p_2,\mathrm{X},\mathrm{V}}(1_\mathrm{X}))) \circ \mathsf{I}^{g'}(\Phi(\mathrm{V})) && \text{(by definition of } \chi_2(\mathrm{V})\text{)}\\
&= \eta^{!^{-1}}_{p'_1,\Phi(\mathrm{X}),\Phi(\mathrm{V})}(\mathrm{D}^{g'}(\Phi(\mathrm{X}), \Phi(\mathrm{V}))(\mathbf{\Phi}_2^2(\eta^!_{p_2,\mathrm{X},\mathrm{V}}(1_\mathrm{X})))) && \text{(by Lemma 2.2.5)}\\
&= \eta^{!^{-1}}_{p'_1,\Phi(\mathrm{X}),\Phi(\mathrm{V})}(\mathbf{\Phi}_1^2(\mathrm{D}^g(\mathrm{X}, \mathrm{V})(\eta^!_{p_2,\mathrm{X},\mathrm{V}}(1_\mathrm{X})))) && \text{(by Lemma 3.1.2)}\\
&= \eta^{!^{-1}}_{p'_1,\Phi(\mathrm{X}),\Phi(\mathrm{V})}(\mathbf{\Phi}_1^2(\eta^!_{p_1,\mathrm{X},\mathrm{V}}(1_\mathrm{X} \circ \mathsf{I}^g(\mathrm{V})))) && \text{(by Lemma 2.2.5)}\\
&= \eta^{!^{-1}}_{p'_1,\Phi(\mathrm{X}),\Phi(\mathrm{V})}(\mathbf{\Phi}_1^2(\eta^!_{p_1,\mathrm{X},\mathrm{V}}(\mathsf{I}^g(\mathrm{V}))))
\end{aligned}$$

It remains to show that

$$\Phi(\mathsf{I}^g(\mathrm{V})) \circ \chi_1(\mathrm{V}) = \eta^{!^{-1}}_{p'_1,\Phi(\mathrm{X}),\Phi(\mathrm{V})}(\mathbf{\Phi}_1^2(\eta^!_{p_1,\mathrm{X},\mathrm{V}}(\mathsf{I}^g(\mathrm{V}))))$$

Let a be any element of $Hom(\mathsf{I}_{p_2}(\mathrm{V}), \mathsf{I}_{p_1}(\mathrm{V}))$. It will suffice to show that

$$\Phi(a) \circ \chi_1(\mathrm{V}) = \eta^{!^{-1}}_{p'_1, \Phi(\mathrm{X}), \Phi(\mathrm{V})}(\mathbf{\Phi}^2_1(\eta^{!}_{p_1, \mathrm{X}, \mathrm{V}}(a)).$$

We have

$$\begin{aligned}
&\Phi(a) \circ \chi_1(\mathrm{V}) \\
&= \Phi(a) \circ \eta^{!^{-1}}_{p'_1, \Phi(\mathsf{I}_{p_1}(\mathrm{V})), \Phi(\mathrm{V})}(\mathbf{\Phi}^2_1(\eta^{!}_{p_1, \mathsf{I}_{p_1}(\mathrm{V}), \mathrm{V}}(1_{\mathsf{I}_{p_1}(\mathrm{V})}))) \quad \text{(by def'n of } \chi_1(\mathrm{V})) \\
&= \eta^{!^{-1}}_{p'_1, \Phi(\mathsf{I}_{p_2}(\mathrm{V})), \Phi(\mathrm{V})}(\mathsf{D}_{p'_1}(\Phi(a), \Phi(\mathrm{V}))(\mathbf{\Phi}^2_1(\eta^{!}_{p_1, \mathsf{I}_{p_1}(\mathrm{V}), \mathrm{V}}(1_{\mathsf{I}_{p_1}(\mathrm{V})}))) \\
&\hfill \text{(by naturality of } \eta^{!^{-1}}) \\
&= \eta^{!^{-1}}_{p'_1, \Phi(\mathsf{I}_{p_2}(\mathrm{V})), \Phi(\mathrm{V})}(\mathbf{\Phi}^2_1(\mathsf{D}_{p_1}(a, \mathrm{V})(\eta^{!}_{p_1, \mathsf{I}_{p_1}(\mathrm{V}), \mathrm{V}}(1_{\mathsf{I}_{p_1}(\mathrm{V})}))) \\
&\hfill \text{(by [18, Lemma 5.3])} \\
&= \eta^{!^{-1}}_{p'_1, \Phi(\mathsf{I}_{p_2}(\mathrm{V})), \Phi(\mathrm{V})}(\mathbf{\Phi}^2_1(\eta^{!}_{p_1, \mathsf{I}_{p_2}(\mathrm{V}), \mathrm{V}}(a \circ 1_{\mathsf{I}_{p_1}(\mathrm{V})})) \qquad \text{(by naturality of } \eta^{!^{-1}}) \\
&= \eta^{!^{-1}}_{p'_1, \Phi(\mathsf{I}_{p_2}(\mathrm{V})), \Phi(\mathrm{V})}(\mathbf{\Phi}^2_1(\eta^{!}_{p_1, \mathsf{I}_{p_2}(\mathrm{V}), \mathrm{V}}(a)) \\
&= \eta^{!^{-1}}_{p'_1, \Phi(\mathrm{X}), \Phi(\mathrm{V})}(\mathbf{\Phi}^2_1(\eta^{!}_{p_1, \mathrm{X}, \mathrm{V}}(a))
\end{aligned}$$

This finishes the proof of Lemma 3.1.3. ☐

Consider the morphisms

$$\zeta_i : \Phi(\mathsf{I}_{p_i}(\mathcal{U})) \to \mathsf{I}_{p'_i}(\mathcal{U}')$$

given by $\zeta_i := \chi_i(\mathcal{U}) \circ \mathsf{I}_{p'_i}(\phi)$ and

$$\widetilde{\zeta}_i : \Phi(\mathsf{I}_{p_i}(\widetilde{\mathcal{U}}_1)) \to \mathsf{I}_{p'_i}(\widetilde{\mathcal{U}}'_1)$$

given by $\widetilde{\zeta}_i := \chi_i(\widetilde{\mathcal{U}}_1) \circ \mathsf{I}_{p'_i}(\widetilde{\phi}_1)$. (Recall from [18, §6] the morphisms $\xi_{\mathbf{\Phi}} : \Phi(\mathsf{I}_p(\mathcal{U})) \to \mathsf{I}_{p'})(\mathcal{U}')$ and $\widetilde{\xi}_{\mathbf{\Phi}} : \Phi(\mathsf{I}_p(\widetilde{\mathcal{U}})) \to \mathsf{I}_{p'}(\widetilde{\mathcal{U}}')$ introduced, for a universe category functor $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi})$, by defining $\xi_{\mathbf{\Phi}} := \chi_{\mathbf{\Phi}}(\mathcal{U}) \circ \mathsf{I}_{p'}(\phi)$ and $\widetilde{\xi}_{\mathbf{\Phi}} := \chi_{\mathbf{\Phi}}(\widetilde{\mathcal{U}}) \circ \mathsf{I}_{p'}(\widetilde{\phi})$. Note that $\zeta_i = \xi_{\mathbf{\Phi}_i}$ and $\widetilde{\zeta}_1 = \widetilde{\xi}_{\mathbf{\Phi}_1}$, but $\widetilde{\zeta}_2 \neq \widetilde{\xi}_{\mathbf{\Phi}_2}$.)

The following theorem will be used to formulate the condition of compatibility of a universe functor with full J-structures.

Theorem **3.1.4.** — *Under the previous assumptions and notation the morphisms* $\zeta_1, \zeta_2, \widetilde{\zeta}_1, \widetilde{\zeta}_2$ *form a morphism from the square*

$$\begin{tikzcd}
\Phi(\mathsf{I}_{p_2}(\widetilde{\mathcal{U}}_1)) \arrow[r, "\Phi(\mathsf{I}^g(\widetilde{\mathcal{U}}_1))"] \arrow[d, "\Phi(\mathsf{I}_{p_2}(p_2))"'] & \Phi(\mathsf{I}_{p_1}(\widetilde{\mathcal{U}}_1)) \arrow[d, "\Phi(\mathsf{I}_{p_1}(p_2))"] \\
\Phi(\mathsf{I}_{p_2}(\mathcal{U})) \arrow[r, "\Phi(\mathsf{I}^g(\mathcal{U}))"] & \Phi(\mathsf{I}_{p_1}(\mathcal{U}))
\end{tikzcd}$$

to the square

$$\begin{tikzcd}
\mathsf{I}_{p'_2}(\widetilde{\mathcal{U}}'_1) \arrow[r, "\mathsf{I}^{g'}(\widetilde{\mathcal{U}}'_1)"] \arrow[d, "\mathsf{I}_{p'_2}(p'_2)"'] & \mathsf{I}_{p'_1}(\widetilde{\mathcal{U}}'_1) \arrow[d, "\mathsf{I}_{p'_1}(p'_2)"] \\
\mathsf{I}_{p'_2}(\mathcal{U}') \arrow[r, "\mathsf{I}^{g'}(\mathcal{U}')"] & \mathsf{I}_{p'_1}(\mathcal{U}')
\end{tikzcd}$$

Proof. — We need to prove commutativity of the outer squares of the following four diagrams:

$$\begin{tikzcd}
\Phi(\mathsf{I}_{p_2}(\widetilde{\mathcal{U}}_1)) \arrow[r, "\chi_2(\widetilde{\mathcal{U}}_1)"] \arrow[d, "\mathsf{I}^{g'}(\Phi(\widetilde{\mathcal{U}}_1))"'] & \mathsf{I}_{p'_2}(\Phi(\widetilde{\mathcal{U}}_1)) \arrow[r, "\mathsf{I}_{p'_2}(\widetilde{\phi}_1)"] \arrow[d, "\mathsf{I}^{g'}(\Phi(\widetilde{\mathcal{U}}_1))"] & \mathsf{I}_{p'_2}(\widetilde{\mathcal{U}}'_1) \arrow[d, "\mathsf{I}^{g'}(\widetilde{\mathcal{U}}'_1)"] \\
\Phi(\mathsf{I}_{p_1}(\widetilde{\mathcal{U}}_1)) \arrow[r, "\chi_1(\widetilde{\mathcal{U}}_1)"] & \mathsf{I}_{p'_1}(\Phi(\widetilde{\mathcal{U}}_1)) \arrow[r, "\mathsf{I}_{p'_1}(\widetilde{\phi}_1)"] & \mathsf{I}_{p'_1}(\widetilde{\mathcal{U}}'_1)
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi(\mathsf{I}_{p_2}(\mathcal{U}_1)) \arrow[r, "\chi_2(\mathcal{U}_1)"] \arrow[d, "\Phi(\mathsf{I}^g(\mathcal{U}_1))"'] & \mathsf{I}_{p'_2}(\Phi(\mathcal{U}_1)) \arrow[r, "\mathsf{I}_{p'_2}(\widetilde{\phi}_1)"] \arrow[d, "\mathsf{I}^{g'}(\Phi(\mathcal{U}_1))"] & \mathsf{I}_{p'_2}(\mathcal{U}'_1) \arrow[d, "\mathsf{I}^{g'}(\mathcal{U}'_1)"] \\
\Phi(\mathsf{I}_{p_1}(\mathcal{U}_1)) \arrow[r, "\chi_1(\mathcal{U}_1)"] & \mathsf{I}_{p'_1}(\Phi(\mathcal{U}_1)) \arrow[r, "\mathsf{I}_{p'_1}(\widetilde{\phi}_1)"] & \mathsf{I}_{p'_1}(\mathcal{U}'_1)
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi(\mathsf{I}_{p_2}(\widetilde{\mathcal{U}}_1)) \arrow[r, "\chi_2(\widetilde{\mathcal{U}}_1)"] \arrow[d, "\Phi(\mathsf{I}_{p_2}(p_1))"'] & \mathsf{I}_{p'_2}(\Phi(\widetilde{\mathcal{U}}_1)) \arrow[r, "\mathsf{I}_{p'_2}(\widetilde{\phi}_1)"] \arrow[d, "\mathsf{I}_{p'_2}(\Phi(p_1))"] & \mathsf{I}_{p'_2}(\widetilde{\mathcal{U}}'_1) \arrow[d, "\mathsf{I}_{p'_2}(p'_1)"] \\
\Phi(\mathsf{I}_{p_2}(\mathcal{U}_1)) \arrow[r, "\chi_2(\mathcal{U}_1)"] & \mathsf{I}_{p'_2}(\Phi(\mathcal{U}_1)) \arrow[r, "\mathsf{I}_{p'_2}(\widetilde{\phi}_1)"] & \mathsf{I}_{p'_2}(\mathcal{U}'_1)
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi(\mathsf{I}_{p_1}(\widetilde{\mathcal{U}}_1)) \arrow[r, "\chi_1(\widetilde{\mathcal{U}}_1)"] \arrow[d, "\Phi(\mathsf{I}_{p_1}(p_1))"'] & \mathsf{I}_{p'_1}(\Phi(\widetilde{\mathcal{U}}_1)) \arrow[r, "\mathsf{I}_{p'_1}(\widetilde{\phi}_1)"] \arrow[d, "\mathsf{I}_{p'_1}(\Phi(p_1))"] & \mathsf{I}_{p'_2}(\widetilde{\mathcal{U}}'_1) \arrow[d, "\mathsf{I}_{p'_1}(p'_1)"] \\
\Phi(\mathsf{I}_{p_1}(\mathcal{U}_1)) \arrow[r, "\chi_1(\mathcal{U}_1)"] & \mathsf{I}_{p'_1}(\Phi(\mathcal{U}_1)) \arrow[r, "\mathsf{I}_{p'_1}(\widetilde{\phi}_1)"] & \mathsf{I}_{p'_1}(\mathcal{U}'_1)
\end{tikzcd}$$

The left squares in the first and the second diagram are commutative by Lemma 3.1.3.

The left squares in the third and the fourth diagram are commutative by [18, Lemma 5.7].

The right hand side squares in the first and second diagram commute by Lemma 2.2.4.

The right hand side squares of the third and the fourth diagram commute because $\mathsf{I}_{p'_i}$ are functorial and therefore take commutative squares to commutative squares. ☐

3.2. *Universe category functors compatible with J-structures*

Let us define now conditions on functors of universe categories that reflect the idea of compatibility with the J0- J1- and J2-structures on the universes. For any functor $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi}) : (\mathcal{C}, p, pt) \to (\mathcal{C}', p', pt')$ of universe categories (the notion was recalled in the previous section), for any $X \in \mathcal{C}$, and for any $F : X \to \mathcal{U}$, the morphism

$$\Phi(\mathsf{p}_{X,F}) * (\Phi(\mathsf{Q}(F)) \circ \widetilde{\phi}) : \Phi((X; F)) \to (\Phi(X); \Phi(F) \circ \phi)'$$

is an isomorphism, and it will be denoted by $\Phi_{X,F}$. (This isomorphism appears in (3.3) above, as the inverse $\Phi_{X,F} = \iota^{-1}_{\mathbf{\Phi},X,F}$.) Let $\Phi\widetilde{\mathcal{U}}p$ be the composition

$$\Phi((\widetilde{\mathcal{U}}; p)) \xrightarrow{\Phi_{\widetilde{\mathcal{U}},p}} (\Phi(\widetilde{\mathcal{U}}); \Phi(p) \circ \phi) = (\Phi(\widetilde{\mathcal{U}}); \widetilde{\phi} \circ p') \xrightarrow{\mathsf{Q}'(\widetilde{\phi},p')} (\widetilde{\mathcal{U}}'; p')$$

We have another description of this morphism given by the following lemma.

Lemma **3.2.1.** — *One has:*

$$\Phi\widetilde{\mathcal{U}}p = (\Phi(\mathsf{p}_{\widetilde{\mathcal{U}},p}) \circ \widetilde{\phi}) * (\Phi(\mathsf{Q}(p)) \circ \widetilde{\phi})$$

Proof. — One has

$$\Phi\widetilde{\mathcal{U}}p \circ p'_{\widetilde{\mathcal{U}}',p'} = \Phi_{\widetilde{\mathcal{U}},p} \circ \mathsf{Q}'(\widetilde{\phi}, p') \circ p'_{\widetilde{\mathcal{U}}',p'} = \Phi_{\widetilde{\mathcal{U}},p} \circ \mathsf{p}_{\Phi(\widetilde{\mathcal{U}}),\widetilde{\phi}\circ p'} \circ \widetilde{\phi} = \Phi(\mathsf{p}_{\widetilde{\mathcal{U}},p}) \circ \widetilde{\phi},$$

where the second equality is by definition of $\mathsf{Q}'(-,-)$ and the third equality is by definition of $\Phi_{\widetilde{\mathcal{U}},p}$. Then

$$\begin{aligned}\Phi\widetilde{\mathcal{U}}p \circ \mathsf{Q}'(p') &= \Phi_{\widetilde{\mathcal{U}},p} \circ \mathsf{Q}'(\widetilde{\phi}, p') \circ \mathsf{Q}'(p') \\ &= \Phi_{\widetilde{\mathcal{U}},p} \circ \mathsf{Q}(\widetilde{\phi} \circ p')\end{aligned}$$

$$
\begin{aligned}
&= \Phi_{\widetilde{\mathcal{U}},p} \circ \mathsf{Q}(\Phi(p) \circ \phi) \\
&= \Phi(\mathsf{Q}(p)) \circ \widetilde{\phi},
\end{aligned}
$$

where again the second equality is by definition of $\mathsf{Q}(-,-)$ and the fourth equality is by definition of $\Phi_{\widetilde{\mathcal{U}},p}$. ☐

Lemma **3.2.2.** — *For $s, s' : \mathrm{Y} \to \widetilde{\mathcal{U}}$ such that $s \circ p = s' \circ p$ one has*

$$\Phi(s * s') \circ \Phi\widetilde{\mathcal{U}}p = \Phi(s \circ \widetilde{\phi}) * \Phi(s' \circ \widetilde{\phi})$$

and thus

$$\Phi(\Delta) \circ \Phi\widetilde{\mathcal{U}}p = \widetilde{\phi} * \widetilde{\phi}.$$

Proof. — Using Lemma 3.2.1 we have

$$\Phi(s * s') \circ \Phi\widetilde{\mathcal{U}}p \circ p'_{\widetilde{\mathcal{U}}',p'} = \Phi(s * s') \circ \Phi(\mathsf{p}_{\widetilde{\mathcal{U}},p}) \circ \widetilde{\phi} = s \circ \widetilde{\phi}$$

and

$$\Phi(s * s') \circ \Phi\widetilde{\mathcal{U}}p \circ \mathsf{Q}'(p') = \Phi(s * s') \circ \Phi(\mathsf{Q}(p)) \circ \widetilde{\phi} = s' \circ \widetilde{\phi}$$

The particular case of Δ follows from the fact that $\Delta = 1_{\widetilde{\mathcal{U}}} * 1_{\widetilde{\mathcal{U}}}$. ☐

Lemma **3.2.3.** — *The square*

$$
\begin{tikzcd}
\Phi((\widetilde{\mathcal{U}}; p)) \arrow[r, "\Phi\widetilde{\mathcal{U}}p"] \arrow[d, "\Phi(\mathsf{p}_{\widetilde{\mathcal{U}},p})"'] & (\widetilde{\mathcal{U}}'; p') \arrow[d, "p'_{\widetilde{\mathcal{U}}',p'}"] \\
\Phi(\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}"] & \widetilde{\mathcal{U}}'
\end{tikzcd}
\tag{\textbf{3.6}}
$$

is a pullback square.

Proof. — This square is equal to the composition of two squares

$$
\begin{tikzcd}
\Phi((\widetilde{\mathcal{U}}; p)) \arrow[r, "\Phi_{\widetilde{\mathcal{U}},p}"] \arrow[d, "\Phi(\mathsf{p}_{\widetilde{\mathcal{U}},p})"'] & (\Phi(\widetilde{\mathcal{U}}); \widetilde{\phi} \circ p') \arrow[r, "\mathsf{Q}'(\widetilde{\phi},p')"] \arrow[d, "\mathsf{p}_{\Phi(\widetilde{\mathcal{U}}),\widetilde{\phi}\circ p'}"'] & (\widetilde{\mathcal{U}}'; p') \arrow[d, "p'_{\widetilde{\mathcal{U}}',p'}"] \\
\Phi(\widetilde{\mathcal{U}}) \arrow[r, "="] & \Phi(\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}"] & \widetilde{\mathcal{U}}'
\end{tikzcd}
$$

The right hand side square is a pullback square (2.16). The left hand side square is a pullback square as a commutative square whose sides are isomorphisms. We conclude that the composition of these two squares is a pullback square. ☐

Definition **3.2.4.** — *Let* $\mathrm{E}q$ *be a* $\mathcal{J}0$*-structure on* p *and* $\mathrm{E}q'$ *a* $\mathcal{J}0$*-structure on* p'*. A universe category functor* $(\Phi, \phi, \widetilde{\phi})$ *is said to be compatible with* $\mathrm{E}q$ *and* $\mathrm{E}q'$ *if the square*

(**3.7**)
$$\begin{tikzcd}
\Phi((\widetilde{\mathcal{U}};p)) \arrow[r, "\Phi(\mathrm{E}q)"] \arrow[d, "\Phi\widetilde{\mathcal{U}}p"'] & \Phi(\mathcal{U}) \arrow[d, "\phi"] \\
(\widetilde{\mathcal{U}}';p') \arrow[r, "\mathrm{E}q'"] & \mathcal{U}'
\end{tikzcd}$$

commutes.

Let $\mathrm{E}q$, $\mathrm{E}q'$ be as above. Let $(\Phi, \phi, \widetilde{\phi})$ be a universe functor compatible with $\mathrm{E}q$ and $\mathrm{E}q'$. Define a morphism

$$\widetilde{\phi}_{\mathrm{E}} : \Phi(\mathrm{E}\widetilde{\mathcal{U}}) \to \mathrm{E}\widetilde{\mathcal{U}}' = ((\widetilde{\mathcal{U}}';p'), \mathrm{E}q')$$

as

$$\widetilde{\phi}_{\mathrm{E}} := (\Phi(\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q}) \circ \Phi\widetilde{\mathcal{U}}p) * (\Phi(\mathsf{Q}(\mathrm{E}q)) \circ \widetilde{\phi}).$$

Lemma **3.2.5.** — *Let* $\mathrm{E}q$, $\mathrm{E}q'$ *be as above. Let* $(\Phi, \phi, \widetilde{\phi})$ *be a universe functor compatible with* $\mathrm{E}q$*, and* $\mathrm{E}q'$*. Then the square*

$$\begin{tikzcd}
\Phi(\mathrm{E}\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}_{\mathrm{E}}"] \arrow[d, "\Phi(\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q})"'] & \mathrm{E}\widetilde{\mathcal{U}}' \arrow[d, "\mathsf{p}_{(\widetilde{\mathcal{U}}';p'),\mathrm{E}q'}"] \\
\Phi((\widetilde{\mathcal{U}};p)) \arrow[r, "\Phi\widetilde{\mathcal{U}}p"] & (\widetilde{\mathcal{U}}';p')
\end{tikzcd}$$

is a pullback square.

Proof. — Consider the diagram

(**3.8**)
$$\begin{tikzcd}
\Phi(\mathrm{E}\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}_{\mathrm{E}}"] \arrow[d, "\Phi(\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q})"'] & \mathrm{E}\widetilde{\mathcal{U}}' \arrow[r, "\mathsf{Q}(\mathrm{E}q')"] \arrow[d, "\mathsf{p}_{(\widetilde{\mathcal{U}}';p'),\mathrm{E}q'}"'] & \widetilde{\mathcal{U}}' \arrow[d, "p'"] \\
\Phi((\widetilde{\mathcal{U}};p)) \arrow[r, "\Phi\widetilde{\mathcal{U}}p"] & (\widetilde{\mathcal{U}}';p') \arrow[r, "\mathrm{E}q'"] & \mathcal{U}'
\end{tikzcd}$$

The outer square of this diagram is equal to the outer square of the diagram

$$\begin{tikzcd}
\Phi(\mathrm{E}\widetilde{\mathcal{U}}) \arrow[r, "\Phi(\mathsf{Q}(\mathrm{E}q))"] \arrow[d, swap, "\Phi(\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q})"] & \Phi(\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}"] \arrow[d, swap, "\Phi(p)"] & \widetilde{\mathcal{U}}' \arrow[d, "p'"] \\
\Phi((\widetilde{\mathcal{U}};p)) \arrow[r, "\Phi(\mathrm{E}q)"] & \Phi(\mathcal{U}) \arrow[r, "\phi"] & \mathcal{U}',
\end{tikzcd} \tag{3.9}$$

where the equality of the lower horizontal arrows follows from the commutativity of the square (3.7). The left hand side square of this diagram is a pullback square because Φ takes canonical squares to pullback squares. The right hand side square is a pullback square by definition of a functor of universe categories. Therefore the outer square is a pullback square. The right hand side square of (3.8) is a canonical square and therefore a pullback square. We conclude that the left hand square of (3.8) is a pullback square. ☐

Lemma **3.2.6.** — *Let* $\mathrm{E}q$, $\mathrm{E}q'$ *be as above. Let* $(\Phi, \phi, \widetilde{\phi})$ *be a functor of universe categories compatible with* $\mathrm{E}q$, *and* $\mathrm{E}q'$. *Then the square*

$$\begin{tikzcd}
\Phi(\mathrm{E}\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}_{\mathrm{E}}"] \arrow[d, swap, "\Phi(p\mathrm{E}\widetilde{\mathcal{U}})"] & \mathrm{E}\widetilde{\mathcal{U}}' \arrow[d, "p\mathrm{E}\widetilde{\mathcal{U}}'"] \\
\Phi(\mathcal{U}) \arrow[r, "\phi"] & \mathcal{U}'
\end{tikzcd} \tag{3.10}$$

is a pullback square.

Proof. — It follows from the fact that the square (3.10) is equal to the vertical composition of the squares of Lemmas 3.2.5 and 3.2.3 with the square (3.6). ☐

Definition **3.2.7.** — *Let* $\mathrm{E}q$, $\mathrm{E}q'$ *be as above and let* Ω, Ω' *be* $\mathcal{J}1$*-structures over* $\mathrm{E}q$ *and* $\mathrm{E}q'$ *respectively. A universe category functor* $(\Phi, \phi, \widetilde{\phi})$ *is said to be compatible with* Ω *and* Ω' *if the square*

$$\begin{tikzcd}
\Phi(\widetilde{\mathcal{U}}) \arrow[r, "\Phi(\Omega)"] \arrow[d, swap, "\widetilde{\phi}"] & \Phi(\widetilde{\mathcal{U}}) \arrow[d, "\widetilde{\phi}"] \\
\widetilde{\mathcal{U}}' \arrow[r, "\Omega'"] & \widetilde{\mathcal{U}}'
\end{tikzcd}$$

commutes.

Lemma **3.2.8.** — *Let* $\mathrm{E}q$, Ω *and* $\mathrm{E}q'$, Ω' *be as above and let* $\mathbf{\Phi}$ *be a universe category functor compatible with* $\mathrm{E}q$, $\mathrm{E}q'$ *and* Ω, Ω'. *Then the square*

$$\begin{tikzcd}
\Phi(\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}"] \arrow[d, "\Phi(\omega)"'] & \widetilde{\mathcal{U}}' \arrow[d, "\omega'"] \\
\Phi(\mathrm{E}\widetilde{\mathcal{U}}) \arrow[r, "\widetilde{\phi}_{\mathrm{E}}"] & \mathrm{E}\widetilde{\mathcal{U}}'
\end{tikzcd}$$

commutes.

Proof. — Since $\mathrm{E}\widetilde{\mathcal{U}}' = ((\widetilde{\mathcal{U}}'; p'); \mathrm{E}q')$ it is sufficient to verify that the compositions of the two paths in the square with $\mathsf{p}_{(\widetilde{\mathcal{U}}';p'),\mathrm{E}q'}$ and $\mathsf{Q}(\mathrm{E}q')$ coincide. We have:

$$\widetilde{\phi} \circ \omega' \circ \mathsf{Q}(\mathrm{E}q') = \widetilde{\phi} \circ \Omega'$$

by definition of ω'. On the other hand

$$\Phi(\omega) \circ \widetilde{\phi}_{\mathrm{E}} \circ \mathsf{Q}(\mathrm{E}q') = \Phi(\omega) \circ \Phi(\mathsf{Q}(\mathrm{E}q)) \circ \widetilde{\phi} = \Phi(\Omega) \circ \widetilde{\phi},$$

where the first equation holds by definition of $\widetilde{\phi}_{\mathrm{E}}$. The proof follows now from the assumption that $\mathbf{\Phi}$ is compatible with Ω and Ω'. ☐

To formulate the condition of compatibility of a universe functor with full J-structures on $\mathcal{C}$ and $\mathcal{C}'$ we will use Theorem 3.1.4.

Let $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi}) : (\mathcal{C}, p, pt) \to (\mathcal{C}', p', pt')$ be a functor of universe categories. In view of Lemma 3.2.6, if Φ is compatible with $\mathrm{E}q$ and $\mathrm{E}q'$ then the triple $\mathbf{\Phi}_{\mathrm{E}} := (\Phi, \phi, \widetilde{\phi}_{\mathrm{E}})$ is a functor of universe categories as well, from $(\mathcal{C}, p\mathrm{E}\widetilde{\mathcal{U}}, pt)$ to $(\mathcal{C}', p\mathrm{E}\widetilde{\mathcal{U}}', pt')$. If, in addition, $\mathbf{\Phi}$ is compatible with Ω and Ω' then, by Lemma 3.2.8, the morphisms ω and ω' satisfy the conditions on morphisms g and g' of Section 3.1.

Let

$$\xi_{\mathbf{\Phi}} : \Phi(\mathsf{I}_p(\mathcal{U})) \to \mathsf{I}_{p'}(\mathcal{U}')$$
$$\widetilde{\xi}_{\mathbf{\Phi}} : \Phi(\mathsf{I}_p(\widetilde{\mathcal{U}})) \to \mathsf{I}_{p'}(\widetilde{\mathcal{U}}')$$

denote the compositions $\chi_{\mathbf{\Phi}}(\mathcal{U}) \circ \mathsf{I}_{p'}(\phi)$ and $\chi_{\mathbf{\Phi}}(\widetilde{\mathcal{U}}) \circ \mathsf{I}_{p'}(\widetilde{\phi})$, respectively,[10] and let

$$\zeta_{\mathbf{\Phi}} : \Phi(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U})) \to \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\mathcal{U}')$$
$$\widetilde{\zeta}_{\mathbf{\Phi}} : \Phi(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}})) \to \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\mathcal{U}}')$$

---

[10] These maps were introduced in [18, §6], and were recalled above.

be given by the following compositions.

$$\zeta_{\mathbf{\Phi}} := \chi_{\mathbf{\Phi}_{\mathrm{E}}}(\mathcal{U}) \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\phi) \tag{\textbf{3.11}}$$

$$\widetilde{\zeta}_{\mathbf{\Phi}} := \chi_{\mathbf{\Phi}_{\mathrm{E}}}(\widetilde{\mathcal{U}}) \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\phi}) \tag{\textbf{3.12}}$$

Don't confuse these maps with ζ_i and $\widetilde{\zeta}_i$, which were introduced above. Also note that $\zeta_{\mathbf{\Phi}} = \xi_{\mathbf{\Phi}_{\mathrm{E}}}$, but $\widetilde{\zeta}_{\mathbf{\Phi}}$ is different from $\widetilde{\xi}_{\mathbf{\Phi}_{\mathrm{E}}}$, since the latter is equal to the composition $\chi_{\mathbf{\Phi}_{\mathrm{E}}}(\mathrm{E}\widetilde{\mathcal{U}}) \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\phi}_{\mathrm{E}})$.

Applying Theorem 3.1.4 in this context we get the following.

Theorem **3.2.9.** — *Let* $\mathbf{\Phi}$ *be a functor of universe categories compatible with the $\mathcal{J}1$-structures* $(\mathrm{E}q, \Omega)$ *and* $(\mathrm{E}q', \Omega')$ *on p and p' respectively. Then the morphisms* $\xi_{\mathbf{\Phi}}, \widetilde{\xi}_{\mathbf{\Phi}}, \zeta_{\mathbf{\Phi}}, \widetilde{\zeta}_{\mathbf{\Phi}}$ *form a morphism from the square*

$$\begin{tikzcd}
\Phi(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}})) \arrow[r, "\Phi(\mathsf{I}^{\omega}(\widetilde{\mathcal{U}}))"] \arrow[d, "\Phi(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(p))"'] & \Phi(\mathsf{I}_p(\widetilde{\mathcal{U}})) \arrow[d, "\Phi(\mathsf{I}_p(p))"] \\
\Phi(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U})) \arrow[r, "\Phi(\mathsf{I}^{\omega}(\mathcal{U}))"] & \Phi(\mathsf{I}_p(\mathcal{U}))
\end{tikzcd}$$

to the square

$$\begin{tikzcd}
\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\mathcal{U}}') \arrow[r, "\mathsf{I}^{\omega'}(\widetilde{\mathcal{U}}')"] \arrow[d, "\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(p')"'] & \mathsf{I}_{p'}(\widetilde{\mathcal{U}}') \arrow[d, "\mathsf{I}_{p'}(p')"] \\
\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\mathcal{U}') \arrow[r, "\mathsf{I}^{\omega}(\mathcal{U}')"] & \mathsf{I}_{p'}(\mathcal{U}')
\end{tikzcd}$$

Let $\mathrm{R}_{\mathbf{\Phi}}$ denote the composite map

$$\Phi((\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^{\omega}(\mathcal{U})) \times_{\mathsf{I}_p(\mathcal{U})} (\mathsf{I}_p(\widetilde{\mathcal{U}}), \mathsf{I}_p(p))) \tag{\textbf{3.13}}$$

$$\to \Phi(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^{\omega}(\mathcal{U})) \times_{\Phi(\mathsf{I}_p(\mathcal{U}))} \Phi(\mathsf{I}_p(\widetilde{\mathcal{U}}), \mathsf{I}_p(p)) \tag{\textbf{3.14}}$$

$$\to (\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\mathcal{U}'), \mathsf{I}^{\omega'}(\mathcal{U}')) \times_{\mathsf{I}_{p'}(\mathcal{U}')} (\mathsf{I}_{p'}(\widetilde{\mathcal{U}}'), \mathsf{I}_{p'}(p')), \tag{\textbf{3.15}}$$

where the second arrow is defined by $\xi_{\mathbf{\Phi}}, \widetilde{\xi}_{\mathbf{\Phi}}$ and $\zeta_{\mathbf{\Phi}}$ in view of Theorem 3.2.9.

Definition **3.2.10.** — *Let* $\mathrm{E}q$, $\mathrm{E}q'$, Ω *and* Ω' *be as above. Let* $\mathrm{J}p$ *and* $\mathrm{J}p'$ *be $\mathcal{J}2$-structures over* $(\mathrm{E}q, \Omega)$ *and* $(\mathrm{E}q', \Omega')$ *respectively. A universe category functor* $(\Phi, \phi, \widetilde{\phi})$ *is said to be* compatible *with* $\mathrm{J}p$ *and* $\mathrm{J}p'$ *if it is compatible with* $\mathrm{E}q$, $\mathrm{E}q'$ *and* Ω, Ω' *in the sense of Definitions 3.2.4 and 3.2.7* *respectively, the square*

$$\begin{tikzcd}
\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\mathcal{U}}') \arrow[r, "\mathsf{I}^{\omega'}(\widetilde{\mathcal{U}}')"] \arrow[d, "\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(p')"'] & \mathsf{I}_{p'}(\widetilde{\mathcal{U}}') \arrow[d, "\mathsf{I}_{p'}(p')"] \\
\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\mathcal{U}') \arrow[r, "\mathsf{I}^{\omega}(\mathcal{U}')"] & \mathsf{I}_{p'}(\mathcal{U}')
\end{tikzcd}$$

commutes, and the square

$$\begin{tikzcd}
\Phi((\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^{\omega}(\mathcal{U})) \times_{\mathsf{I}_p(\mathcal{U})} (\mathsf{I}_p(\widetilde{\mathcal{U}}), \mathsf{I}_p(p))) \arrow[r, "\mathrm{R}_\Phi"] \arrow[d, "\Phi(\mathrm{J}p)"'] & (\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\mathcal{U}'), \mathsf{I}^{\omega'}(\mathcal{U}')) \times_{\mathsf{I}_{p'}(\mathcal{U}')} (\mathsf{I}_{p'}(\widetilde{\mathcal{U}}'), \mathsf{I}_{p'}(p')) \arrow[d, "\mathrm{J}p'"] \\
\Phi(\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}})) \arrow[r, "\widetilde{\zeta}_\Phi"] & \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\mathcal{U}}')
\end{tikzcd}$$

commutes.

In Lemma 3.4.1 we will use the definition above to show that universe category functors compatible with J-structures give rise to homomorphisms of C-systems with J-structures.

3.3. *Homomorphisms of C-systems compatible with J-structures*

Definition **3.3.1.** — *Let* $\mathrm{H} : \mathbf{C} \to \mathbf{C}'$ *be a homomorphism of C-systems.*

1. *Let* Id *and* Id' *be J0-structures on* $\mathbf{C}$ *and* $\mathbf{C}'$ *respectively. Then* H *is called a homomorphism of C-systems with J0-structures* $(\mathbf{C}, \mathsf{Id}) \to (\mathbf{C}, \mathsf{Id}')$ *if for each* $\Gamma \in \mathrm{O}b(\mathbf{C})$ *and* $o, o' \in \widetilde{\mathrm{O}b}_1(\Gamma)$ *such that* $\partial(o) = \partial(o')$*, one has*

$$\mathrm{H}(\mathsf{Id}_\Gamma(o, o')) = \mathsf{Id}'_{\mathrm{H}(\Gamma)}(\mathrm{H}(o), \mathrm{H}(o'))$$

   (the right hand side of the equality makes sense because H *commutes with* ∂*).*
2. *Let* $\mathsf{Id}, \mathsf{Id}'$ *be as above and let* $\mathsf{refl}, \mathsf{refl}'$ *be J1-structures over* Id *and* Id' *respectively. A homomorphism of C-systems with J0-structures* $\mathrm{H} : (\mathbf{C}, \mathsf{Id}) \to (\mathbf{C}', \mathsf{Id}')$ *is called a homomorphism of C-systems with J1-structures*

$$(\mathbf{C}, \mathsf{Id}, \mathsf{refl}) \to (\mathbf{C}', \mathsf{Id}', \mathsf{refl}')$$

   if for all $\Gamma \in \mathrm{O}b(\mathbf{C})$ *and* $o \in \widetilde{\mathrm{O}b}_1(\Gamma)$ *one has*

$$\mathrm{H}(\mathsf{refl}(o)) = \mathsf{refl}'(\mathrm{H}(o))$$

For a C-system $\mathbf{C}$ with a J0-structure Id and a J1-structure refl over Id define $\mathrm{J}dom(\mathbf{C}, \mathsf{Id}, \mathsf{refl})$ as the set of quadruples $(\Gamma, \mathrm{T}, \mathrm{P}, s0)$, where $\Gamma \in \mathrm{O}b$, $\mathrm{T} \in \mathrm{O}b_1(\Gamma)$,

$P \in Ob_1(\mathsf{Id}_3(T))$ and $s0 \in \widetilde{Ob}(\mathsf{rf}_T^*(P))$. Equivalently we can say that $Jdom(\mathbf{C}, \mathsf{Id}, \mathsf{refl})$ is the subset in $Ob \times Ob \times Ob \times \widetilde{Ob}$ that consists of quadruples $(\Gamma, T, P, s0)$, where $\mathsf{ft}(T) = \Gamma$, $\mathsf{ft}(P) = \mathsf{Id}_3(T)$ and $\partial(s0) = \mathsf{rf}_T^*(P)$. Then a J2-structure is defined by a map $Jdom \to \widetilde{Ob}$ with some properties.

Lemma **3.3.2.** — *Let* $H : \mathbf{C} \to \mathbf{C}'$ *be a homomorphism of C-systems. Let* $\Gamma, X, Y \in Ob(\mathbf{C})$, $m, n \in \mathbf{N}$ *and suppose that* $\mathsf{ft}^m(X) = \mathsf{ft}^n(Y) = \Gamma$. *Let* $f : X \to Y$ *be a morphism over* Γ *and let* $F : \Gamma' \to \Gamma$ *be a morphism. Then*

$$H(F^*(f)) = H(F)^*(H(f))$$

Proof. — This is easy to show from the defining properties of $F^*(f)$ and $H(F)^*(H(f))$. ☐

Lemma **3.3.3.** — *Let* Id, Id', refl *and* refl' *be as in Definition 3.3.1 and let*

$$H : (\mathbf{C}, \mathsf{Id}, \mathsf{refl}) \to (\mathbf{C}', \mathsf{Id}', \mathsf{refl}')$$

be a homomorphism of C-systems with J1-structures. Then for all elements $(\Gamma, T, P, s0)$ *of* $Jdom(\mathsf{Id}, \mathsf{refl})$ *one has* $(H(\Gamma), H(T), H(P), H(s0)) \in Jdom(\mathsf{Id}', \mathsf{refl}')$.

Proof. — We have $\mathsf{ft}(H(T)) = H(\mathsf{ft}(T)) = H(\Gamma)$ and $\mathsf{ft}(H(P)) = H(\mathsf{ft}(P)) = H(\mathsf{Id}_3(T))$. We also have $\partial(H(s0)) = H(\partial(s0)) = H(\mathsf{rf}_T^*(P))$. By Lemma 3.3.2 we further have

$$H(\mathsf{rf}_T^*(P)) = H(\mathsf{rf}_T)^*(H(P))$$

It remains to show that the following two equations.

$$H(\mathsf{Id}_3(T)) = \mathsf{Id}'_3(H(T)) \tag{3.16}$$

$$H(\mathsf{rf}_T) = \mathsf{rf}'_{H(T)} \tag{3.17}$$

They follow by a lengthy computation from the defining equations (2.9) and (2.11), which we omit. ☐

Definition **3.3.4.** — *Let* Id, Id', refl *and* refl' *be as in Definition 3.3.1 and let* J, J' *be J2-structures over* $(\mathsf{Id}, \mathsf{refl})$ *and* $(\mathsf{Id}', \mathsf{refl}')$ *respectively. A homomorphism of C-systems with J1-structures*

$$H : (\mathbf{C}, \mathsf{Id}, \mathsf{refl}) \to (\mathbf{C}', \mathsf{Id}', \mathsf{refl}')$$

is called a homomorphism of C-systems with J-structures

$$(\mathbf{C}, \mathsf{Id}, \mathsf{refl}, \mathsf{J}) \to (\mathbf{C}', \mathsf{Id}', \mathsf{refl}', \mathsf{J}')$$

if for all $\Gamma \in Ob(\mathbf{C})$, $T \in Ob_1(\Gamma)$, $P \in Ob_1(\mathsf{Id}_3(T))$ *and* $s0 \in \widetilde{Ob}(\mathsf{rf}_T^*(P))$ *one has*

$$H(\mathsf{J}(\Gamma, T, P, s0)) = \mathsf{J}'(H(\Gamma), H(T), H(P), H(s0)),$$

where the right hand side of the equation makes sense by Lemma 3.3.3.

3.4. *Functoriality of the $\mathcal{J}$-structures* $(\mathsf{Id}_{Eq}, \mathsf{refl}_\Omega, \mathsf{J}_{Jp})$

Let us first recall that by [17, Construction 3.3] any universe category functor

$$\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi}) : (\mathcal{C}, p, pt) \to (\mathcal{C}', p', pt')$$

defines a homomorphism of C-systems

$$H = H(\mathbf{\Phi}) : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p').$$

To define H on objects, one defines by induction on n, for all $\Gamma \in Ob_n(CC(\mathcal{C}, p))$, pairs $(H(\Gamma), \psi_\Gamma)$, where $H(\Gamma) \in Ob(CC(\mathcal{C}', p'))$ and ψ_Γ is a morphism

$$\psi_\Gamma : int'(H(\Gamma)) \to \Phi(int(\Gamma)),$$

as follows. For $n = 0$ one has $H(()) = ()$ and $\psi_{()} : pt' \to \Phi(pt)$ is the unique morphism to a final object $\Phi(pt)$. For $T = (\Gamma, F) \in Ob_{n+1}$ one has

$$H(T) = (H(\Gamma), \psi_\Gamma \circ \Phi(F) \circ \phi),$$

in other words, that

$$u_1(H(T)) = \psi_{\mathsf{ft}(T)} \circ \Phi(u_1(T)) \circ \phi \tag{\textbf{3.18}}$$

and

$$\mathsf{ft}(H(T)) = H(\mathsf{ft}(T)). \tag{\textbf{3.19}}$$

Moreover, $\psi_{(\Gamma,F)}$ is the unique morphism $int'(H(\Gamma, F)) \to \Phi(int(\Gamma, F))$ such that

$$\psi_{(\Gamma,F)} \circ \Phi(\mathsf{Q}(F)) \circ \widetilde{\phi} = \mathsf{Q}'(\psi_\Gamma \circ \Phi(F) \circ \phi) \tag{\textbf{3.20}}$$

and

$$\psi_{(\Gamma,F)} \circ \Phi(\mathsf{p}_{\Gamma,F}) = \mathsf{p}_{H((\Gamma,F))} \circ \psi_\Gamma \tag{\textbf{3.21}}$$

Observe that ψ_Γ is automatically an isomorphism. The action of H on morphisms is given, for $f : \Gamma \to \Gamma'$, by

$$H(f) = \psi_\Gamma \circ \Phi(f) \circ \psi_{\Gamma'}^{-1}$$

Lemma **3.4.1.** — *Let* $\mathbf{\Phi}$ *be a universe category functor as above that is compatible (as defined in Definition 3.2.4) with the* $\mathcal{J}0$*-structures* $\mathrm{E}q$ *and* $\mathrm{E}q'$ *on* p *and* p' *respectively. Then the homomorphism of C-systems* $\mathrm{H} = \mathrm{H}(\mathbf{\Phi})$ *is a homomorphism of C-systems with* $\mathcal{J}0$*-structures relative to* $\mathsf{Id}_{\mathrm{E}q}$ *and* $\mathsf{Id}_{\mathrm{E}q'}$.

Proof. — Let $\mathsf{Id} = \mathsf{Id}_{\mathrm{E}q}$ and $\mathsf{Id}' = \mathsf{Id}_{\mathrm{E}q'}$. We need to check that for all $\Gamma \in \mathrm{O}b(\mathrm{CC}(\mathcal{C}, p))$ and $o, o' \in \widetilde{\mathrm{O}b}_1(\Gamma)$ such that $\partial(o) = \partial(o')$ one has

$$\mathrm{H}(\mathsf{Id}(o, o')) = \mathsf{Id}'(\mathrm{H}(o), \mathrm{H}(o'))$$

Since

$$\partial(\mathrm{H}(\mathsf{Id}(o, o')) = \mathrm{H}(\Gamma) = \mathrm{H}(\mathsf{ft}(\partial(o))) = \mathsf{ft}(\partial(\mathrm{H}(o))) = \partial(\mathsf{Id}'(\mathrm{H}(o), \mathrm{H}(o')))$$

it suffices to check that

$$u_1(\mathrm{H}(\mathsf{Id}(o, o'))) = u_1(\mathsf{Id}'(\mathrm{H}(o), \mathrm{H}(o'))).$$

We have:

$$\begin{aligned}
&u_1(\mathrm{H}(\mathsf{Id}(o, o')))\\
&= \psi_\Gamma \circ \Phi(u_1(\mathsf{Id}(o, o'))) \circ \phi && \text{(by [18, Lemma 6.1(1)])}\\
&= \psi_\Gamma \circ \Phi((\widetilde{u}_1(o) * \widetilde{u}_1(o')) \circ \mathrm{E}q) \circ \phi && \text{(by (2.43))}\\
&= \psi_\Gamma \circ \Phi(\widetilde{u}_1(o) * \widetilde{u}_1(o')) \circ \Phi(\mathrm{E}q) \circ \phi && \text{(by functoriality of } \Phi\text{)}\\
&= \psi_\Gamma \circ \Phi(\widetilde{u}_1(o) * \widetilde{u}_1(o')) \circ \Phi\widetilde{\mathcal{U}}p \circ \mathrm{E}q' && \text{(by (3.7))}\\
&= \psi_\Gamma \circ ((\Phi(\widetilde{u}_1(o)) \circ \widetilde{\phi}) * (\Phi(\widetilde{u}_1(o')) \circ \widetilde{\phi})) \circ \mathrm{E}q' && \text{(by Lemma 3.2.2)}\\
&= ((\psi_\Gamma \circ \Phi(\widetilde{u}_1(o)) \circ \widetilde{\phi}) * (\psi_\Gamma \circ \Phi(\widetilde{u}_1(o')) \circ \widetilde{\phi})) \circ \mathrm{E}q' && \text{(by (2.15))}\\
&= (\widetilde{u}_1(\mathrm{H}(o)) * \widetilde{u}_1(\mathrm{H}(o'))) \circ \mathrm{E}q' && \text{(by [18, Lemma 6.1(2)])}\\
&= u_1(\mathsf{Id}'(\mathrm{H}(o), \mathrm{H}(o))) && \text{(by (2.43))} \quad \square
\end{aligned}$$

Lemma **3.4.2.** — *Let* $\mathbf{\Phi}$ *be a universe category functor as above that is compatible with the* $(\mathcal{J}0, \mathcal{J}1)$*-structures* $(\mathrm{E}q, \Omega)$ *and* $(\mathrm{E}q', \Omega')$ *on* p *and* p' *respectively. Then the homomorphism of C-systems* $\mathrm{H} = \mathrm{H}(\mathbf{\Phi})$ *is a homomorphism of C-systems with* $(\mathcal{J}0, \mathcal{J}1)$*-structures relative to* $(\mathsf{Id}_{\mathrm{E}q}, \mathsf{refl}_\Omega)$ *and* $(\mathsf{Id}_{\mathrm{E}q'}, \mathsf{refl}_{\Omega'})$.

Proof. — Let $\mathsf{refl} = \mathsf{refl}_\Omega$ and $\mathsf{refl}' = \mathsf{refl}_{\Omega'}$. The compatibility condition is

$$\Phi(\Omega) \circ \widetilde{\phi} = \widetilde{\phi} \circ \Omega' \tag{3.22}$$

We need to check that for $\Gamma \in \mathrm{O}b(\mathrm{CC}(\mathcal{C}, p))$ and $s \in \widetilde{\mathrm{O}b}_1(\Gamma)$ one has

$$\mathrm{H}(\mathsf{refl}(s)) = \mathsf{refl}'(\mathrm{H}(s))$$

We have

$$\begin{aligned}
\mathrm{H}(\mathsf{refl}(s)) &= \mathrm{H}(\widetilde{u}_1^{-1}(\widetilde{u}_1(s)\circ\Omega)) && \text{(by (2.48))}\\
&= \widetilde{u}_1'^{-1}(\psi_\Gamma\circ\Phi(\widetilde{u}_1(s)\circ\Omega)\circ\widetilde{\phi}) && \text{(by [18, Lemma 6.1(2)])}\\
&= \widetilde{u}_1'^{-1}(\psi_\Gamma\circ\Phi(\widetilde{u}_1(s))\circ\Phi(\Omega)\circ\widetilde{\phi}) && \text{(by functoriality of }\Phi)\\
&= \widetilde{u}_1'^{-1}(\psi_\Gamma\circ\Phi(\widetilde{u}_1(s))\circ\widetilde{\phi}\circ\Omega') && \text{(by (3.22))}\\
&= \widetilde{u}_1'^{-1}(\widetilde{u}_1(\mathrm{H}(s))\circ\Omega') && \text{(by [18, Lemma 6.1(2)])}\\
&= \mathsf{refl}'(\mathrm{H}(s)) && \text{(by (2.48))} \quad \square
\end{aligned}$$

To prove the functoriality of the full J-structures we will need some lemmas first.

Recall that in [20] we let $\mathsf{p}_{\Gamma,n}:\Gamma\to\mathsf{ft}^n(\Gamma)$ denote the composition $\mathsf{p}_\Gamma\circ\cdots\circ\mathsf{p}_{\mathsf{ft}^{n-1}(\Gamma)}$ of n canonical projections.

Lemma **3.4.3.** — *Let* $\mathbf{\Phi}$ *be a universe category functor and* $\Gamma\in\mathrm{O}b(\mathrm{CC}(\mathcal{C},p))$ *be such that* $l(\Gamma)\geq n$. *Then the square*

$$\begin{tikzcd}
int'(\mathrm{H}(\Gamma)) \arrow[r, "\psi_\Gamma"] \arrow[d, "p_{\mathrm{H}(\Gamma),n}"'] & \Phi(int(\Gamma)) \arrow[d, "\Phi(\mathsf{p}_{\Gamma,n})"] \\
int'(\mathrm{H}(\mathsf{ft}^n(\Gamma))) \arrow[r, "\psi_{\mathsf{ft}^n(\Gamma)}"] & \Phi(int(\mathsf{ft}^n(\Gamma)))
\end{tikzcd}$$

commutes.

Proof. — It follows by induction from the defining relation $\psi_\Gamma\circ\Phi(\mathsf{p}_\Gamma)=\mathsf{p}_{\mathrm{H}(\Gamma)}\circ\psi_{\mathsf{ft}(\Gamma)}$ of ψ_Γ. ☐

Lemma **3.4.4.** — *Let* $\mathrm{E}q$ *and* $\mathrm{E}q'$ *be* $\mathcal{J}0$*-structures on* $(\mathcal{C},p)$ *and* $(\mathcal{C}',p')$ *respectively; let* $\mathbf{\Phi}:(\mathcal{C},p,pt)\to(\mathcal{C}',p',pt')$ *be a universe category functor compatible with* $\mathrm{E}q$ *and* $\mathrm{E}q'$ *(as defined in Definition 3.2.4); let* Id_3' *be the analogue, for the category* $\mathrm{CC}(\mathcal{C}',p')$*, of* Id_3*; let* $\mathbf{\Phi}^2_{\mathrm{E}}$ *be the analogue, for the universe* $p\mathrm{E}\widetilde{\mathcal{U}}$*, of* $\mathbf{\Phi}^2$ *(as defined in (3.4)); and let* $\mathrm{H}=\mathrm{H}(\mathbf{\Phi})$. *Then for all* $\Gamma\in\mathrm{O}b(\mathrm{CC}(\mathcal{C},p))$, $\mathrm{T}\in\mathrm{O}b_1(\Gamma)$, $\mathrm{P}\in\mathrm{O}b_1(\mathsf{Id}_3(\mathrm{T}))$*, and* $o\in\widetilde{\mathrm{O}b}(\mathrm{P})$*, one has:*

1. $(u'_{1,\mathrm{H}(\Gamma)}(\mathrm{H}(\mathrm{T})),u'_{1,\mathsf{Id}_3'(\mathrm{H}(\mathrm{T}))}(\mathrm{H}(\mathrm{P})))$ *is a well defined element of* $\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\Phi(int(\Gamma)),\mathcal{U}')$*, and it is equal to* $\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\psi_\Gamma,\widetilde{\mathcal{U}}')(\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(int'(\mathrm{H}(\Gamma)),\phi)(\mathbf{\Phi}^2_{\mathrm{E}}(u_{1,\Gamma}(\mathrm{T}),u_{1,\mathsf{Id}_3(\mathrm{T})}(\mathrm{P}))))$*; and*
2. $(u'_{1,\mathrm{H}(\Gamma)}(\mathrm{H}(\mathrm{T})),\widetilde{u}'_{1,\mathsf{Id}_3'(\mathrm{H}(\mathrm{T}))}(\mathrm{H}(o)))$ *is a well defined element of* $\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\Phi(int(\Gamma)),\widetilde{\mathcal{U}}')$*, and it is equal to* $\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\psi_\Gamma,\widetilde{\mathcal{U}}')(\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(int'(\mathrm{H}(\Gamma)),\widetilde{\phi})(\mathbf{\Phi}^2_{\mathrm{E}}(u_{1,\Gamma}(\mathrm{T}),\widetilde{u}_{1,\mathsf{Id}_3(\mathrm{T})}(o))))$.

Remark **3.4.5.** — Since

$$(\mathbf{3.23}) \qquad u_2(\mathrm{T}) := (u_1(\mathsf{ft}(\mathrm{T})), u_1(\mathrm{T})) \in \mathsf{D}_p(int(\Gamma), \mathcal{U}) \qquad \text{and}$$

$$(\mathbf{3.24}) \qquad \widetilde{u}_2(s) := (u_1(\mathsf{ft}(\partial(s))), \widetilde{u}_1(s)) \in \mathsf{D}_p(int(\Gamma), \widetilde{\mathcal{U}}),$$

this lemma is very similar to [18, Lemma 6.1(3, 4)], but its proof is more involved because of the interaction of the two universe functors. (For the introduction of u_2 and $\widetilde{u}_2$ see [18, Problem 3.5].)

Proof. — We will consider only the second assertion; the proof of the first one is similar and simpler.

To prove that the pair

$$(u'_{1,\mathrm{H}(\Gamma)}(\mathrm{H}(\mathrm{T})), \widetilde{u}'_{1,\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))}(\mathrm{H}(o)))$$

is a well defined element of $\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\Phi(int(\Gamma)), \widetilde{\mathcal{U}}')$ we need to show that $\mathsf{ft}(\partial(\mathrm{H}(o))) = \mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))$ and that the source of $\widetilde{u}'_{1,\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))}(\mathrm{H}(o)))$ equals $(int'(\mathrm{H}(\Gamma)); u'_1(\mathrm{H}(\mathrm{T})))_{\mathrm{E}}$, i.e., that

$$int'(\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))) = (int'(\mathrm{H}(\Gamma)); u'_1(\mathrm{H}(\mathrm{T})))_{\mathrm{E}}$$

The former is a corollary of our assumptions and Lemma 3.4.1, and the latter is a corollary of [18, Problem 3.3(1)] and the first equation of Lemma 2.4.3.

Let $\mathrm{X} = int(\Gamma)$, $\mathrm{F} = u_{1,\Gamma}(\mathrm{T})$, and $\widetilde{\mathrm{G}} = \widetilde{u}_{1,\mathsf{Id}_3(\mathrm{T})}(o)$, so that $(\mathrm{F}, \widetilde{\mathrm{G}}) \in \mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{X}, \widetilde{\mathcal{U}})$. By the definitions we have

$$\begin{aligned}
&\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\psi_\Gamma, \_)(\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\_, \widetilde{\phi})(\mathbf{\Phi}^2_{\mathrm{E}}(\mathrm{F}, \widetilde{\mathrm{G}})))\\
&= \mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\psi_\Gamma, \_)(\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\_, \widetilde{\phi})(\Phi(\mathrm{F}) \circ \phi, \iota \circ \Phi(\widetilde{\mathrm{G}})))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by def'n of } \mathbf{\Phi}^2_{\mathrm{E}}\text{; cf. (3.4))}\\
&= \mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\psi_\Gamma, \_)(\Phi(\mathrm{F}) \circ \phi, \iota \circ \Phi(\widetilde{\mathrm{G}}) \circ \widetilde{\phi})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by def'n of } \mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'} \text{ on morphisms)}\\
&= (\psi_\Gamma \circ \Phi(\mathrm{F}) \circ \phi, \mathsf{Q}(\psi_\Gamma, \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \iota \circ \Phi(\widetilde{\mathrm{G}}) \circ \widetilde{\phi}),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by def'n of } \mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'} \text{ on morphisms)}
\end{aligned}$$

where

$$\iota : (\Phi(\mathrm{X}); \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \to \Phi((\mathrm{X}; \mathrm{F})_{\mathrm{E}})$$

is the unique morphism such that the following two equations are satisfied.

$$\iota \circ \Phi(\mathsf{p}^{\mathrm{E}}_{\mathrm{X},\mathrm{F}}) = p^{\mathrm{E}'}_{\Phi(\mathrm{X}),\Phi(\mathrm{F})\circ\phi} \tag{3.25}$$

$$\iota \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \widetilde{\phi}_{\mathrm{E}} = \mathsf{Q}(\Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \tag{3.26}$$

(This map ι is analogous to the one defined in (3.3).)

On the other hand, by [18, Lemma 6.1(1)],

$$u_{1,\mathrm{H}(\Gamma)}(\mathrm{H}(\mathrm{T})) = \psi_{\Gamma} \circ \Phi(u_{1,\Gamma}(\mathrm{T})) \circ \phi$$

and

$$\begin{aligned}\widetilde{u}_{1,\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))}(\mathrm{H}(o)) &= \widetilde{u}_{1,\mathrm{H}(\mathsf{Id}_3(\mathrm{T}))}(\mathrm{H}(o)) \\ &= \psi_{\mathrm{H}(\mathsf{Id}_3(\mathrm{T}))} \circ \Phi(\widetilde{u}_{1,\mathsf{Id}_3(\mathrm{T})}(o)) \circ \widetilde{\phi}\end{aligned}$$

by [18, Lemma 6.1(1,2)]. Applying that and (3.18), we see that to prove the lemma it is sufficient to show that

$$\psi_{\mathsf{Id}_3(\mathrm{T})} = \mathsf{Q}(\psi_{\Gamma}, \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \iota.$$

Both sides are morphisms with codomain

$$\Phi(\mathit{int}(\mathsf{Id}_3(\mathrm{T}))) = \Phi((\mathrm{X};\mathrm{F})_{\mathrm{E}}),$$

and since $\mathbf{\Phi}_{\mathrm{E}}$ is a universe category functor it is sufficient to show that the compositions of the two sides with $\Phi(\mathsf{p}^{\mathrm{E}}_{\mathrm{X},\mathrm{F}})$ and $\Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \widetilde{\phi}_{\mathrm{E}}$ are the same, i.e., that the following two equations hold.

$$\psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{p}^{\mathrm{E}}_{\mathrm{X},\mathrm{F}}) = \mathsf{Q}(\psi_{\Gamma}, \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \iota \circ \Phi(\mathsf{p}^{\mathrm{E}}_{\mathrm{X},\mathrm{F}}) \tag{3.27}$$

$$\psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \widetilde{\phi}_{\mathrm{E}} = \mathsf{Q}(\psi_{\Gamma}, \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \iota \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \widetilde{\phi}_{\mathrm{E}} \tag{3.28}$$

We establish equation (3.27) as follows.

$$\begin{aligned}&\mathsf{Q}(\psi_{\Gamma}, \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \iota \circ \Phi(\mathsf{p}^{\mathrm{E}}_{\mathrm{X},\mathrm{F}}) \\ &\quad = \mathsf{Q}(\psi_{\Gamma}, \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \mathsf{p}^{\mathrm{E}'}_{\Phi(\mathrm{X}),\Phi(\mathrm{F})\circ\phi} && \text{(by (3.25))} \\ &\quad = \mathsf{p}^{\mathrm{E}'}_{\mathit{int}'(\mathrm{H}(\Gamma)),\psi_{\Gamma}\circ\Phi(\mathrm{F})\circ\phi} \circ \psi_{\Gamma} && \text{(by commutativity of (2.27))} \\ &\quad = \mathsf{p}_{\mathrm{H}(\mathsf{Id}_3(\mathrm{X},\mathrm{F})),3} \circ \psi_{\Gamma} && \text{(by (2.45))} \\ &\quad = \mathsf{p}_{\mathrm{H}(\mathsf{Id}_3(\mathrm{T})),3} \circ \psi_{\Gamma} && \text{(by def'n of X and F)} \\ &\quad = \psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{p}_{\mathsf{Id}_3(\mathrm{T}),3}) && \text{(by (3.4.3))} \\ &\quad = \psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{p}_{\mathsf{Id}_3(\mathrm{X},\mathrm{F}),3}) && \text{(by def'n of X and F)} \\ &\quad = \psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{p}^{\mathrm{E}}_{\mathrm{X},\mathrm{F}}). && \text{(by (2.46))}\end{aligned}$$

Since

$$\begin{aligned} &\mathsf{Q}(\psi_{\Gamma}, \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \iota \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \widetilde{\phi}_{\mathrm{E}} \\ &\qquad = \mathsf{Q}(\psi_{\Gamma}, \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \mathsf{Q}(\Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} && \text{(by (3.26))} \\ &\qquad = \mathsf{Q}(\psi_{\Gamma} \circ \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'}, && \text{(by (2.2.9))} \end{aligned}$$

equation (3.28) reduces to

$$\psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \widetilde{\phi}_{\mathrm{E}} = \mathsf{Q}(\psi_{\Gamma} \circ \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'}. \tag{\textbf{3.29}}$$

We have

$$\widetilde{\phi}_{\mathrm{E}} = (\Phi(\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q}) \circ \Phi\widetilde{\mathcal{U}}p) * (\Phi(\mathsf{Q}(\mathrm{E}q)) \circ \widetilde{\phi}).$$

Therefore (3.29) is equivalent to two equations:

$$\psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \Phi(\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q}) \circ \Phi\widetilde{\mathcal{U}}p = \mathsf{Q}(\psi_{\Gamma} \circ \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \mathsf{p}_{(\widetilde{\mathcal{U}}';p'),\mathrm{E}q'} \tag{\textbf{3.30}}$$

$$\psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \Phi(\mathsf{Q}(\mathrm{E}q)) \circ \widetilde{\phi} = \mathsf{Q}(\psi_{\Gamma} \circ \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \mathsf{Q}'(\mathrm{E}q'). \tag{\textbf{3.31}}$$

The first equality we will have to decompose further into two using the fact that by Lemma 3.2.1 we have

$$\Phi\widetilde{\mathcal{U}}p = (\Phi(\mathsf{p}_{\widetilde{\mathcal{U}},p}) \circ \widetilde{\phi}) * (\Phi(\mathsf{Q}(p)) \circ \widetilde{\phi}).$$

Therefore (3.30) is equivalent to two equations:

$$\begin{aligned} &\psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \Phi(\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q}) \circ \Phi(\mathsf{p}_{\widetilde{\mathcal{U}},p}) \circ \widetilde{\phi} \\ &\quad = \mathsf{Q}(\psi_{\Gamma} \circ \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \mathsf{p}_{(\widetilde{\mathcal{U}}';p'),\mathrm{E}q'} \circ \mathsf{p}_{\widetilde{\mathcal{U}}',p'} \end{aligned} \tag{\textbf{3.32}}$$

$$\begin{aligned} &\psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \Phi(\mathsf{p}_{(\widetilde{\mathcal{U}};p),\mathrm{E}q}) \circ \Phi(\mathsf{Q}(p)) \circ \widetilde{\phi} \\ &\quad = \mathsf{Q}(\psi_{\Gamma} \circ \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \mathsf{p}_{(\widetilde{\mathcal{U}}';p'),\mathrm{E}q'} \circ \mathsf{Q}'(p'). \end{aligned} \tag{\textbf{3.33}}$$

To prove (3.31) observe first two useful equalities:

$$u_1(\mathsf{Id}_3(\mathrm{T})) = \mathsf{Q}(\mathsf{Q}(\mathrm{F}), p) \circ \mathrm{E}q \tag{\textbf{3.34}}$$

$$\mathsf{Q}(u_1(\mathsf{Id}_3(\mathrm{T}))) = \mathsf{Q}(\mathrm{F})_{\mathrm{E}} \circ \mathsf{Q}(\mathrm{E}q), \tag{\textbf{3.35}}$$

where the first follows from the proof of Lemma 2.4.3, and the second is the combination of the first with the third equality of the same lemma.

Now we have:

$$\begin{aligned} &\psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}}) \circ \Phi(\mathsf{Q}(\mathrm{E}q)) \circ \widetilde{\phi} \\ &\qquad = \psi_{\mathsf{Id}_3(\mathrm{T})} \circ \Phi(\mathsf{Q}(\mathrm{F})_{\mathrm{E}} \circ \mathsf{Q}(\mathrm{E}q)) \circ \widetilde{\phi} && \text{(by functoriality of } \Phi) \end{aligned}$$

$$
\begin{aligned}
&= \psi_{\mathsf{Id}_3(T)} \circ \Phi(\mathsf{Q}(u_1(\mathsf{Id}_3(T)))) \circ \widetilde{\phi} && \text{(by (3.35))}\\
&= \mathsf{Q}'(\psi_\Gamma \circ \Phi(u_1(\mathsf{Id}_3(T))) \circ \phi) && \text{(by (3.20))}\\
&= \mathsf{Q}'(u_1(H(\mathsf{Id}_3(T)))) && \text{(by (3.18))}\\
&= \mathsf{Q}'(u_1(\mathsf{Id}'_3(H(T)))) && \text{(by (3.16))}\\
&= \mathsf{Q}(u_1(H(T)))_{E'} \circ \mathsf{Q}'(Eq') && \text{(by (3.35))}\\
&= \mathsf{Q}(\psi_\Gamma \circ \Phi(F) \circ \phi)_{E'} \circ \mathsf{Q}'(Eq'). && \text{(by (3.18))}
\end{aligned}
$$

The equality (3.31) is proved.

To prove (3.32) observe two equalities:

$$\mathsf{Q}(F)_E \circ \mathsf{p}_{(\widetilde{\mathcal{U}};p),Eq} = \mathsf{p}_{\mathsf{Id}_3(T)} \circ \mathsf{Q}(\mathsf{Q}(F),p) \tag{\textbf{3.36}}$$

$$\mathsf{Q}(\mathsf{Q}(F),p) \circ \mathsf{p}_{\widetilde{\mathcal{U}},p} = \mathsf{p}_{\mathsf{ft}(\mathsf{Id}_3(T)} \circ \mathsf{Q}(F). \tag{\textbf{3.37}}$$

The same equalities hold for $F' := \psi_\Gamma \circ \Phi(F) \circ \phi = u'_1(H(T))$, and Equation (3.32) becomes

$$
\begin{aligned}
&\psi_{\mathsf{Id}_3(T)} \circ \Phi(\mathsf{p}_{\mathsf{Id}_3(T)}) \circ \Phi(\mathsf{p}_{\mathsf{ft}(\mathsf{Id}_3(T))}) \circ \Phi(\mathsf{Q}(F)) \circ \widetilde{\phi}\\
&\quad = \mathsf{p}_{\mathsf{Id}'_3(H(T))} \circ \mathsf{p}_{\mathsf{ft}(\mathsf{Id}'_3(H(T)))} \circ \mathsf{Q}(F')
\end{aligned}
$$

Using the defining equations for ψ we rewrite the left hand side as follows:

$$
\begin{aligned}
&\psi_{\mathsf{Id}_3(T)} \circ \Phi(\mathsf{p}_{\mathsf{Id}_3(T)}) \circ \Phi(\mathsf{p}_{\mathsf{ft}(\mathsf{Id}_3(T))}) \circ \Phi(\mathsf{Q}(F)) \circ \widetilde{\phi}\\
&\quad = \mathsf{p}_{H(\mathsf{Id}_3(T))} \circ \mathsf{p}_{\mathsf{ft}(H(\mathsf{Id}_3(T)))} \circ \psi_T \circ \Phi(\mathsf{Q}(F)) \circ \widetilde{\phi}.
\end{aligned}
$$

It remains to show that

$$\mathsf{Q}(\psi_\Gamma \circ \Phi(F) \circ \phi) = \psi_{(\Gamma,F)} \circ \Phi(\mathsf{Q}(F)) \circ \widetilde{\phi}, \tag{\textbf{3.38}}$$

which is the defining equation of $\psi_{(\Gamma,F)}$.

We prove (3.33) as follows.

$$
\begin{aligned}
&\psi_{\mathsf{Id}_3(T)} \circ \Phi(\mathsf{Q}(F)_E) \circ \Phi(\mathsf{p}_{(\widetilde{\mathcal{U}};p),Eq}) \circ \Phi(\mathsf{Q}(p)) \circ \widetilde{\phi}\\
&= \psi_{\mathsf{Id}_3(T)} \circ \Phi(\mathsf{Q}(F)_E \circ \mathsf{p}_{(\widetilde{\mathcal{U}};p),Eq}) \circ \Phi(\mathsf{Q}(p)) \circ \widetilde{\phi} && \text{(by functoriality of } \Phi)\\
&= \psi_{\mathsf{Id}_3(T)} \circ \Phi(\mathsf{p}_{\mathsf{Id}_3(T)} \circ \mathsf{Q}(\mathsf{Q}(F),p)) \circ \Phi(\mathsf{Q}(p)) \circ \widetilde{\phi} && \text{(by commutativity}\\
& && \text{of (2.22))}\\
&= \psi_{\mathsf{Id}_3(T)} \circ \Phi(\mathsf{p}_{\mathsf{Id}_3(T)}) \circ \Phi(\mathsf{Q}(\mathsf{Q}(F),p) \circ \mathsf{Q}(p)) \circ \widetilde{\phi} && \text{(by functoriality of } \Phi)\\
&= \psi_{\mathsf{Id}_3(T)} \circ \Phi(\mathsf{p}_{\mathsf{Id}_3(T)}) \circ \Phi(\mathsf{Q}(\mathsf{Q}(F) \circ p)) \circ \widetilde{\phi} && \text{(by commutativity}\\
& && \text{of (2.16))}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathsf{p}_{\mathrm{H}(\mathsf{Id}_3(\mathrm{T}))} \circ \psi_{\mathsf{ft}(\mathsf{Id}_3(\mathrm{T}))} \circ \Phi(\mathsf{Q}(\mathsf{Q}(\mathrm{F}) \circ p)) \circ \widetilde{\phi} && \text{(by (3.21))}\\
&= \mathsf{p}_{\mathrm{H}(\mathsf{Id}_3(\mathrm{T}))} \circ \psi_{((\Gamma,\mathrm{F}),\mathsf{Q}(\mathrm{F})\circ p)} \circ \Phi(\mathsf{Q}(\mathsf{Q}(\mathrm{F}) \circ p)) \circ \widetilde{\phi} && \text{(by [18, Lemma 3.2])}\\
&= \mathsf{p}_{\mathrm{H}(\mathsf{Id}_3(\mathrm{T}))} \circ \mathsf{Q}'(\psi_{(\Gamma,\mathrm{F})} \circ \Phi(\mathrm{Q}(\mathrm{F}) \circ p) \circ \phi) && \text{(by (3.20))}\\
&= \mathsf{p}_{\mathrm{H}(\mathsf{Id}_3(\mathrm{T}))} \circ \mathsf{Q}'(u_1(\mathrm{H}(\mathsf{ft}(\mathsf{Id}_3(\mathrm{T}))))) && \text{(by (3.18))}\\
&= \mathsf{p}_{\mathrm{H}(\mathsf{Id}_3(\mathrm{T}))} \circ \mathsf{Q}'(u_1(\mathsf{ft}(\mathrm{H}(\mathsf{Id}_3(\mathrm{T}))))) && \text{(by (3.19))}\\
&= \mathsf{p}_{\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))} \circ \mathsf{Q}'(u_1(\mathsf{ft}(\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))))) && \text{(by (3.16))}\\
&= \mathsf{p}_{\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))} \circ \mathsf{Q}'(\mathsf{Q}'(u_1(\mathrm{H}(\mathrm{T}))) \circ p') && \text{(by (2.44))}\\
&= \mathsf{p}_{\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))} \circ \mathsf{Q}'(\mathsf{Q}'(u_1(\mathrm{H}(\mathrm{T}))), p') \circ \mathsf{Q}'(p') && \text{(by commutativity}\\
& && \text{of (2.16))}\\
&= \mathsf{Q}(u_1(\mathrm{H}(\mathrm{T})))_{\mathrm{E}'} \circ \mathsf{p}_{(\widetilde{\mathcal{U}}';p'),\mathrm{E}q'} \circ \mathsf{Q}'(p') && \text{(by commutativity}\\
& && \text{of (2.2.8))}\\
&= \mathsf{Q}(\psi_{\Gamma} \circ \Phi(\mathrm{F}) \circ \phi)_{\mathrm{E}'} \circ \mathsf{p}_{(\widetilde{\mathcal{U}}';p'),\mathrm{E}q'} \circ \mathsf{Q}'(p') && \text{(by (3.18))}
\end{aligned}
$$

That finishes the proof of Lemma 3.4.4. ☐

Lemma **3.4.6.** — *Let* $\mathbf{\Phi}$ *be a universe category functor as above that is compatible with the* $(\mathrm{J}0,\mathrm{J}1,\mathrm{J}2)$*-structures* $(\mathrm{E}q, \Omega, \mathrm{J}p)$ *and* $(\mathrm{E}q', \Omega', \mathrm{J}p)$ *on* p *and* p' *respectively. Then the homomorphism of C-systems* $\mathrm{H} = \mathrm{H}(\mathbf{\Phi})$ *is a homomorphism of C-systems with* $(\mathrm{J}0,\mathrm{J}1,\mathrm{J}2)$*-structures relative to* $(\mathsf{Id}_{\mathrm{E}q}, \mathsf{refl}_{\Omega}, \mathsf{J}_{\mathrm{J}p})$ *and* $(\mathsf{Id}_{\mathrm{E}q'}, \mathsf{refl}_{\Omega'}, \mathsf{J}_{\mathrm{J}p'})$.

Proof. — Let $\mathsf{Id} = \mathsf{Id}_{\Omega}$, $\mathsf{Id}' = \mathsf{Id}_{\Omega'}$, $\mathsf{refl} = \mathsf{refl}_{\Omega}$, $\mathsf{refl}' = \mathsf{refl}_{\Omega'}$, $\mathsf{J} = \mathsf{J}_{\mathrm{J}p}$ and $\mathsf{J}' = \mathsf{J}_{\mathrm{J}p'}$.

We need to verify that for all $\Gamma \in Ob(\mathrm{CC}(\mathcal{C}, p))$, $\mathrm{T} \in Ob_1(\Gamma)$, $\mathrm{P} \in Ob_1(\mathsf{Id}_3(\mathrm{T}))$ and $s0 \in \widetilde{Ob}(\mathsf{rf}^*_{\mathrm{T}}(\mathrm{P}))$ one has

$$\mathrm{H}(\mathsf{J}(\Gamma, \mathrm{T}, \mathrm{P}, s0)) = \mathsf{J}'(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0))$$

The defining equation for J' is

$$\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(u'_1(\mathrm{H}(\mathrm{T})), \widetilde{u}'_{1,\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))}(\mathsf{J}'))) = \phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0)) \circ \mathrm{J}p' \tag{\textbf{3.39}}$$

and to prove the lemma we need to show that $\mathrm{H}(\mathsf{J})$ satisfies this equation.

Using Lemma 3.4.4 we have

$$
\begin{aligned}
&\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(u'_1(\mathrm{H}(\mathrm{T})), \widetilde{u}'_{1,\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))}(\mathrm{H}(\mathsf{J})))) \\
&\qquad = \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\psi_{\Gamma}, \_)(\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\_, \widetilde{\phi})(\Phi^2_{\mathrm{E}}(u_1(\mathrm{T}), \widetilde{u}_{1,\mathsf{Id}_3(\mathrm{T})}(\mathsf{J}))))) \\
&\qquad = \psi_{\Gamma} \circ \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\Phi^2_{\mathrm{E}}(u_1(\mathrm{T}), \widetilde{u}_{1,\mathsf{Id}_3(\mathrm{T})}(\mathsf{J}))) \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\phi})
\end{aligned}
$$

We further have:

$$\begin{aligned}
&\psi_\Gamma \circ \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\Phi^2_\mathrm{E}(u_1(\mathrm{T}), \widetilde{u}_{1,\mathsf{Id}_3(\mathrm{T})}(\mathsf{J}))) \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\phi})\\
&= \psi_\Gamma \circ \Phi(\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(u_1(\mathrm{T}), \widetilde{u}_{1,\mathsf{Id}_3(\mathrm{T})}(\mathsf{J}))) \circ \chi_{\mathbf{\Phi}_\mathrm{E}}(\widetilde{\mathcal{U}}) \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\phi})\\
&\hspace{20em}\text{(by [18, Lemma 5.8])}\\
&= \psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathrm{J}p) \circ \chi_{\mathbf{\Phi}_\mathrm{E}}(\widetilde{\mathcal{U}}) \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\widetilde{\phi}) \qquad \text{(by (2.4.11))}\\
&= \psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathrm{J}p) \circ \widetilde{\zeta}_{\mathbf{\Phi}}. \qquad \text{(by def'n of } \widetilde{\zeta}_{\mathbf{\Phi}} \text{ in (3.12))}
\end{aligned}$$

It remains to show that

$$\psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathrm{J}p) \circ \widetilde{\zeta}_{\mathbf{\Phi}} = \phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0)) \circ \mathrm{J}p'$$

By the compatibility condition of Definition 3.2.10 we see that it is sufficient to prove that

$$\psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0)) \circ \mathrm{R}_{\mathbf{\Phi}} = \phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0))$$

Let

$$\begin{aligned}
&\mathsf{pr}_1 : (\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^\omega) \times_{\mathsf{I}_p(\mathcal{U})} (\mathsf{I}_p(\widetilde{\mathcal{U}}), \mathsf{I}_p(p)) \to \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U})\\
&\mathsf{pr}_2 : (\mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathcal{U}), \mathsf{I}^\omega) \times_{\mathsf{I}_p(\mathcal{U})} (\mathsf{I}_p(\widetilde{\mathcal{U}}), \mathsf{I}_p(p)) \to \mathsf{I}_p(\widetilde{\mathcal{U}})
\end{aligned}$$

be the projections and let pr'_1, pr'_2 be their analogues in $\mathcal{C}'$. Then one has the following two equations.

$$\mathrm{R}_\Phi \circ \mathsf{pr}'_1 = \Phi(\mathsf{pr}_1) \circ \zeta_\Phi \tag{\textbf{3.40}}$$

$$\mathrm{R}_\Phi \circ \mathsf{pr}'_2 = \Phi(\mathsf{pr}_2) \circ \widetilde{\xi}_\Phi. \tag{\textbf{3.41}}$$

On the other hand, the defining relations of $\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0)$ are

$$\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathsf{pr}_1 = \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{F}, \mathrm{G}) \tag{\textbf{3.42}}$$

$$\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathsf{pr}_2 = \eta^{!-1}_{p}(\mathrm{F}, \widetilde{\mathrm{H}}), \tag{\textbf{3.43}}$$

and the defining relations of $\phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0))$ are

$$\phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0)) \circ \mathsf{pr}'_1 = \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\mathrm{F}', \mathrm{G}') \tag{\textbf{3.44}}$$

$$\phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0)) \circ \mathsf{pr}'_2 = \eta^{!-1}_{p'}(\mathrm{F}', \widetilde{\mathrm{H}}'), \tag{\textbf{3.45}}$$

where

$$\mathrm{F} = u_{1,\Gamma}(\mathrm{T}) \qquad\qquad \mathrm{F}' = u'_{1,\mathrm{H}(\Gamma)}(\mathrm{H}(\mathrm{T}))$$

$$\begin{aligned} &\mathrm{G} = u_{1,\mathsf{Id}_3(\mathrm{T})}(\mathrm{P}) && \mathrm{G}' = u'_{1,\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))}(\mathrm{H}(\mathrm{P}))\\ &\widetilde{\mathrm{H}} = \widetilde{u}_{1,\mathrm{T}}(s0), && \widetilde{\mathrm{H}}' = \widetilde{u}'_{1,\mathrm{H}(\mathrm{T})}(\mathrm{H}(s0)) \end{aligned}$$

We need to prove

$$\psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0)) \circ \mathrm{R}_\Phi \circ \mathsf{pr}'_1 = \phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0)) \circ \mathsf{pr}'_1 \tag{\textbf{3.46}}$$

and

$$\psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0)) \circ \mathrm{R}_\Phi \circ \mathsf{pr}'_2 = \phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0)) \circ \mathsf{pr}'_2 \tag{\textbf{3.47}}$$

We establish (3.46) as follows.

$$\begin{aligned}
&\psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0)) \circ \mathrm{R}_\Phi \circ \mathsf{pr}'_1 \\
&= \psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0)) \circ \Phi(\mathsf{pr}_1) \circ \zeta_\Phi && \text{(by (3.40))}\\
&= \psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathsf{pr}_1) \circ \zeta_\Phi && \text{(by functoriality of } \Phi)\\
&= \psi_\Gamma \circ \Phi(\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{F}, \mathrm{G})) \circ \zeta_\Phi && \text{(by (3.42))}\\
&= \psi_\Gamma \circ \Phi(\eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\mathrm{F}, \mathrm{G})) \circ \chi_{\boldsymbol{\Phi}_\mathrm{E}}(\mathrm{U}) \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\phi) && \text{(by def'n of } \zeta_\Phi \text{ in (3.11))}\\
&= \psi_\Gamma \circ \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\boldsymbol{\Phi}^2_\mathrm{E}(\mathrm{F}, \mathrm{G})) \circ \mathsf{I}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\phi), && \text{(by [18, Lemma 5.8])}\\
&= \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\psi_\Gamma, \_)(\mathsf{D}_{p\mathrm{E}\widetilde{\mathcal{U}}}(\_, \phi)(\boldsymbol{\Phi}^2(\mathrm{F}, \mathrm{G})))) && \text{(by naturality of } \eta^!_{p\mathrm{E}\widetilde{\mathcal{U}}'})\\
&= \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}'}(u'_{1,\mathrm{H}(\Gamma)}(\mathrm{H}(\mathrm{T})), u_{1,\mathsf{Id}'_3(\mathrm{H}(\mathrm{T}))}(\mathrm{H}(\mathrm{P}))), && \text{(by Lemma 3.4.4(1))}\\
&= \eta^{!-1}_{p\mathrm{E}\widetilde{\mathcal{U}}'}((\mathrm{F}', \mathrm{G}')) && \text{(by def'n of F}' \text{ and G}')\\
&= \phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0)) \circ \mathsf{pr}'_1 && \text{(by (3.44))}
\end{aligned}$$

For (3.47), we proceed as follows.

$$\begin{aligned}
&\psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0)) \circ \mathrm{R}_\Phi \circ \mathsf{pr}'_2 \\
&= \psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0)) \circ \Phi(\mathsf{pr}_2) \circ \widetilde{\xi}_\Phi && \text{(by (3.41))}\\
&= \psi_\Gamma \circ \Phi(\phi(\Gamma, \mathrm{T}, \mathrm{P}, s0) \circ \mathsf{pr}_2) \circ \widetilde{\xi}_\Phi && \text{(by functoriality of } \Phi)\\
&= \psi_\Gamma \circ \Phi(\eta^{!-1}_{p}((\mathrm{F}, \widetilde{\mathrm{H}}))) \circ \widetilde{\xi}_\Phi && \text{(by (3.43))}\\
&= \psi_\Gamma \circ \Phi(\eta^{!-1}_{p}((u_1(\mathrm{T}), \widetilde{u}_1(s0))) \circ \widetilde{\xi}_\Phi && \text{(by def'n of F and } \widetilde{\mathrm{H}})\\
&= \psi_\Gamma \circ \Phi(\eta^{!-1}_{p}((u_1(\mathsf{ft}(\partial(s0))), \widetilde{u}_1(s0))) \circ \widetilde{\xi}_\Phi && \text{(by the type of } s0)\\
&= \psi_\Gamma \circ \Phi(\eta^{!-1}_{p}(\widetilde{u}_{2,\Gamma}(s0))) \circ \widetilde{\xi}_\Phi && \text{(by def'n of } \widetilde{u}_2 \text{ in (3.24))}
\end{aligned}$$

$$
\begin{aligned}
&= \eta^{!-1}_{p'}(\widetilde{u}'_{2,\mathrm{H}(\Gamma)}(\mathrm{H}(s0))) && \text{(by [18, Lemma 6.2(2)])}\\
&= \eta^{!-1}_{p'}((u'_1(\mathsf{ft}(\partial(\mathrm{H}(s0)))), \widetilde{u}'_1(\mathrm{H}(s0)))) && \text{(by def'n of } \widetilde{u}_2 \text{ in (3.24))}\\
&= \eta^{!-1}_{p'}((u'_1(\mathsf{ft}(\partial(\mathrm{H}(s0)))), \widetilde{\mathrm{H}}')) && \text{(by def'n of } \widetilde{\mathrm{H}}'\text{)}\\
&= \eta^{!-1}_{p'}((u'_1(\mathsf{ft}(\mathrm{H}(\partial(s0)))), \widetilde{\mathrm{H}}')) && \text{(by functoriality of H)}\\
&= \eta^{!-1}_{p'}((u'_1(\mathrm{H}(\mathsf{ft}(\partial(s0)))), \widetilde{\mathrm{H}}')) && \text{(by (3.19))}\\
&= \eta^{!-1}_{p'}((u'_1(\mathrm{H}(\mathrm{T}))), \widetilde{\mathrm{H}}')) && \text{(by the type of } s0\text{)}\\
&= \eta^{!-1}_{p'}((\mathrm{F}', \widetilde{\mathrm{H}}')) && \text{(by def'n of F}'\text{)}\\
&= \phi(\mathrm{H}(\Gamma), \mathrm{H}(\mathrm{T}), \mathrm{H}(\mathrm{P}), \mathrm{H}(s0)) \circ \mathsf{pr}'_2 && \text{(by (3.45))}
\end{aligned}
$$

This finishes the proof of Lemma 3.4.6. □

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