TWO DIMENSIONAL NEIGHBORHOODS OF ELLIPTIC CURVES: ANALYTIC CLASSIFICATION IN THE TORSION CASE

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ABSTRACT

We investigate the analytic classification of two dimensional neighborhoods of an elliptic curve with torsion normal bundle. We provide the complete analytic classification for those neighborhoods in the simplest formal class and we indicate how to generalize this construction to general torsion case.

CONTENTS

1. Introduction and results

Let C be a smooth elliptic curve: $\mathrm{C} = \mathbf{C}/\Gamma_\tau$, where $\Gamma_\tau = \mathbf{Z} + \tau\mathbf{Z}$, with $\Im(\tau) > 0$. Given an embedding $\iota : \mathrm{C} \hookrightarrow \mathrm{U}$ of C into a smooth complex surface U, we would like to understand the germ $(\mathrm{U}, \iota(\mathrm{C}))$ of neighborhood of $\iota(\mathrm{C})$ in U. Precisely, we will say that two embeddings $\iota, \iota' : \mathrm{C} \hookrightarrow \mathrm{U}, \mathrm{U}'$ are (formally/analytically) equivalent if there is a (formal/analytic) isomorphism $\Psi : (\mathrm{U}, \iota(\mathrm{C})) \to (\mathrm{U}', \iota'(\mathrm{C}))$ between germs of neighborhoods making commutative the following diagram

$$\begin{tikzcd} \mathrm{C} \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"'] & \mathrm{U} \arrow[d, "\Psi"] \\ \mathrm{C} \arrow[r, "\iota'"] & \mathrm{U}' \end{tikzcd} \tag{1.1}$$

By abuse of notation, we will still denote by C the image $\iota(\mathrm{C})$ of its embedding in U, and we will simply denote by (U, C) the germ of neighborhood.

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1.1. *Some historical background.* — The problem of analytic classification of neighborhoods of compact complex curves in complex surfaces goes back at least to the celebrated work of Grauert [8]. There, he considered the normal bundle N_C of the curve in U. The neighborhood of the zero section in the total space of N_C, that we denote $(N_C, 0)$, can be viewed[1] as the linear part of (U, C). A coarse invariant is given by the degree $\deg N_C$ which is also the self-intersection $C \cdot C$ of the curve. In this paper [8], Grauert proved that the germ of neighborhood is "linearizable", i.e. analytically equivalent to the germ of neighborhood $(N_C, 0)$, provided that $\deg(N_C)$ is negative enough, namely $\deg(N_C) < 4 - 4g$ for a curve of genus $g > 0$, and $\deg(N_C) < 0$ for a rational curve $g = 0$. It was also clear from his work that even the formal classification was much more complicated when $\deg(N_C) > 0$. At the same period, Kodaira investigated the deformation of compact submanifolds of complex manifolds in [12]. His result, in the particular case of curves in surfaces, says that the curve can be deformed provided that $\deg(N_C)$ is positive enough, namely $\deg(N_C) > 2g - 2$ for a curve of genus $g > 0$, and $\deg(N_C) \geq 0$ for a rational curve $g = 0$. Using these deformations, it is possible to provide a complete set of invariants for analytic classification for $g = 0$: (U, C) is linearizable when $\deg(N_C) \leq 0$ (Grauert for $C \cdot C < 0$ and Savelev [25] for $C \cdot C = 0$), and there is a functional moduli[2] when $\deg(N_C) > 0$ following Mishustin [19] (see also [6]). Also, when $g > 0$ and $\deg(N_C) > 2g - 2$, the analytic classification has been carried out by Ilyashenko [11] and Mishustin [20]. In all these results, it is important to notice that formally equivalent neighborhoods are also analytically equivalent: the two classifications coincide for such neighborhoods. Such a rigidity property is called *the formal principle* (see the recent works [10, 23] on this topic).

The case of an elliptic[3] curve $g = 1$ with $\deg(N_C) = 0$, which is still open today, has been investigated by Arnold [1] in another celebrated work. In this case, the normal bundle N_C belongs to the Jacobian curve $\mathrm{Jac}(C) \simeq C = \mathbf{C}/\Gamma_\tau$ and can be torsion[4] or not. Torsion points correspond to the image of $\mathbf{Q} + \tau\mathbf{Q} \subset C$ in the curve. Arnold investigated the non torsion case and proved in that case

- if N_C is non torsion, then (U, C) is formally linearizable;
- if N_C is generic[5] enough in Jac(C), then (U, C) is analytically linearizable;
- for non generic (and still non torsion) N_C, there is a huge[6] moduli space for the analytic classification.

[1] Strictly speaking, the linear part is more complicated in general, as it needs not fiber over the curve, as it is the case for the neighborhood of a conic in $\mathbf{P}^2$.

[2] The moduli space is comparable with the ring of convergent power series $\mathbf{C}\{X, Y\}$.

[3] Elliptic means $g = 1$ and that we have moreover fixed a (zero) point on the curve, to avoid considering automorphisms of the curve in our study.

[4] Torsion means that some iterate for the group law $\otimes$ is the trivial bundle $\mathcal{O}_C$.

[5] I.e. belongs to some subset of total Lebesgue measure defined by a certain diophantine condition.

[6] Thanks to the works of Yoccoz [39] and Perez-Marco [24], we can embed at least $\mathbf{C}\{X\}$ in the moduli space with a huge degree of freedom.

However, we are still far, nowadays, to expect a complete description of the analytic classification in the non torsion case. It is the first case where the divergence between formal and analytic classification arises. Also, it is interesting to note that the study of neighborhoods of elliptic curves in the case $\deg(\mathrm{N}_{\mathrm{C}}) = 0$ has strong reminiscence with the classification of germs of diffeomorphisms up to conjugacy. It will be more explicit later when describing the torsion case.

The goal of this paper is to investigate the analytic classification when the normal bundle is torsion, and show that we can expect to provide a complete description of the moduli space in that case. More precisely, the formal classification of such neighborhoods has been achieved in [15]; we provide the analytic classification inside the simplest formal class, and we explain in Sections 10 and 11 how we think it should extend to all other formal classes, according to the same "principle".

1.2. *Formal classification.* — An important formal invariant has been introduced by Ueda [22, 33] in the case $\deg(\mathrm{N}_{\mathrm{C}}) = 0$ and $g > 0$. There, among other results, he investigates the obstruction for the curve to be the fiber of a fibration (as it would be in the linear case $(\mathrm{N}_{\mathrm{C}}, 0)$ when N_{C} is torsion). The Ueda type $k \in \mathbf{Z}_{>0} \cup \{\infty\}$ is the largest integer for which the aforementioned fibration[7] of N_{C} can be extended to the k^{th} infinitesimal neighborhood of C. In other words, $k \geq k_0$ if the neighborhood can be covered by coordinate-charts $(x_i, y_i) : \mathrm{U}_i \to \mathbf{C}$ such that $\mathrm{C} \cap \mathrm{U}_i = \{y_i = 0\}$ and $y_i = a_{ij} y_j + o(y_j^{k_0})$ on intersections $\mathrm{U}_i \cap \mathrm{U}_j$ with $a_{ij}^m = 1$ (m is torsion); see [3, section 2] for a more detailed exposition. When $k = \infty$, then we have a formal fibration, that can be proved to be analytic; the classification in that case goes back to the works of Kodaira, in particular in the elliptic case $g = 1$.

Inspired by Ueda's approach, it has been proved by Claudon, Pereira and the two first named authors of this paper (see [3]) that a formal neighborhood (U, C) with $\deg(\mathrm{N}_{\mathrm{C}}) = 0$ carries many regular (formal) foliations such that C is a compact leaf. This construction has been improved in [15, 31] showing that one can choose two of these foliations in a canonical way and use them to produce a complete set of formal invariants. In the elliptic case $g = 1$, there are $\frac{k}{m} + 1$ independent formal invariants for finite fixed Ueda type k where m is the torsion order $\mathrm{N}_{\mathrm{C}}^{\otimes m} = \mathcal{O}_{\mathrm{C}}$ (see [15]); for $g > 1$ and $\mathrm{N}_{\mathrm{C}} = \mathcal{O}_{\mathrm{C}}$ (the trivial bundle), Thom founds infinitely many independent formal invariants in [31].

In this paper, we only consider the case $g = 1$ where N_{C} is a torsion bundle. Most of the paper deals with the simplest formal class for which $\mathrm{N}_{\mathrm{C}} = \mathcal{O}_{\mathrm{C}}$ is the trivial bundle, and Ueda type $k = 1$ is minimal. We explain in Sections 10 and 11 how to generalize to the more general case of torsion normal bundle $\mathrm{N}_{\mathrm{C}}^{\otimes m} = \mathcal{O}_{\mathrm{C}}$, and finite Ueda type $k < \infty$.

In the specific situation where $k = 1$, there are two scalar formal invariants $\nu, \mu \in \mathbf{C}$. We associate a germ of neighborhood $(\mathrm{U}_{1,\nu,\mu}, \mathrm{C})$ as follows. Writing C as a quotient of

[7] More generally, in the non torsion case, we may try to extend the foliation defined by the unitary connection on N_{C}.

$\mathbf{C}^*$ by a contraction:

$$\mathrm{C} = \mathbf{C}^*_z / < z \mapsto qz > \quad \text{with } z = e^{2i\pi x} \text{ and } q = e^{2i\pi\tau}, \ (|q| < 1)$$

we similarly define $(\mathrm{U}_{1,\nu,\mu}, \mathrm{C})$ as the quotient of the germ of neighborhood

$$(\mathbf{C}^*_z \times \mathbf{C}_y, \{y = 0\})$$

by the germ of diffeomorphism

$$\mathrm{F}_{1,\nu,\mu} = \exp(v_0 + v_\infty), \quad \text{where } \begin{cases} v_0 = \frac{y^2}{1+\nu y}\partial_y + 2i\pi\tau\mu\frac{y}{1+\nu y}z\partial_z \\ v_\infty = 2i\pi\tau z\partial_z \end{cases}$$

The two vector fields v_0 and v_∞ span a commutative Lie algebra, and therefore an infinitesimal $\mathbf{C}^2$-action on the quotient neighborhood. By duality, we have a 2-dimensional vector space of closed meromorphic 1-forms spanned by

$$-\omega_0 = \frac{dy}{y^{k+1}} + \nu\frac{dy}{y} \quad \text{and} \quad \omega_\infty = \frac{1}{2i\pi\tau}\frac{dz}{z} - \mu\frac{dy}{y}.$$

In particular, we get a pencil of foliations $\mathcal{F}_t$, $t \in \mathbf{P}^1$, by considering

- either the phase portrait of the vector fields $v_t = tv_0 + v_\infty$,
- or $\omega_t = 0$ where $\omega_t = \omega_0 + t\omega_\infty$.

When $\mu = 0$, $\mathcal{F}_\infty$ defines a fibration transversal to the curve C and the neighborhood is the suspension[8] of a representation $\varrho : \pi_1(\mathrm{C}) \to \mathrm{Diff}(\mathbf{C}, 0)$ taking values into the one-parameter group generated by $v_0 = \frac{y^{k+1}}{1+\nu y^k}\partial_y$. For $t \in \mathbf{C}$ finite, $\mathcal{F}_t$ is always (smooth) tangent to C, i.e. C is a compact leaf; when $\mu \neq 0$, the same holds for $\mathcal{F}_\infty$. For $m + \tau n \in \Gamma \setminus \{0\}$, $\mathcal{F}_{\frac{\tau n}{m+\tau n}}$ is the unique foliation of the pencil whose holonomy along the corresponding loop $m + \tau n$ in $\pi_1(\mathrm{C}) \sim \Gamma$ is trivial (see Section 7). As proved in [15, Theorem 1.3], the neighborhoods $(\mathrm{U}_{1,\nu,\mu}, \mathrm{C})$ span all formal classes of neighborhoods with minimal Ueda class $k = 1$. For general $k < \infty$ with $\mathrm{N_C} = \mathcal{O}_\mathrm{C}$, the scalar invariant μ is replaced by a polynomial $\mathrm{P} \in \mathbf{C}[\mathrm{X}]$ of degree $< k$, and the corresponding models $(\mathrm{U}_{k,\nu,\mathrm{P}}, \mathrm{C})$ are described at the beginning of Section 10; the case of torsion $\mathrm{N_C}$ is settled in Section 11.

As explained in [15, Theorem 1.5], the moduli space of those neighborhoods with two convergent foliations in a given formal class up to analytic conjugacy is infinite dimensional,[9] comparable with $\mathbf{C}\{\mathrm{X}\}$. *A contrario*, if a third foliation is convergent, then the neighborhood is analytically equivalent to its formal model $(\mathrm{U}_{k,\nu,\mathrm{P}}, \mathrm{C})$. However, an example of a neighborhood without convergent foliation is given by Mishustin in [21], and it is expected to be a generic property. In this paper, we describe the analytic classification

[8] In the sense of foliations.

[9] Isomorphic to Écalle-Voronin moduli spaces.

of those neighborhoods formally conjugated to the simplest model $(\mathrm{U}_{1,0,0},\mathrm{C})$. As we shall see, the moduli space is comparable with $\mathbf{C}\{\mathrm{X},\mathrm{Y}\}$. We also provide in Sections 10 and 11 some evidence to the fact that a similar result holds more generally for torsion normal bundle $\mathrm{N}_{\mathrm{C}}^{\otimes m}=\mathcal{O}_{\mathrm{C}}$, and finite Ueda type $k<\infty$.

1.3. *The fundamental isomorphism.* — In order to explain our classification result, it is convenient to recall the following classical construction. For the simplest formal type $(k,\nu,\mathrm{P})=(1,0,0)$, the neighborhood $(\mathrm{U}_{1,0,0},\mathrm{C})$ actually embeds into a ruled surface $\mathrm{S}_0\to\mathrm{C}$, namely one of the two indecomposable ruled surfaces over C after Atiyah [2]. Indeed, setting $y=1/\xi$, the ruled surface is defined as the quotient

$$\mathrm{S}_0=\tilde{\mathrm{U}}_0/<\mathrm{F}_{1,0,0}>\quad\text{where } \tilde{\mathrm{U}}_0=\mathbf{C}_z^*\times\overline{\mathbf{C}}_\xi \text{ and } \mathrm{F}_{1,0,0}(z,\xi)=(qz,\xi-1)$$

and the infinity section $\xi=\infty$ defines the embedding of the curve $\mathrm{C}\subset\mathrm{S}_0$.

There is an analytic isomorphism between complement of the curve $\mathrm{S}_0\setminus\mathrm{C}$ and $\mathbf{C}^*\times\mathbf{C}^*$ explicitly given by

$$(\mathbf{1.2})\qquad \Pi:\mathrm{S}_0\setminus\mathrm{C}\overset{\sim}{\longrightarrow}\mathbf{C}_\mathrm{X}^*\times\mathbf{C}_\mathrm{Y}^*\ ;\quad (z,\xi)\mapsto(e^{2i\pi\xi},ze^{2i\pi\tau\xi}).$$

In this sense, we can view S_0 and $\mathbf{P}^1_\mathrm{X}\times\mathbf{P}^1_\mathrm{Y}\supset\mathbf{C}_\mathrm{X}^*\times\mathbf{C}_\mathrm{Y}^*$ as two non algebraically equivalent compactifications of the same analytic variety. In fact, the algebraic structures of the two open sets are different as $\mathbf{C}_\mathrm{X}^*\times\mathbf{C}_\mathrm{Y}^*$ is affine, while $\mathrm{S}_0\setminus\mathrm{C}$ is not. Isomorphism (1.2) is therefore transcendental; we call it *Serre isomorphism*.

Remark **1.1.** — The complement of the curve $\mathrm{S}_0\setminus\mathrm{C}$ is known to be isomorphic to the moduli space of flat line bundles[10] over the elliptic curve, and has the structure of an affine bundle. The Riemann-Hilbert correspondence provides an analytic isomorphism with the space of characters $\mathrm{Hom}(\pi_1(\mathrm{C}),\mathrm{GL}_1(\mathbf{C}))$, which is isomorphic to $\mathbf{C}^*\times\mathbf{C}^*$. This construction, due to Serre, provides an example of a Stein quasiprojective variety which is not affine (see [9, page 232], see also [32]). Precisely, it is shown that there is no non constant regular function on $\mathrm{S}_0\setminus\mathrm{C}$.

Denote by $\mathrm{D}\subset\mathbf{P}^1_\mathrm{X}\times\mathbf{P}^1_\mathrm{Y}$ the compactifying divisor, union of four projective lines:

$$\mathrm{D}=\mathrm{L}_1\cup\mathrm{L}_2\cup\mathrm{L}_3\cup\mathrm{L}_4\quad\text{with}$$

$$\mathrm{L}_1:\{\mathrm{Y}=0\},\quad \mathrm{L}_2:\{\mathrm{X}=\infty\},\quad \mathrm{L}_3:\{\mathrm{Y}=\infty\}\quad\text{and}\quad \mathrm{L}_4:\{\mathrm{X}=0\}$$

Logarithmic one-forms with poles supported on D correspond to the space of closed one-forms considered above via the isomorphism (see Fig. 1):

$$\begin{cases}\omega_0=d\xi\\ \omega_\infty=\frac{1}{2i\pi\tau}\frac{dz}{z}\end{cases}\quad\text{and}\quad\begin{cases}\frac{1}{2i\pi}\frac{d\mathrm{X}}{\mathrm{X}}=\omega_0\\ \frac{1}{2i\pi}\frac{d\mathrm{Y}}{\mathrm{Y}}=\tau(\omega_0+\omega_\infty)\end{cases}$$

[10] I.e. lines bundles together with a holomorphic connection.

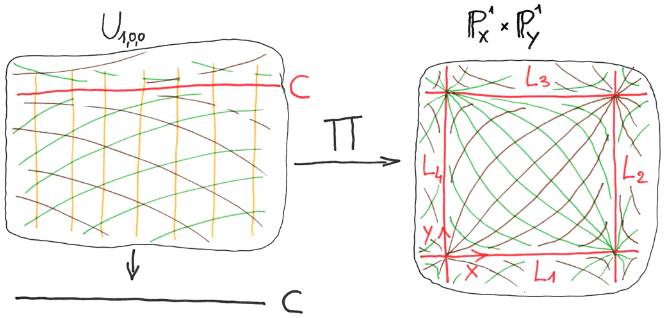

FIG. 1. — Serre isomorphism

Therefore, at the level of foliations, we have the following correspondence

$$\textbf{(1.3)}\qquad \mathcal{F}_t : \{\omega_0 + t\omega_\infty = 0\} \quad \longleftrightarrow \quad (1-t)\tau\frac{d\mathrm{X}}{\mathrm{X}} + t\frac{d\mathrm{Y}}{\mathrm{Y}} = 0.$$

In particular, for $m+\tau n \in \Gamma \sim \pi_1(\mathrm{C})$ in the lattice, the unique foliation with trivial holonomy along $m+\tau n$ corresponds to the one with rational first integral $\mathrm{X}^m\mathrm{Y}^n$:

$$\mathcal{F}_{\frac{\tau n}{m+\tau n}} \quad \longleftrightarrow \quad m\frac{d\mathrm{X}}{\mathrm{X}} + n\frac{d\mathrm{Y}}{\mathrm{Y}} = 0$$

and the ruling corresponds to a foliation with transcendental leaves:

$$\mathcal{F}_\infty \quad \longleftrightarrow \quad \tau\frac{d\mathrm{X}}{\mathrm{X}} - \frac{d\mathrm{Y}}{\mathrm{Y}} = 0.$$

Let us now study the isomorphism $\Pi : \mathrm{S}_0 \setminus \mathrm{C} \to \mathbf{P}^1 \times \mathbf{P}^1 \setminus \mathrm{D}$ near the compactifying divisors (compare [28]). Denote by V_i a tubular neighborhood of L_i in $\mathbf{P}^1_\mathrm{X} \times \mathbf{P}^1_\mathrm{Y}$, of the form $\mathrm{L}_i \times$ disc say, and $\mathrm{V} = \mathrm{V}_1 \cup \mathrm{V}_2 \cup \mathrm{V}_3 \cup \mathrm{V}_4$ the corresponding neighborhood of D. Denote by $\mathrm{V}^* = \mathrm{V} \setminus \mathrm{D}$ the complement of D in V. On may think of $\mathrm{U}_{1,0,0} \setminus \mathrm{C}$ as $\Pi^{-1}(\mathrm{V}^*)$.[11] Similarly, define $\mathrm{V}_i^* := \mathrm{V}_i \setminus \mathrm{D}$ the complement of the divisor in V_i and by $\mathrm{U}_i = \Pi^{-1}(\mathrm{V}_i^*)$ the preimage: we have a decomposition neighborhood $\mathrm{U}_{1,0,0} \setminus \mathrm{C} = \mathrm{U}_1 \cup \mathrm{U}_2 \cup \mathrm{U}_3 \cup \mathrm{U}_4$. One can show that $\mathrm{U}_i's$ look like sectorial domains of opening π in the variable y saturated by variable z (see Section 3.1 and Fig. 2). Our main result is that this sectorial decomposition together with isomorphisms $\Pi_i : \mathrm{U}_i \to \mathrm{V}_i^*$ persists for general neighborhoods (U, C) in the formal class $(\mathrm{U}_{1,0,0}, \mathrm{C})$; we conjecture and actually give the strategy to prove that a similar result holds true for all formal types, whenever N_C is torsion (see Section 10).

[11] Actually, this correspondence is meaningful in the germified setting in the sense that a basis of neighborhood (V^α) of D gives rise to a basis of neighborhoods (U^α) of C where $\mathrm{U}^\alpha = \Pi^{-1}(\mathrm{V}^\alpha \setminus \mathrm{D}) \cup \mathrm{C}$.

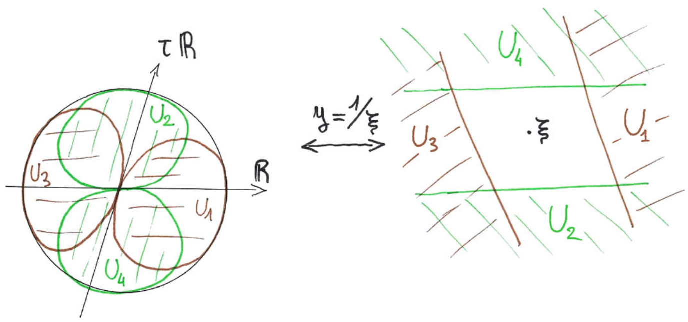

FIG. 2. — Neighborhoods correspondence

1.4. *Analytic classification: main result.* — A general neighborhood (U,C) formally conjugated to $(\mathrm{U}_{1,0,0},\mathrm{C})$ can be described as quotient (see Proposition 2.3)

$$\mathrm{U}=\tilde{\mathrm{U}}/<\mathrm{F}>\quad \text{where } \tilde{\mathrm{U}}\subset \mathbf{C}_z^*\times\mathbf{C}_y$$

is a neighborhood of the zero section $\tilde{\mathrm{C}}=\{y=0\}$, and

$$\mathrm{F}(z,y)=(qz+\mathrm{O}(y^2),y+y^2+y^3+\mathrm{O}(y^4)).$$

There is a formal isomorphism

$$\hat{\Psi}=\left(z+\sum_{m\geq 2}a_m(z)y^m,y+\sum_{n\geq 4}b_n(z)y^n\right)$$

such that $\hat{\Psi}\circ\mathrm{F}=\mathrm{F}_{1,0,0}\circ\hat{\Psi}$; we have $a_m,b_n\in\mathcal{O}(\mathbf{C}_z^*)$ and no convergence assumption in y-variable. We can also consider $\hat{\Psi}$ as a formal diffeomorphism $(\mathrm{U},\mathrm{C})\to(\mathrm{U}_{1,0,0},\mathrm{C})$. The main ingredient of our classification result, proved in Section 9, is the

Lemma **A** *(Sectorial normalization). — Denote* $\varpi=\arg\tau$. *For each interval*

$$(\mathbf{1.4})\qquad \mathrm{I}_1=]\varpi-\pi,\varpi[,\qquad \mathrm{I}_2=]0,\pi[,\qquad \mathrm{I}_3=\mathrm{I}_1+\pi,\qquad \mathrm{I}_4=\mathrm{I}_2+\pi$$

there is a transversely sectorial domain[12] $\mathrm{U}_i\subset\mathrm{U}$ *of opening* I_i *and a diffeomorphism*

$$\Psi_i:\mathrm{U}_i\to\mathrm{U}_{1,0,0}$$

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[12] Given an interval $\mathrm{I}=[\theta_1,\theta_2]\subset\mathbf{R}$, an open subset $\mathrm{U}_0\subset\mathrm{U}$ is said to be a *transversely sectorial of opening* I if the lift $\tilde{\mathrm{U}}_0\subset\tilde{\mathrm{U}}\subset\mathbf{C}_z^*\times\mathbf{C}_y$ contains, for arbitrary large and relatively compact open set $\mathcal{C}\Subset\mathbf{C}^*$ and arbitrary small $\epsilon>0$, a sector $\mathcal{C}\times\mathcal{S}(\mathrm{I}_\varepsilon,r)$ where

$$\mathcal{S}(\mathrm{I}_\varepsilon,r)=\{y\in\mathbf{C}\ ;\ \arg(y)\subset\mathrm{I}_\varepsilon,\ 0<|y|<r\},\quad \mathrm{I}_\varepsilon=]\theta_1+\varepsilon,\theta_2-\varepsilon[$$

for some $r>0$ (See also Definitions 3.1 and 3.4).

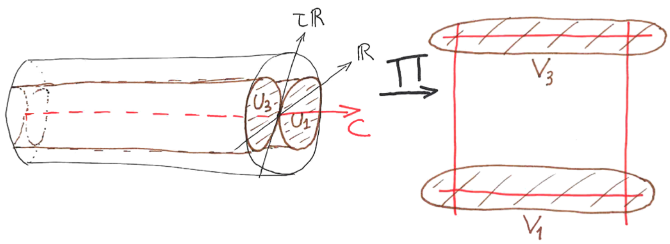

FIG. 3. — Sectorial normalization

(onto its image) having $\hat{\Psi}$ *as asymptotic expansion*[13] *along* C, *satisfying*

$$\Psi_i \circ F = F_{1,0,0} \circ \Psi_i.$$

After composition with the fundamental isomorphism $\Pi : U_{1,0,0} \to \mathbf{C}_X^* \times \mathbf{C}_Y^*$, we get (see Fig. 3)

Corollary **B.** — *The composition* $\Pi_i = \Pi \circ \Psi_i$ *provides an isomorphism germ*

$$\Pi_i : (U_i, C) \to (V_i^*, L_i) \quad \textit{such that } \Pi_i = \varphi_{i,i+1} \circ \Pi_{i+1} \textit{ on } U_i \cap U_{i+1}$$

for some diffeomorphism germs $\varphi_{i,i+1} \in \mathrm{Diff}(V_{i,i+1}, p_{i,i+1})$.[14]

After patching copies of germs $(V_i, L_i) \simeq (\mathbf{P}_X^1 \times \mathbf{P}_Y^1, L_i)$ by the $\varphi_{i,i+1} : (V_{i+1}, p_{i,i+1}) \to (V_i, p_{i,i+1})$, we get a new neighborhood germ (V_φ, D) of the divisor D, where $\varphi = (\varphi_{i,i+1})_{i \in \mathbf{Z}_4}$, together with a diffeomorphism germ

$$\Pi : (U \setminus C, C) \xrightarrow{\sim} (V_\varphi \setminus D, D)$$

which does not depend on the choice of sectorial normalisations Ψ_i.

More generally, consider a neighborhood (V, D) in which each component $L_i \subset D$ has zero self-intersection. Then after [25], the neighborhood (V, L_i) is trivial (a product $L_i \times$ disc). After identification with our model $\psi_i : (V, L_i) \xrightarrow{\sim} (\mathbf{P}_X^1 \times \mathbf{P}_Y^1, L_i)$, we get that V takes the form V_φ for a convenient 4-uple of diffeomorphisms φ. The gluing data φ is not unique as we can compose each embedding ψ_i by an automorphism germ $\varphi_i \in \mathrm{Diff}(V_i, L_i)$.[15] Therefore, it is natural to introduce the following equivalence relation

$$\varphi \sim \varphi' \quad \Leftrightarrow \quad \exists\, (\varphi_i \in \mathrm{Diff}(V_i, L_i))_{i \in \mathbf{Z}_4}$$
$$\text{such that } \varphi_i \circ \varphi'_{i,i+1} = \varphi_{i,i+1} \circ \varphi_{i+1}.$$

[13] The diffeomorphism $\Psi_i : U_i \to U_{1,0,0}$ admits $\hat{\Psi}_i$ as an asymptotic expansion along C if the entries of its lift $\tilde{\Psi}_i : \tilde{U}_i \to \mathbf{C}_z^* \times \mathbf{C}_y$ admit the entries of $\hat{\Psi}$ as asymptotic expansion on each open subset $\mathcal{C} \times \mathcal{S}(I_\varepsilon, r)$ (see Section 3.1).

[14] These are germs of diffeomorphisms on $V_i \cap V_{i+1}$ at $p_{i,i+1} = L_i \cap L_{i+1}$ preserving restrictions of each L_j. See Section 3.3.

[15] These are germs of diffeomorphisms on V_i along L_i preserving restrictions of L_{i-1} and L_{i+1}. See Section 3.3.

Clearly, the moduli space $\mathcal{V}$ of neighborhoods (V, D) up to analytic equivalence[16] identifies with the set of equivalence classes for $\sim$. Notice that each equivalence class contains a representative φ such that $\varphi_{1,2}, \varphi_{2,3}, \varphi_{3,4}$ are tangent to the identity, and the linear part

$$\varphi_{4,1}(\mathrm{X}, \mathrm{Y}) = (a\mathrm{X} + \cdots, b\mathrm{Y} + \cdots)$$

does not depend on the choice of such representative φ. Therefore, $a, b \in \mathbf{C}^*$ are invariants for the equivalence relation, and we denote by $\mathcal{V}_{a,b}$ the moduli space of those triples. With this in hand, we are able to prove:

Theorem **C.** — *We have a one-to-one correspondence between*

$$\mathcal{U}_{1,0,0} \leftrightarrow \mathcal{V}_{1,1}$$

- *the moduli space* $\mathcal{U}_{1,0,0}$ *of neighborhoods* (U, C) *formally equivalent to* $(\mathrm{U}_{1,0,0}, \mathrm{C})$ *up to analytic equivalence*[17]
- *the moduli space* $\mathcal{V}_{1,1}$ *of neighborhoods* $(\mathrm{V}_\varphi, \mathrm{D})$ *with all* $\varphi_{i,i+1}$ *tangent to the identity.*

Remark **1.2.** — A thorough look to the proof of the Sectorial Normalization Lemma (9) may prove that the correspondence is analytic in the sense that analytic families of neighborhoods $t \mapsto (\mathrm{U}_t, \mathrm{C})$ correspond to analytic families of cocycles $t \mapsto \varphi_t$. As the freedom lies in the choice of (essentially) one-dimensional diffeomorphisms φ_i, it is quite clear that the moduli space is essentially parametrized by two-dimensional diffeomorphisms, and therefore quite huge.

In a similar vein, it is reasonable to expect that the analytic moduli space $\mathcal{U}_{1,\nu,\mu}$ of neighborhoods (U, C) formally equivalent to $(\mathrm{U}_{1,\nu,\mu}, \mathrm{C})$ is in one to one correspondence with $\mathcal{V}_{a,b}$ with $a = e^{-4\pi^2\nu}$ and $b = e^{-4\pi^2\tau(\nu+\mu)}$. Actually, we explain in Section 10 how to construct an embedding $\mathcal{V}_{a,b} \hookrightarrow \mathcal{U}_{1,\nu,\mu}$, but the surjectivity needs to adapt our Sectorial Normalization Lemma. This creates additional issues (of purely technical nature) and we just indicate briefly how to proceed. Actually, one directly addresses in loc.cit the general case Ueda type $= k$, where we inherit $4k$ sectors with opening $\frac{\pi}{k}$ and the moduli space would be then equivalent to the moduli of neighborhoods of cycles of $4k$ rational curves (the model must be thought as a degree k cyclic étale cover of (V, D)). A precise statement, summarizing the structure of the analytic moduli space when $\mathrm{N}_\mathrm{C} \simeq \mathcal{O}_\mathrm{C}$, is given in Section 10, Theorem E. With this in hand, it is not difficult to undertake the analytic classification of neighborhood when N_C is torsion. The idea consists in reducing to the case of trivial normal bundle by an appropriate cyclic cover. This is settled in Section 11.

[16] One requires each component of the cycle to be preserved.

[17] More precisely, we allow for this statement analytic isomorphisms inducing translations on C; see Proposition 4.3 for a more precise statement.

1.5. *Foliations.* — A neighborhood (U, C) formally conjugated to $(U_{1,0,0}, C)$ admits a pencil of formal foliations $\hat{\mathcal{F}}_t$ corresponding to $\mathcal{F}_t$ in (1.3) via the formal normalization $\hat{\Psi}$.

Theorem **D.** — *The foliation $\hat{\mathcal{F}}_t$ is convergent if, and only if, there exists a representative φ in the corresponding equivalence class such that each $\varphi_{i,i+1}$ preserves the foliation*

$$(1-t)\tau\frac{dX}{X} + t\frac{dY}{Y} = 0.$$

In that case, these two foliations are conjugated via the isomorphism $U \setminus C \to V \setminus D$.

When $\mathcal{F}_t$ is not of rational type, i.e. $\tau(1-\frac{1}{t}) \notin \mathbf{Q} \cup \{\infty\}$, then $\mathcal{F}_t$ is defined by a closed meromorphic 1-form and the logarithmic 1-form of the statement is also preserved by all $\varphi_{i,i+1}$ and defines a global logarithmic 1-form on (V, D). On the other hand, in the rational case, Écalle-Voronin moduli of the holonomy provide obstruction to define the foliation by a closed meromorphic 1-form. For instance, when $\mathcal{F}_0$ is convergent, Martinet-Ramis cocycle are given by the X-coordinate of $\varphi_{1,2} \circ \varphi_{2,3}$ and $\varphi_{3,4} \circ \varphi_{4,1}$ (see Section 7.9 for details).

In [15], the two first authors with O. Thom provided the analytic classification of neighborhoods with 2 convergent foliations. In Section 7, we provide examples of neighborhoods with only one foliation, and also without foliation which is the generic case. An example without foliations has been given by Mishustin in [21] few years ago and it would be nice to understand what is the corresponding invariant φ.

In Section 8, we investigate the automorphism group of neighborhood germs. We prove in Theorem 8.1 that it can be of three types: finite (the generic case), one dimensional and we get an holomorphic vector field (and in particular a convergent foliation), or two dimensional only in the Serre example.

1.6. $\mathrm{SL}_2(\mathbf{Z})$ *action.* — The analytic classification of resonant diffeomorphism germs of one variable is reminiscent in our classification result. However, there are strong differences like the fact that the sectorial trivialization is not unique in our case. Indeed, our sectorial decomposition $U \setminus C = U_1 \cup U_2 \cup U_3 \cup U_4$ has been imposed by our choice of a basis for the lattice $\Gamma = \mathbf{Z} + \tau\mathbf{Z}$. It comes from the sectorial decomposition of the holonomy maps of the two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ having cyclic holonomy, trivial along 1 and τ respectively. If we change for another basis $(m + \tau n, m' + \tau n')$, with

$$\begin{pmatrix} m & m' \\ n & n' \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

then the change of coordinates

$$x' = \frac{x}{m+\tau n}, \quad \xi' = (m+\tau n)\xi + nx \quad \rightsquigarrow z' = e^{2i\pi x'} = z^{\frac{1}{m+\tau n}}$$

gives $(\mathrm{S}_0, \mathrm{C})$ as the quotient of $\mathbf{C}^*_{z'} \times \overline{\mathbf{C}}_{\xi'}$[18] by the transformation

$$(z', \xi') \mapsto (q'z', \xi' - 1), \quad q' = e^{2i\pi\tau'}, \ \tau' = \frac{m' + \tau n'}{m + \tau n}.$$

The new isomorphism is related to the previous one by a monomial transformation

$$(\mathrm{X}', \mathrm{Y}') = (e^{2i\pi\xi'}, z' e^{2i\pi\tau'\xi'}) = (\mathrm{X}^m \mathrm{Y}^n, \mathrm{X}^{m'} \mathrm{Y}^{n'}).$$

Using sectorial normalization for a general neighborhood (U, C) with this new basis gives a new compactification $(\mathrm{V}', \mathrm{D})$ which is bimeromorphically equivalent to (V, D).

1.7. *Concluding remarks.* — Contrary to the diophantine case (non torsion normal bundle), the classification of neighborhoods of elliptic curves with torsion normal bundle can be completely described, as shown in Theorem E. A naive reading of Arnold's work [1] might suggest that classification of neighborhoods of elliptic curves with topologically trivial normal bundle could be similar to that of germs of one dimensional diffeomorphisms. In fact, the suspension of a representation $\pi_1(\mathrm{C}) \to \mathrm{Diff}(\mathbf{C}, 0)$ permits to embed the moduli space of diffeomorphisms into that of neighborhoods. However, this latter one turns out to be much more complicated, even if the general approach by sectorial normalization and classifying cocycle is still in the spirit of Écalle-Voronin classification for resonant diffeomorphisms, or Martinet-Ramis' version. For instance, an unexpected phenomenon in the case of neighborhoods is that the sectorial covering is not unique, due to the $\mathrm{SL}_2(\mathbf{Z})$-action. We can expect that the sectorial normalizations involve resurgent functions with lattice of singularities isomorphic to the lattice of the elliptic curve. Recall that the lattice of resurgence for resonant diffeomorphism has rank one. It would be interesting to better understand this phenomenon.

An important motivation to study neighborhoods was initially raised by Arnold: there is a close link with the study of germs of analytic diffeomorphisms of $(\mathbf{C}^2, 0)$. Indeed, if we consider our model $\mathrm{F}_{1,0,0}(z, y) = (qz, \frac{y}{1-y})$ at the neighborhood of $(z, y) = (0, 0)$, then we get a semi-hyperbolic map whose space of orbits (when deleting $z = 0$) is obviously the neighborhood $(\mathrm{U}_{1,0,0}, \mathrm{C})$ where $\mathrm{C} = \mathbf{C}^* / < qz >$. One can investigate the analytic classification of small perturbations $\mathrm{F} := \mathrm{F}_{1,0,0} + \cdots$ where dots are vanishing at sufficiently high order at the origin. Then, it is a classical fact that F has also an invariant manifold in the contracting direction $|q| < 1$, and the space of orbits gives rise to a neighborhood (U, C) formally equivalent to $(\mathrm{U}_{1,0,0}, \mathrm{C})$. The analytic classification of these germs of semi-hyperbolic maps has been done by the last author with P. A. Fomina-Shaĭkhullina (see [27, 38]) and comparing the two moduli shows that moduli of maps embed in moduli of neighborhoods but is infinite codimensional. In fact, the analytic extension of the map F to the origin imposes strong restrictions on the corresponding invariants φ defined in Section 1.4. It is interesting to consider the following hierarchy:

[18] It may be useful to think of $\mathbf{C}^*_{z'} \times \overline{\mathbf{C}}_{\xi'}$ as the cyclic cover of S_0 associated to the subgroup $\langle m + n\tau \rangle$ of Γ (see 9.1).

1. one dimensional resonant diffeomorphisms in $(\mathbf{C}, 0)$,
2. singular points of foliation in $(\mathbf{C}^2, 0)$ of resonant-saddle or saddle-node type,
3. singular points of vector fields in $(\mathbf{C}^2, 0)$ of resonant-saddle or saddle-node type,
4. singular points of diffeomorphisms in $(\mathbf{C}^2, 0)$ of resonant-saddle or saddle-node (i.e. semi-hyperbolic) type,
5. neighborhoods of elliptic curves with torsion normal bundle.

The first occurrence gives rise to Écalle-Voronin moduli (see [4, 5, 36], and also [16]). One dimensional resonant diffeomorphisms also occur as monodromy map of those foliations arising in case (2). These latter ones have been classified by Martinet-Ramis in [17, 18] and the classification on resonant-saddles and their monodromy map (1) turns out to be equivalent; however, saddle-node impose strong restriction to the invariants of its holonomy map (we can realize half of the moduli only). See [16–18] for details. Classification of vector fields has been done by the third author with Meshcheryakova (see [37] for instance) and independently by Teyssier [30]. This gives rise to twice the moduli space of foliations: the classification of vector fields with same underlying foliation is solved by the linearization of the conjugacy equation for foliations. Still, the moduli space is parametrized by finitely many copies of $\mathbf{C}\{x\}$ (power-series in one variable). There is a huge step when we pass to diffeomorphisms in $(\mathbf{C}^2, 0)$ as the moduli space is now parametrized by copies of $\mathbf{C}\{x, y\}$. Diffeomorphisms occurring in (4) are actually one-time-map of formal vector fields of type (3), but divergent as a rule. We expect that resonant-saddle diffeomorphisms (4) have same classification as neighborhoods (5), but classification of former ones looks somehow more delicate. It would also be nice to understand how Ueda's results [34, 35] can be related to our work from that point of view. Also, diffeomorphisms of $(\mathbf{C}^2, 0)$ arise as monodromy map of reduced singular foliations by curves in $(\mathbf{C}^3, 0)$ and we can expect that the analytic classification is similar under generic conditions on the spectrum.

One might expect to investigate higher dimensional neighborhood of elliptic curve with trivial normal bundle by mimicking what has been done in [15] for the formal classification, and in the present paper regarding analytic classification. We haven't considered this direction at all. However, it might be interesting to note that Ueda's Theory has been generalized to higher codimension by Koike in [13]. What would be the higher dimensional analogue of Serre's isomorphism?

2. Preliminary remarks

Recall that $\mathrm{C} = \mathbf{C}^*/ < q >$, and we denote by $\tilde{\mathrm{C}} \simeq \mathbf{C}^* \to \mathrm{C}$ the corresponding cyclic cover. Denote $\tilde{\mathrm{U}} = \mathbf{C}^*_z \times \mathbf{C}_y$ and $\tilde{\mathrm{C}} = \{y = 0\} \subset \tilde{\mathrm{U}}$. The following is already mentioned by Arnol'd [1].

Lemma **2.1.** — *Any germ of neighborhood* (U,C) *with* $\mathrm{C}^2=0$ *is biholomorphic to a germ of the form* $(\tilde{\mathrm{U}},\tilde{\mathrm{C}})/<\mathrm{F}>$ *where*

$$\mathrm{F}(z,y)=(qz+yf(z,y),\lambda(z)y+y^2g(z,y)) \tag{2.1}$$

with f, g *holomorphic on a neighborhood of* $\{y=0\}$, *where* $q=e^{2i\pi\tau}$ *and* $\lambda\in\mathcal{O}^*(\mathbf{C}^*_z)$.

Proof. — The self-intersection $\mathrm{C}\cdot\mathrm{C}$ determines topologically the germ of neighborhood. Then, by taking a suitable small representative, U is homeomorphic to a product $\mathbf{D}\times\mathrm{C}$. So, one can consider the cyclic covering $\tilde{\mathrm{U}}\to\mathrm{U}$ extending the cyclic cover $\tilde{\mathrm{C}}\to\mathrm{C}$. This gives rise to a neighborhood $\tilde{\mathrm{U}}$ of $\tilde{\mathrm{C}}\simeq\mathbf{C}^*$. Following Siu [29], the germ of this neighborhood along $\tilde{\mathrm{C}}$ is isomorphic to the germ of a neighborhood of the zero section $\{y=0\}$ in the normal bundle $\mathrm{N}_{\tilde{\mathrm{C}}}\simeq\mathbf{C}^*_z\times\mathbf{C}_y$. The deck transformation of the (germ of) covering takes the form F of the statement. □

Definition/Proposition **2.2.** — [19] *Any two quotients* $(\tilde{\mathrm{U}},\tilde{\mathrm{C}})/<\mathrm{F}>$ *and* $(\tilde{\mathrm{U}},\tilde{\mathrm{C}})/<\mathrm{F}'>$ *are analytically (resp. formally) equivalent, and we note*

$$(\mathrm{U},\mathrm{C})\overset{an}{\sim}(\mathrm{U}',\mathrm{C})\quad \textit{(resp. } (\mathrm{U},\mathrm{C})\overset{for}{\sim}(\mathrm{U}',\mathrm{C})\textit{)},$$

if there is a germ of analytic (resp. formal) diffeomorphism

$$\Psi(z,y)=\left(z+\sum_{n=1}^{\infty}a_n(z)y^n,\sum_{n=1}^{\infty}b_n(z)y^n\right)\quad \textit{such that } \Psi\circ\mathrm{F}=\mathrm{F}'\circ\Psi. \tag{2.2}$$

Although the formal classification is already done in [15], we need the following formulation and give some basic steps.

Proposition **2.3.** — *A germ of neighborhood* (U,C) *is formally equivalent to*

$$(\mathrm{U}_0,\mathrm{C})=(\tilde{\mathrm{U}},\tilde{\mathrm{C}})/<\mathrm{F}_{1,0,0}>,\quad \mathrm{F}_{1,0,0}(z,y)=\left(qz,\frac{y}{1-y}\right)$$

if, and only if, it is biholomorphic to a germ of the form $(\tilde{\mathrm{U}},\tilde{\mathrm{C}})/<\mathrm{F}>$ *where*

$$\mathrm{F}(z,y)=(qz+y^2f(z,y),y+y^2+y^3+y^4g(z,y)). \tag{2.3}$$

In that case, there exists a formal diffeomorphism (tangent to the identity along C*)*

$$\hat{\Psi}(z,y)=\left(z+\sum_{n>0}a_n(z)y^n,y+\sum_{n>1}b_n(z)y^n\right) \tag{2.4}$$

[19] Actually, one can, as usually, define analytic/formal conjugations between both neighborhoods in terms involving only the structural analytic/formal sheaves along C. The two definitions obviously coincide.

with a_n, $b_n \in \mathcal{O}(\mathbf{C}_z^*)$ (and no convergence condition on y), such that

$$\hat{\Psi} \circ \mathrm{F} = \mathrm{F}_{1,0,0} \circ \hat{\Psi}. \tag{2.5}$$

Moreover, any other formal diffeomorphism $\hat{\Psi}'$ of the form (2.4) satisfying (2.5) writes

$$\hat{\Psi}' = \Phi \circ \hat{\Psi} \quad \textit{where } \Phi(z,y) = \left(z, \frac{y}{1-ty}\right) = (z, y + ty^2 + \cdots), \ t \in \mathbf{C}. \tag{2.6}$$

Proof. — Let $g = \sum_{n\in\mathbf{Z}} g_n z^n$ be holomorphic on $\mathbf{C}_z^*$. The functional equation

$$\phi(qz) - \phi(z) = g(z) \tag{2.7}$$

admits a solution ϕ holomorphic on $\mathbf{C}_z^*$ if, and only if, $g_0 = 0$; then ϕ is unique up to the choice of $\phi(0)$. Indeed, if we write $\phi(z) = \sum_{n\in\mathbf{Z}} \phi_n z^n$, then equation (2.7) writes $\phi_n(q^n - 1) = g_n$ for all n.

Let f be a holomorphic non vanishing function on $\mathbf{C}_z^*$. The functional equation

$$\varphi(qz)/\varphi(z) = f(z) \tag{2.8}$$

admits a solution φ holomorphic and non vanishing on $\mathbf{C}_z^*$ if, and only if,

– $f : \mathbf{C}^* \to \mathbf{C}^*$ has topological index 0 so that $g = \log(f)$ is well-defined,
– the coefficient g_0 of $g = \sum_{n\in\mathbf{Z}} g_n z^n$ vanishes.

Indeed, topological index is multiplicative and those of $\varphi(qz)$ and $\varphi(z)$ are equal and cancel each other. Then we can solve the corresponding equation (2.8) for g and set $\varphi = \exp(\phi)$, which is unique up to a multiplicative constant. Note that, if $g_0 \neq 0$, then we can solve

$$\varphi(qz)/\varphi(z) = \frac{f(z)}{a} \tag{2.9}$$

for $a = \exp(g_0)$.

Let us start with F like in (2.1). The change of coordinate $\Psi_1(z,y) = (z, \varphi(z)y)$ yields

$$\Psi_1^{-1} \circ \mathrm{F} \circ \Psi_1(z,y) = \left(qz + \mathrm{O}(y), \frac{\varphi(z)}{\varphi(qz)} \lambda(z) y + \mathrm{O}(y^2)\right).$$

We can easily check that the coefficient f in F defines the normal bundle $\mathrm{N_C}$ in the quotient, and its topological index coincides with $\deg(\mathrm{N_C})$ which is zero in our case. Then we can find $\varphi \in \mathcal{O}^*(\mathbf{C}_z^*)$ satisfying (2.9) and get

$$\mathrm{F}_1(z,y) = \Psi_1^{-1} \circ \mathrm{F} \circ \Psi_1(z,y) = (qz + \mathrm{O}(y), ay + \mathrm{O}(y^2)).$$

Moreover, φ is unique up to a multiplicative constant. The coefficient a can be interpreted as a flat connection on N_C with trivial monodromy along the loop $1 \in \Gamma$ and monodromy a along the loop $\tau \in \Gamma$. In our case, $N_C = \mathcal{O}_C$ and $a = 1$ and we can write

$$F_1(z, y) = (qz + O(y), y + g(z)y^2 + O(y^3)).$$

Now the change of coordinate $\Psi_2(z, y) = (z, y + \phi(z)y^2)$ gives

$$\Psi_2^{-1} \circ F_1 \circ \Psi_2(z, y) = (qz + O(y), y + [g(z) + \phi(z) - \phi(qz)]y^2 + O(y^3)).$$

Solving equation (2.7), we get

$$F_2(z, y) = \Psi_2^{-1} \circ F_1 \circ \Psi_2(z, y) = (qz + O(y), y + by^2 + O(y^3)).$$

In our case, $b \neq 0$ (i.e. Ueda type $k = 1$). By using a change $(z, \lambda y)$ (freedom in the choice of φ above) we can set $b = 1$ and write

$$F_2(z, y) = (qz + qzf(z)y + O(y^2), y + y^2 + g(z)y^3 + O(y^4)).$$

The change of coordinate $\Psi_3(z, y) = (z + z\varphi(z)y, y + \phi(z)y^3)$ gives

$$\begin{aligned} F_3(z, y) &= \Psi_3^{-1} \circ F_2 \circ \Psi_3(z, y) \\ &= (qz + qz[f(z) + \varphi(z) - \varphi(qz)]y + O(y^2), y + y^2 \\ &\quad + [g(z) + \phi(z) - \phi(qz)]y^3 + O(y^4)). \end{aligned}$$

Solving twice equation (2.7), we get

$$F_3(z, y) = (qz + \alpha zy + O(y^2), y + y^2 + \beta y^3 + O(y^4)).$$

Here, we have no freedom and α, β are formal invariants corresponding to μ, ν in the end of Section 1.4: in the formal class $U_{1,0,0}$ we get $\alpha = 0$ and $\beta = 1$. Then, we can kill-out all higher order terms in F by a formal change of coordinate, or better normalize it to $F_{1,0,0}$. Indeed, at the N^{th} step, we get

$$\begin{aligned} F_N(z, y) = (&qz + qzf(z)y^{N-1} + O(y^N), y + y^2 + y^3 + \cdots \\ &+ g(z)y^{N+1} + O(y^{N+2})); \end{aligned}$$

the coordinate change $\Psi_{N+1}(z, y) = (z + azy^{N-2} + z\varphi(z)y^{N-1}, y + by^N + \phi(z)y^{N+1})$ gives

$$\begin{aligned} F_{N+1}(z, y) &= \Psi_{N+1}^{-1} \circ F_N \circ \Psi_{N+1}(z, y) \\ &= (qz + qz[f(z) + \varphi(z) - \varphi(qz) - (N-2)a]y^{N-1} + O(y^N), \\ y + y^2 + y^3 + \cdots &+ [g(z) + \phi(z) - \phi(qz) - (N-4)b]y^{N+1} + O(y^{N+2})). \end{aligned}$$

We can clearly normalize the two coefficients into brackets by a constant, and can even choose the constant by means of a, b.

The composition of all changes of coordinates $\hat{\Psi}^{-1} := \Psi_1 \circ \Psi_2 \circ \Psi_3 \circ \cdots$ converges in the formal topology as a formal diffeomorphism satisfying (2.5). For any other formal diffeomorphism $\hat{\Psi}'$ of the form (2.4) satisfying (2.5), we have that $\hat{\Phi} := \hat{\Psi}' \circ \hat{\Psi}^{-1}$ is an automorphism of $(\mathrm{U}_0, \mathrm{C})$ inducing the identity on C. As we shall see in Lemma 3.9, $\hat{\Phi}$ is necessarily convergent and of the form (2.6). □

3. Sectorial decomposition and sectorial symmetries

In this section, we introduce the sectorial decomposition of U by transversely sectorial domains $\mathrm{U}_i = \Pi^{-1}(\mathrm{V}_i^*)$ and compare spaces of functions on both sides. From now on, we work in the variable $\xi = 1/y$, at the neighborhood of $\xi = \infty$; this is much more convenient for computations. Notations are as in Section 1.3.

3.1. *Some sheaves of functions on the circle of directions.*

— Let $\mathbf{S}^1 := \mathbf{R}/2\pi\mathbf{Z}$ and I be an open interval of $\mathbf{R}$ (regarded as the universal covering of $\mathbf{S}^1$).

Definition **3.1.** — *For* $(c, \mathrm{R}) \in]0, +\infty] \times [0, +\infty[$, *denote by*

$$\mathrm{S}(\mathrm{I}, \mathrm{R}; c) = \{(z, \xi) \in \mathbf{C}_z^* \times \mathbf{C}_\xi \ ; \ \arg(\xi) \subset \mathrm{I},\ \mathrm{R} < |\xi|,\ e^{-c} < |z| < e^c\}.$$

A sector of opening I *is an open subset* $\Sigma_\mathrm{I} \subset \mathrm{S}(\mathrm{I}, 0; \infty)$ *such that for all* $c \gg 0$, *there exists* $\mathrm{R}_c > 0$ *such that*

$$\mathrm{S}(\mathrm{I}, \mathrm{R}_c; c) \subset \Sigma_\mathrm{I}.$$

Let Σ_I be an open sector as above. Then, $\mathcal{O}(\Sigma_\mathrm{I})$ contains the subalgebra $\mathcal{A}(\Sigma_\mathrm{I})$ of holomorphic functions admitting an asymptotic expansion along $\mathbf{C}_z^*$ in the sense defined below:

Definition **3.2.** — *Let* m *be a positive integer. A function* $f \in \mathcal{O}(\Sigma_\mathrm{I})$ *belongs to* $\mathcal{A}^m(\Sigma_\mathrm{I})$ *if there exists a polynomial* $\mathrm{P}_m(f) = \sum_{0\le k\le m} a_k \xi^{-k} \in \mathcal{O}(\mathbf{C}_z^*)[\xi^{-1}]$ *such that* $\forall\, c \gg 0$, $\exists\, \mathrm{C}_c, \mathrm{R}_c > 0$ *such that* $\forall (z, \xi) \in \mathrm{S}(\mathrm{I}, \mathrm{R}_c; c) \subset \Sigma_\mathrm{I}$, *we have*

$$\left| f(z, \xi) - \mathrm{P}_m(f)(z, \xi) \right| \le \frac{\mathrm{C}_c}{|\xi^{m+1}|}. \tag{3.1}$$

Note that $\mathrm{P}_m(f)$ *is necessarily unique.*

Define

$$\mathcal{A}(\Sigma_\mathrm{I}) = \bigcap_m \mathcal{A}^m(\Sigma_\mathrm{I}).$$

Then one can associate to f *its asymptotic expansion along* $\{\xi = \infty\}$*. This is a formal power series* $\hat{f} \in \mathcal{O}(\mathbf{C}_z^*)[[\xi^{-1}]]$ *whose truncation at order* m *coincides with* $\mathrm{P}_m(f)$*. The asymptotic expansion is then unique, and we have a well-defined morphism of* $\mathbf{C}$*-algebras*

$$\mathcal{A}(\Sigma_{\mathrm{I}}) \to \mathcal{O}(\mathbf{C}_z^*)[[\xi]]\ ;\quad f \mapsto \hat{f},$$

whose kernel, denoted $\mathcal{A}^\infty(\Sigma_{\mathrm{I}})$*, consists of flat functions.*

When fixing only I and taking inductive limits associated to restriction maps induced by the inclusion between open sectors, the collection of algebras of the form $\mathcal{O}(\Sigma_{\mathrm{I}})$ defines an algebra of germs $\mathcal{O}_{\mathrm{I}}$. Alternatively one can recover $\mathcal{O}_{\mathrm{I}}$ by considering $\mathcal{O}_{\mathrm{R},c} := \mathcal{O}\left(\mathrm{S}(\mathrm{I},\mathrm{R};c)\right)$ and taking inductive limit with respect to the parameter R and projective limit with respect to the parameter c:

$$\mathcal{O}_{\mathrm{I}} = \varprojlim_{0<c}\left(\varinjlim_{\mathrm{R}\geq 0}\mathcal{O}_{\mathrm{R},c}\right)$$

The presheaf on $\mathbf{S}^1$ defined by $\mathrm{I} \to \mathcal{O}_{\mathrm{I}}$ naturally gives rise to a sheaf on $\mathbf{S}^1$ which we will denote by $\mathcal{O}$. One can define on the same way the sheaves $\mathcal{A}^m$, $\mathcal{A}$, $\mathcal{A}^\infty$ respectively associated to $\mathrm{I} \to \mathcal{A}_{\mathrm{I}}^m$, $\mathrm{I} \to \mathcal{A}_{\mathrm{I}}$, $\mathrm{I} \to \mathcal{A}_{\mathrm{I}}^\infty$ and the last two are sheaves of differential algebras with respect to ∂_z and ∂_ξ. The stability by derivation is indeed a straightforward consequence of Cauchy's formula. As the asymptotic expansion is independent of the representative, we have a morphism of sheaves

$$\mathcal{A} \to \mathcal{O}(\mathbf{C}_z^*)[[\xi]]\ ;\ f \mapsto \hat{f}$$

whose kernel is $\mathcal{A}_{\mathrm{I}}^\infty$ (here $\mathcal{O}(\mathbf{C}_z^*)[[\xi]]$ is viewed as a constant sheaf over $\mathbf{S}^1$).

Remark **3.3.** — Mind that the inclusion $\mathcal{O}_{\mathrm{I}} \to \mathcal{O}(\mathrm{I})$ (resp. $\mathcal{A}_{\mathrm{I}} \to \mathcal{A}(\mathrm{I})$) is strict. For instance, one must think that a section $f \in \mathcal{A}(\mathrm{I})$ can be represented for every interval $\mathrm{J} \Subset \mathrm{I}$ by a function belonging to $\mathcal{A}(\Sigma_{\mathrm{J}})$ for suitable sectors of opening J but does not necessarily admit a representative on a sector of the form Σ_{I}. In other words, the domain of definition of f is a transversely sectorial open set in the following sense.

Definition **3.4.** — *Given an interval* $\mathrm{I} =]\theta_1, \theta_2[\subset \mathbf{R}$*, an open subset* $\Sigma \subset \mathrm{S}(\mathrm{I}, 0; \infty) \subset \mathbf{C}_z^* \times \mathbf{C}_\xi$ *is said transversely sectorial of opening* I *if, for arbitrary large* $c \gg 0$ *and small* $\epsilon > 0$*, there exists* $\mathrm{R}_{c,\epsilon} > 0$ *such that*

$$\mathcal{S}(\mathrm{I}_\epsilon, \mathrm{R}_{c,\epsilon}; c) \subset \Sigma, \quad \textit{where } \mathrm{I}_\epsilon =]\theta_1 + \epsilon, \theta_2 - \epsilon[.$$

Remark **3.5.** — The sheaves $\mathcal{O}$, $\mathcal{A}$ and $\mathcal{A}^\infty$ are invariant under the action of a diffeomorphism F of the form (2.3) (expressed in the (z, ξ) coordinates). Moreover, this

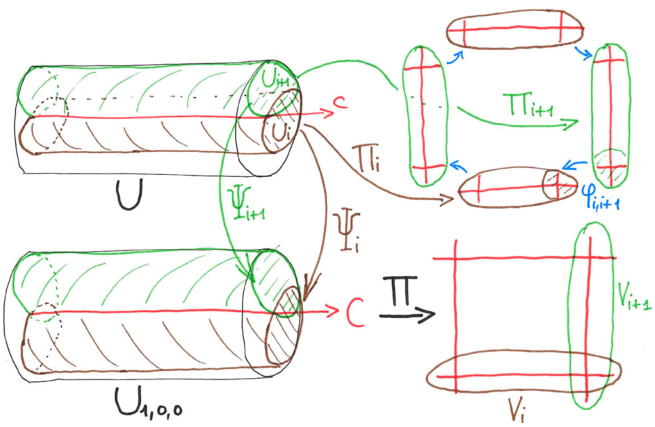

FIG. 4. — Sectorial open sets

action is stalk-preserving due to the fact that F is tangent to the identity along $\tilde{\mathrm{C}}$ on the transversal direction ξ. In particular, they define similar sheaves of sectorial functions on the quotient $(\mathrm{U},\mathrm{C}) = (\tilde{\mathrm{U}},\tilde{\mathrm{C}})/<\mathrm{F}>$ by considering those sections invariant under F. We will denote by $\mathcal{O}[\mathrm{F}]$, $\mathcal{A}[\mathrm{F}]$ and $\mathcal{A}^\infty[\mathrm{F}]$ these latter sheaves. In the next section, we characterize sections of $\mathcal{A}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I})$ for special intervals I.

3.2. *Sectorial decomposition.* — Denote $\varpi = \arg(\tau) \in]0,\pi[$ and let us define[20]

$$\mathrm{I}_1 =]-\varpi, \pi-\varpi[,\quad \mathrm{I}_2 =]-\pi, 0[,\quad \mathrm{I}_3 = \mathrm{I}_1+\pi \quad\text{and}\quad \mathrm{I}_4 = \mathrm{I}_2+\pi.$$

Denote by V_i a (small enough) neighborhood of $\mathrm{L}_i \subset \mathbf{P}^1_{\mathrm{X}} \times \mathbf{P}^1_{\mathrm{Y}}$ where

$$\mathrm{L}_1 : \{\mathrm{Y}=0\},\quad \mathrm{L}_2 : \{\mathrm{X}=\infty\},\quad \mathrm{L}_3 : \{\mathrm{Y}=\infty\}\quad\text{and}\quad \mathrm{L}_4 : \{\mathrm{X}=0\}.$$

Denote $\mathrm{D} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{L}_3 \cup \mathrm{L}_4$, and $\mathrm{V}_i^* = \mathrm{V}_i \setminus \mathrm{D}$. Let $\mathrm{V}_{i,i+1} = \mathrm{V}_i \cap \mathrm{V}_{i+1}$ for $i \in \mathbf{Z}_4$ and $\mathrm{V}_{i,i+1}^* = \mathrm{V}_{i,i+1} \setminus \mathrm{D}$. Recall that

$$\Pi : \mathrm{S}_0 \setminus \mathrm{C} \xrightarrow{\sim} \mathbf{C}^*_{\mathrm{X}} \times \mathbf{C}^*_{\mathrm{Y}}\ ;\quad (z,\xi) \mapsto (e^{2i\pi\xi}, ze^{2i\pi\tau\xi}).$$

Then we have:

Proposition **3.6.** — *The preimage* $\mathrm{U}_i = \Pi^{-1}(\mathrm{V}_i^*)$ *lifts on* $\tilde{\mathrm{U}} = \mathbf{C}^*_z \times \mathbf{C}_\xi$ *as a transversely sectorial open set of opening* I_i *(in the sense of Definition 3.4). Moreover, the lift of* $\Pi^{-1}(\mathrm{V}_{i,i+1}^*) = \mathrm{U}_i \cap \mathrm{U}_{i+1}$ *is transversely sectorial of opening* $\mathrm{I}_i \cap \mathrm{I}_{i+1}$ *(see Fig. 4).*

Proof. — For instance, for $a, b, c > 0$, we easily check that for

$$\mathrm{V}_{4,1} = \{(\mathrm{X},\mathrm{Y}) \in \mathbf{C}^* \times \mathbf{C}^*\ ;\ |\mathrm{X}| < \exp(-a),\ |\mathrm{Y}| < \exp(-b)\},$$

[20] Mind that these intervals for $\arg(\xi)$ correspond to those defined in Lemma A for $\arg(y) = -\arg(\xi)$.

$\Pi^{-1}(\mathrm{V}_{4,1}^*)$ contains the sectorial open set

$$\left\{(z,\xi)\in\mathbf{C}^*\times\mathbf{C}\ ;\ e^{-c}<|z|<e^{c},\ \mathrm{Im}(\xi)>\frac{a}{2\pi},\ \mathrm{Im}(\tau\xi)>\frac{b+c}{2\pi}\right\}.$$

The remaining cases are similar and straightforward. □

Denote $p_{i,i+1}=\mathrm{L}_i\cap\mathrm{L}_{i+1}$. Denote by $\mathcal{O}^0(\mathrm{V}_i,\mathrm{L}_i)$ (resp. $\mathcal{O}^0(\mathrm{V}_{i,i+1},p_{i,i+1})$) the set of germs of holomorphic functions on $(\mathrm{V}_i,\mathrm{L}_i)$ (resp. $(\mathrm{V}_{i,i+1},p_{i,i+1})$) vanishing along L_i (resp. at $p_{i,i+1}$). Denote by $\mathcal{A}^\infty[\mathrm{F}_{1,0,0}]$ the subsheaf of $\mathcal{A}^\infty$ whose sections f are invariant by $\mathrm{F}_{1,0,0}(z,\xi)=(qz,\xi-1)$.

Proposition **3.7.** — *A section $f\in\mathcal{O}(\mathrm{I}_i)$ (resp. $\mathcal{O}(\mathrm{I}_{i,i+1})$) belongs to $\mathcal{A}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_i)$ (resp. $\mathcal{A}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_{i,i+1})$) if, and only if, $f=g\circ\Pi$ with $g\in\mathcal{O}^0(\mathrm{V}_i,\mathrm{L}_i)$ (resp. $\mathcal{O}^0(\mathrm{V}_{i,i+1},p_{i,i+1})$).*

Proof. — As before, we only give the proof for $\mathrm{I}_{4,1}$, the other cases are similar. If $f=g\circ\Pi$ with $g\in\mathcal{O}^0(\mathrm{V}_{4,1},p_{4,1})$, then $g(\mathrm{X},\mathrm{Y})=\mathrm{X}g_1(\mathrm{X},\mathrm{Y})+\mathrm{Y}g_2(\mathrm{X},\mathrm{Y})$ with g_k holomorphic at $p_{4,1}$ (and therefore bounded), so that $f(z,\xi)=e^{2i\pi\xi}f_1(z,\xi)+e^{2i\pi\tau\xi}f_2(z,\xi)$ with f_k bounded: clearly, f is (exponentially) flat at $\xi=\infty$ in restriction to any sector $\mathrm{S}(\mathrm{J},\mathrm{R};c)\subset\mathrm{U}_{4,1}$, with $\mathrm{J}\Subset\mathrm{I}_{4,1}$.

Conversely, let $f\in\mathcal{A}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_{4,1})$, defined on a sectorial open set $\mathrm{U}_{4,1}$ of opening $\mathrm{I}_{4,1}$. Let $\mathrm{U}_{4,1}$ be the domain of definition of f, a transversely sectorial open set of opening $\mathrm{I}_{4,1}$ (see definition 3.4). One can find another one $\mathrm{U}'_{4,1}\subset\mathrm{U}_{4,1}$ such that

$$\begin{aligned}&\forall(z_0,\xi_0)\in\mathrm{U}'_{4,1},\quad\forall(s,t)\in[0,1]\times[0,1],\\&\qquad\text{then }(z,\xi)=(\xi_0+s,e^{2i\pi(t-\tau s)}z_0)\in\mathrm{U}_{4,1}.\end{aligned}$$

If we denote $(\mathrm{X}_0,\mathrm{Y}_0)=\Pi(z_0,\xi_0)$, then the image of (z,ξ) while (s,t) runs over the square is

$$(\mathrm{X},\mathrm{Y})=\Pi(z,\xi)=(e^{2i\pi s}\mathrm{X}_0,e^{2i\pi t}\mathrm{Y}_0)$$

a product of two loops. Therefore, the image $\Pi(\mathrm{U}_{4,1})$ contains an open set W' which is saturated by the toric action of $\mathbf{S}^1\times\mathbf{S}^1$ on $\mathbf{C}_\mathrm{X}^*\times\mathbf{C}_\mathrm{Y}^*$, i.e. a Reinhardt domain (see [26, Chap. 1, sec. 2]), and which contains $\mathrm{U}'_{4,1}$ (just take W' to be the image of all those (z,ξ) like above when (z_0,ξ_0) runs over $\mathrm{U}'_{4,1}$). Since f is invariant under $\mathrm{F}_{1,0,0}$, i.e. $f\circ\mathrm{F}_{1,0,0}=f$, then it factors through Π and, maybe passing to another representative, we have $f=g\circ\Pi$ where $g\in\mathcal{O}(\mathrm{W}')$. Mind that W' (as well as $\Pi(\mathrm{U}_{4,1})$) might not be of the form $\mathrm{W}\setminus(\mathrm{W}\cap\mathrm{D})$ for a neighborhood W of $p_{4,1}$, but we will prove that the holomorphic hull of g is such a neighborhood.

As W' is a Reinhardt domain, let us consider the (convergent) Laurent series of g:

$$g(\mathrm{X},\mathrm{Y})=\sum_{m,n\in\mathbf{Z}}a_{m,n}\mathrm{X}^m\mathrm{Y}^n.$$

The coefficients are given by the integral

$$a_{m,n} = \frac{1}{2i\pi}\int_{\beta_{\xi_0}}(\frac{1}{2i\pi}\int_{\alpha_{\xi_0}} g(\mathrm{X},\mathrm{Y})\mathrm{X}^{-m-1}\mathrm{Y}^{-n-1}d\mathrm{X})d\mathrm{Y}$$

where $\alpha_{\xi_0}(s) = (e^{2i\pi s}\mathrm{X}_0, \mathrm{Y}_0)$ and $\beta_{\xi_0}(t) = (\mathrm{X}_0, e^{2i\pi t}\mathrm{Y}_0)$. This can be rewritten as

$$a_{m,n} = \int_{t=0}^{1}(\int_{s=0}^{1} g(\mathrm{X},\mathrm{Y})\mathrm{X}_0^{-m}\mathrm{Y}_0^{-n}e^{-2i\pi(ms+nt)}ds)dt$$

from which we deduce the estimate

$$|a_{m,n}| \leq \int_{t=0}^{1}(\int_{s=0}^{1}|g(\mathrm{X},\mathrm{Y})\mathrm{X}_0^{-m}\mathrm{Y}_0^{-n}|ds)dt$$

$$|a_{m,n}| \leq \|g(\mathrm{X},\mathrm{Y})\|_{\mathrm{W}'}|z_0|^n|e^{-2i\pi(m+\tau n)\xi_0}|$$

$$|a_{m,n}| \leq \|f\|_{\mathrm{U}_{4,1}}|z_0|^n e^{2\pi\Im\{(m+\tau n)\xi_0\}}.$$

Now, given $m, n \in \mathbf{Z}$, assume that there exists $\theta \in \mathrm{I}_{4,1}$ such that $\Im(e^{i\theta}(m+\tau n)) < 0$. The above inequality promptly implies that $a_{m,n} = 0$ by fixing z_0 and making $\xi_0 \to \infty$ in the direction θ (which is possible in $\mathrm{U}'_{4,1}$ as its opening is $\mathrm{I}_{4,1}$). This is possible if, and only if

$$\arg(m+\tau n) + \mathrm{I}_{4,1} \quad \text{intersects} \quad]-\pi, 0[\mod 2\pi$$

which, since $\mathrm{I}_{4,1} =]0, \pi - \varpi[$, means that

$$\begin{aligned}\arg(m+\tau n) &\in\ ]-\pi,0[\ -\ ]0,\pi-\varpi[\ =\ ]-\pi,0[\ +\ ]\varpi-\pi,0[\\ &=\ ]\varpi-2\pi, 0[.\end{aligned}$$

It promptly follows that the only non zero coefficients $a_{m,n}$ occur when

$$\arg(m+\tau n) \in [0, \varpi]$$

which means that $m, n \geq 0$, and g extends holomorphically at $p_{4,1} : \mathrm{X} = \mathrm{Y} = 0$. Finally, since $f \to 0$ as $\xi \to \infty$, we get that $a_{0,0} = 0$ and $g(0,0) = 0$. □

Remark **3.8.** — The second part of the proof does not use the fact that f is flat (i.e. admits asymptotic expansion zero) along $\tilde{\mathrm{C}}$, but only the fact that it is bounded. As a consequence, any bounded holomorphic function on a transversely sectorial open set U_i or $\mathrm{U}_{i,i+1}$ as above automatically admits a constant as asymptotic expansion along C. We note that bounded functions on U_1, U_3 (resp. U_2, U_4) therefore correspond to first integrals of the foliation $\mathcal{F}_1$ (resp. $\mathcal{F}_0$).

3.3. *Sheaves of sectorial automorphisms.* — Denote by $\mathrm{Aut}(S_0)$ the automorphism group of the ruled surface S_0 whose elements induce translations on C. It preserves the ruling as well as the section $C \subset S_0$, inducing an action on the neighborhood of C. The subgroup $\mathrm{Aut}^C(S_0)$ of elements fixing C point-wise is the one-parameter group generated by the flow of the vector field[21]

$$\partial_\xi = 2i\pi(X\partial_X + \tau Y\partial_Y).$$

We have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^C(S_0) \longrightarrow \mathrm{Aut}(S_0) \longrightarrow \mathrm{Aut}^0(C) \longrightarrow 1 \tag{3.2}$$

where $\mathrm{Aut}^0(C)$ is the translation group on C. The group $\mathrm{Aut}(S_0)$ is connected and generated by the flows of

$$\partial_\xi + 2i\pi\tau z\partial_z = 2i\pi X\partial_X \quad \text{and} \quad -2i\pi z\partial_z = 2i\pi Y\partial_Y \tag{3.3}$$

The full group of automorphisms $\widetilde{\mathrm{Aut}(S_0)}$ must also preserve the ruling and C (this latter being the unique section of vanishing self-intersection). We claim that the restriction morphism

$$\widetilde{\mathrm{Aut}(S_0)} \to \mathrm{Aut}(C)$$

is surjective and induces an isomorphism

$$\widetilde{\mathrm{Aut}(S_0)}/\mathrm{Aut}(S_0) \simeq \mathrm{Aut}(C)/\mathrm{Aut}^0(C)$$

For a generic elliptic curve C, where $\mathrm{Aut}(C)/\mathrm{Aut}^0(C)$ has order 2, this is clear by considering the involution of S_0 $(z,\xi) \to (\frac{1}{z}, -\xi)$.

In full generality, one has $\mathrm{Aut}(C)/\mathrm{Aut}^0(C) \simeq \mathbf{Z}/\kappa\mathbf{Z}$ where $\kappa = 2, 3, 4$ or 6. Fix κ and consider an associated lattice $\langle 1, \tau\rangle \subset \mathbf{C}$ defining C. One can present S_0 as the quotient $\left(\mathbf{C}_x \times \overline{\mathbf{C}_\xi}\right)/G$ where G is the group generated by

$$\begin{cases} \phi_1(x,y) = (x+1\ ,\ \xi) \\ \phi_\tau(x,y) = (x+\tau\ ,\ \xi - 1) \end{cases}$$

Let ω be a primitive κ^{th} root of unit. Then, there exists a matrix $M = \begin{pmatrix} m & n \\ m' & n' \end{pmatrix} \in GL(2,\mathbf{Z})$ such that $\omega = m + n\tau$, $\omega\tau = m' + n'\tau$. Consider now a transformation of $\mathbf{C}_x \times \overline{\mathbf{C}_\xi}$ of the form $h_{\lambda,\mu}(x,\xi) = (\omega x, \lambda\xi + \mu z)$ where $(\lambda,\mu) \in \mathbf{C}^* \times \mathbf{C}$. It is then easy to adjust (λ,μ) in such a way that $h_{\lambda,\mu} \circ \phi_1 \circ h_{\lambda,\mu}^{-1} = \varphi_1^m \circ \varphi_\tau^n$ and $h_{\lambda,\mu} \circ \phi_\tau \circ h_{\lambda,\mu}^{-1} = \varphi_1^{m'} \circ \varphi_\tau^{n'}$. In

[21] See notations of Section 1.3.

particular, $h_{\lambda,\mu}$ lies in the normalizer of G, hence descend to S_0, thus justifying the claim above.

In fact, specializing $\widetilde{\mathrm{Aut}(S_0)}$ to the neighborhood of the curve, we get all analytic, and even formal automorphisms of the neighborhood (S_0, C):

Lemma **3.9.** — *Any formal automorphism* $\hat{\Phi} : \circlearrowright (S_0, C)$ *fixing* C *point-wise is actually convergent and belongs to* $\mathrm{Aut}^C(S_0)$.

Proof. — Recall [15] that the only formal regular foliations on (S_0, C) are those defined by $\omega = 0$ where ω belongs to the vector space of closed 1-forms $E = \mathbf{C}\frac{dz}{z} + \mathbf{C}d\xi$. Moreover, for $\omega \in E \setminus \mathbf{C}\frac{dz}{z}$, $\mathcal{F}_\omega$ does not admit non constant formal meromorphic first integral, and the only formal closed meromorphic 1-forms defining $\mathcal{F}_\omega$ must be a constant multiple of ω, thus belonging to E. If

$$\hat{\Phi}(z,\xi) = \left(z + \sum_{n>0} \frac{a_n(z)}{\xi^n}, \sum_{n\geq -1} \frac{b_n(z)}{\xi^n}\right)$$

is a formal automorphism of (S_0, C) fixing C point-wise (b_{-1} non vanishing), then it must preserve the vector space E. In particular, it must preserve $\mathbf{C}\frac{dz}{z}$ (and z actually as it fixes C point-wise) and sends $d\xi$ to some other element $\alpha\frac{dz}{z} + \beta d\xi$. A straightforward computation shows that $\hat{\Phi}$ writes

$$\hat{\Phi}(x,\xi) = \left(z, \alpha\log(z) + \beta\xi + \gamma\right), \quad \gamma \in \mathbf{C},$$

and we have $\alpha = 0$. Finally, as $\hat{\Phi}$ must commute with $F_{1,0,0}(z,\xi) = (qz, \xi - 1)$, we get $\beta = 1$. □

Corollary **3.10.** — [22] *Any formal automorphism* $\hat{\Phi} : \circlearrowright (S_0, C)$ *is actually convergent and belongs to* $\widetilde{\mathrm{Aut}(S_0)}$.

Proof. — The formal diffeomorphism $\hat{\Phi}$ induces an automorphism of C. Using the surjectivity of the restriction morphism $\widetilde{\mathrm{Aut}(S_0)} \to \mathrm{Aut}(C)$ and after composing $\hat{\Phi}$ by a convenient element of $\widetilde{\mathrm{Aut}(S_0)}$, we can assume that it fixes C point-wise, and then apply Lemma 3.9. □

Definition **3.11.** — *Let us consider the germs of sectorial biholomorphisms in the direction* $\arg(\xi) = \theta$ *of* $(\tilde{U}, \tilde{C})$ *that are tangent to the identity:*

$$\Phi(z,\xi) = (z + \frac{f_1(z,\xi)}{\xi}, \xi + \frac{f_2(z,\xi)}{\xi}), \quad f_1, f_2 \in \mathcal{A}_\theta.$$

[22] We will generalize this result in Section 10.2 using the notion of periods as defined in [15, Section 2.4], and their invariance under automorphisms.

The collection of these germs when varying θ naturally gives rise to a sheaf of groups (with respect to the composition law) on $\mathbf{S}^1$ that will be denoted by $\mathscr{G}^1$. We will consider for further use the subsheaf $\mathscr{G}^\infty$ of $\mathscr{G}^1$ of germs of sectorial biholomorphisms flat to identity, i.e. when $f_1, f_2 \in \mathcal{A}_\theta^\infty$. Denote by $\mathscr{G}^1[\mathrm{F}_{1,0,0}]$ (resp. $\mathscr{G}^\infty[\mathrm{F}_{1,0,0}]$) the subsheaf of $\mathscr{G}^1$ (resp. $\mathscr{G}^\infty$) defined by germs of transformations Φ commuting with $\mathrm{F}_{1,0,0}$: $\Phi \circ \mathrm{F}_{1,0,0} = \mathrm{F}_{1,0,0} \circ \Phi$.

Remark **3.12.** — Note that $\Phi \in \mathscr{G}^1[\mathrm{F}_{1,0,0}]$ implies that its asymptotic expansion $\hat{\Phi}$ also commutes with $\mathrm{F}_{1,0,0}$, i.e. $\hat{\Phi} \circ \mathrm{F}_{1,0,0} = \mathrm{F}_{1,0,0} \circ \hat{\Phi}$. According to the description of the formal centralizer of $\mathrm{F}_{1,0,0}$ in Lemma 3.9, it turns out that $\mathscr{G}^1[\mathrm{F}_{1,0,0}] = \mathscr{G}^\infty[\mathrm{F}_{1,0,0}] \rtimes \mathrm{Aut}^{\mathrm{C}}(\mathrm{S}_0)$ where $\mathrm{Aut}^{\mathrm{C}}(\mathrm{S}_0)$ is regarded as a constant sheaf on $\mathbf{S}^1$.

We would like to apply characterization of $\mathcal{A}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I})$ obtained in the previous section for our special sectors I_i and $\mathrm{I}_{i,i+1}$ to obtain a similar characterization of sections of $\mathscr{G}^\infty[\mathrm{F}_{1,0,0}]$. For this, denote by $\mathrm{Diff}(\mathrm{V}_i, \mathrm{L}_i)$ the group of germs of biholomorphisms of $(\mathrm{V}_i, \mathrm{L}_i)$ which preserves the divisor $(\mathrm{D} \cap \mathrm{V}_i)$, for instance:

$$\mathrm{Diff}(\mathrm{V}_1, \mathrm{L}_1) = \{\varphi(\mathrm{X}, \mathrm{Y}) = (\mathrm{X}a(\mathrm{Y}), \mathrm{Y}b(\mathrm{Y}))\ ;\ a, b \in \mathbf{C}\{\mathrm{Y}\},\ a(0), b(0) \neq 0\} \tag{3.4}$$

and by $\mathrm{Diff}^1(\mathrm{V}_i, \mathrm{L}_i)$ the subgroup of germs tangent to the identity along L_i, i.e. $a(0) = b(0) = 1$ in example (3.4). In a similar way, denote by $\mathrm{Diff}(\mathrm{V}_{i,i+1}, p_{i,i+1})$ the group of germs of biholomorphisms of $(\mathrm{V}_{i,i+1}, p_{i,i+1})$ which preserve the germ of divisor $(\mathrm{D} \cap \mathrm{V}_{i,i+1}, p_{i,i+1})$ and by $\mathrm{Diff}^1(\mathrm{V}_{i,i+1}, p_{i,i+1})$ the subgroup of germs tangent to the identity at $p_{i,i+1}$. For instance:

$$\begin{aligned}\mathrm{Diff}(\mathrm{V}_{4,1}, p_{4,1}) = \{\varphi(\mathrm{X}, \mathrm{Y}) = (\mathrm{X}a(\mathrm{X}, \mathrm{Y}), \mathrm{Y}b(\mathrm{X}, \mathrm{Y}))\ ;\\ a, b \in \mathbf{C}\{\mathrm{X}, \mathrm{Y}\},\ a(0), b(0) \neq 0\},\end{aligned}$$

and $\mathrm{Diff}^1(\mathrm{V}_{4,1}, p_{4,1})$ is characterized by $a(0) = b(0) = 1$.

Proposition **3.13.** — *We have the following characterizations:*

- $\Phi \in \mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_i)$ *if and only if* $\Pi \circ \Phi = \varphi \circ \Pi$ *where* $\varphi \in \mathrm{Diff}^1(\mathrm{V}_i, \mathrm{L}_i)$*;*
- $\Phi \in \mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_{i,i+1})$ *if and only if* $\Pi \circ \Phi = \varphi \circ \Pi$ *where* $\varphi \in \mathrm{Diff}^1(\mathrm{V}_{i,i+1}, p_{i,i+1})$.

Proof. — For any interval I, a section Φ of $\mathscr{G}^\infty(\mathrm{I})$ can be written $\Phi(z, \xi) = (z(1 + f_1), \xi + f_2)$ with $f_1, f_2 \in \mathcal{A}^\infty(\mathrm{I})$. Then Φ belongs to $\mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I})$ if, and only if, f_1, f_2 are invariant by $\mathrm{F}_{1,0,0}$, i.e. $f_1, f_2 \in \mathcal{A}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I})$. Assume now $\mathrm{I} = \mathrm{I}_{4,1}$, say. Then, by Proposition 3.7, one can write $f_k = g_k \circ \Pi$, i.e. $f_k(z, \xi) = g_k(\mathrm{X}, \mathrm{Y})$, with $g_k \in \mathcal{O}^0(\mathrm{V}_{4,1}, p_{4,1})$. Therefore, one can write

$$\Pi \circ \Phi = (\mathrm{X}a(\mathrm{X}, \mathrm{Y}), \mathrm{Y}b(\mathrm{X}, \mathrm{Y})) \quad \text{with} \begin{cases} a = e^{2i\pi g_2(\mathrm{X},\mathrm{Y})}, \\ b = e^{2i\pi\tau g_2(\mathrm{X},\mathrm{Y})}(1 + g_1(\mathrm{X}, \mathrm{Y})). \end{cases}$$

Clearly, a, b are holomorphic at $(\mathrm{X},\mathrm{Y})=(0,0)$ and $a(0,0)=b(0,0)=1$. Conversely, given $\varphi\in\mathrm{Diff}^1(\mathrm{V}_{4,1},p_{4,1})$, thus of the form $\varphi(\mathrm{X},\mathrm{Y})=(\mathrm{X}a(\mathrm{X},\mathrm{Y}),\mathrm{Y}b(\mathrm{X},\mathrm{Y}))$, we recover $f_1,f_2\in\mathcal{A}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_{4,1})$, and $\Phi(z,\xi)=(z(1+f_1),\xi+f_2)$, by setting

$$f_1=\left(\frac{b}{a^\tau}-1\right)\circ\Pi\quad\text{and}\quad f_2=\frac{\log(a)}{2i\pi}\circ\Pi.$$

The description of elements of $\mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_i)$, $\mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_{i,i+1})$ can be carried out exactly along the same line. □

4. Analytic classification: an overview

Here, we would like to detail our main result, namely the analytic classification of all neighborhoods that are formally equivalent to $(\mathrm{U}_{1,0,0},\mathrm{C})$. The most technical ingredient is the sectorial normalization (Lemma A in the introduction) which now reads as follows. Let F be a biholomorphism like in Proposition 2.3

$$\mathrm{F}(z,\xi)=\left(qz+\sum_{n\geq 2}\frac{\alpha_n(z)}{\xi^n},\xi-1+\sum_{n\geq 2}\frac{\beta_n(z)}{\xi^n}\right).$$

In particular, there is a formal diffeomorphism $\hat\Psi$ (that can be assumed to be tangent to the identity along C) conjugating F to $\mathrm{F}_{1,0,0}(z,\xi)=(qz,\xi-1)$, i.e. $\mathrm{F}\circ\hat\Psi=\hat\Psi\circ\mathrm{F}_{1,0,0}$.

Lemma **4.1.** — *Denote* $\varpi=\arg\tau$. *For each interval*

$$(\mathbf{4.1})\qquad \mathrm{I}_1=]-\varpi,\pi-\varpi[,\quad \mathrm{I}_2=]-\pi,0[,\quad \mathrm{I}_3=\mathrm{I}_1+\pi\quad\textit{and}\quad \mathrm{I}_4=\mathrm{I}_2+\pi,$$

there is a section Ψ_i *of* $\mathscr{G}^1(\mathrm{I}_i)$ *(see Definition 3.11) such that*

$$\Psi_i\circ\mathrm{F}=\mathrm{F}_{1,0,0}\circ\Psi_i.$$

Section 9 is devoted to the proof of this lemma. Let us see how to use it in order to provide a complete set of invariants for the neighborhood $(\mathrm{U},\mathrm{C})=(\tilde{\mathrm{U}},\tilde{\mathrm{C}})/<\mathrm{F}>$. First of all, we note that Ψ_i is unique up to left-composition by a section of $\mathscr{G}^1[\mathrm{F}_{1,0,0}](\mathrm{I}_i)$, i.e. the composition of an element of the one-parameter group $\mathrm{Aut}^{\mathrm{C}}(\mathrm{S}_0)$ with a section of $\mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_i)$ (see Remark 3.12). Using this freedom, we may assume that asymptotic expansions coincide:

$$\hat\Psi_i=\hat\Psi_j=\hat\Psi.$$

It follows that, on intersections $\mathrm{I}_{i,i+1}=\mathrm{I}_i\cap\mathrm{I}_{i+1}$, we get sections

$$\Phi_{i,i+1}:=\Psi_i\circ\Psi_{i+1}^{-1}\in\mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_{i,i+1}).$$

Using Proposition 3.13, we have (see Fig. 4)

$$\Pi \circ \Phi_{i,i+1} = \varphi_{i,i+1} \circ \Pi \quad \text{for some } \varphi_{i,i+1} \in \mathrm{Diff}^1(\mathrm{V}_{i,i+1}, p_{i,i+1}).$$

In other words, setting $\Pi_i := \Pi \circ \Psi_i$, we get

$$\Pi_i = \Pi \circ \Psi_i = \Pi \circ \Phi_{i,i+1} \circ \Psi_{i+1} = \varphi_{i,i+1} \circ \Pi \circ \Psi_{i+1} = \varphi_{i,i+1} \circ \Pi_{i+1} \tag{4.2}$$

which proves Corollary B. We have therefore associated to each neighborhood (U, C) formally equivalent to $(\mathrm{U}_{1,0,0}, \mathrm{C})$ a cocycle $\varphi = (\varphi_{i,i+1})_{i \in \mathbf{Z}_4}$ which is unique up to the freedom for the choice of Ψ_i's.

Definition **4.2.** — *We say that two cocycles φ and φ' are equivalent if*

$$\exists t \in \mathbf{C},\ \exists \varphi_i \in \mathrm{Diff}^1(\mathrm{V}_i, \mathrm{L}_i)$$

$$\textit{such that} \quad \varphi'_{i,i+1} = \phi^t \circ \varphi_i \circ \varphi_{i,i+1} \circ \varphi_{i+1}^{-1} \circ \phi^{-t} \tag{4.3}$$

where $\phi^t = (e^{2i\pi t}\mathrm{X}, e^{2i\pi\tau t}\mathrm{Y})$ is the one-parameter group of the vector field $v_\tau = 2i\pi(\mathrm{X}\partial_\mathrm{X} + \tau \mathrm{Y}\partial_\mathrm{Y})$. We will denote this equivalence relation by $\approx$.

Proposition **4.3** *(Proof of Theorem C). — Two neighborhoods* (U, C) *and* $(\mathrm{U}', \mathrm{C})$ *formally equivalent to* $(\mathrm{U}_{1,0,0}, \mathrm{C})$ *are analytically equivalent if, and only if, the corresponding cocycles are equivalent:*

$$(\mathrm{U}, \mathrm{C}) \overset{an}{\sim} (\mathrm{U}', \mathrm{C}) \quad \Leftrightarrow \quad \varphi \approx \varphi'.$$

Proof. — According to the description of $\mathrm{Aut}^\mathrm{C}(\mathrm{S}_0)$, a biholomorphism germ $(\mathrm{U}, \mathrm{C}) \to (\mathrm{U}', \mathrm{C})$ is indeed tangent to identity along C and then lifts-up to a global section $\Psi \in \mathscr{G}^1(\mathbf{S}^1)$ satisfying $\Psi \circ \mathrm{F} = \mathrm{F}' \circ \Psi$. Let (Ψ_i) and (Ψ'_i) be the sectorial normalizations used to compute the invariants φ and φ'. Clearly, $\Psi'_i \circ \Psi$ provides a new collection of sectorial trivializations for (U, C). We can write (using Remark 3.12)

$$\Psi'_i \circ \Psi = \exp(t_i \partial_\xi) \circ \Phi_i \circ \Psi_i \quad \text{with } \Phi_i \in \mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_i).$$

However, as $\hat{\Psi}_i = \hat{\Psi}_j$ and $\hat{\Psi}'_i = \hat{\Psi}'_j$, we have $t_i = t_j =: t$ for all i, j. Therefore, we have

$$\begin{aligned}
\Phi'_{i,i+1} &= (\Psi'_i \circ \Psi) \circ (\Psi'_{i+1} \circ \Psi)^{-1} \\
&= (\exp(t\partial_\xi) \circ \Phi_i \circ \Psi_i) \circ (\exp(t\partial_\xi) \circ \Phi_{i+1} \circ \Psi_{i+1})^{-1} \\
&= \exp(t\partial_\xi) \circ \Phi_i \circ \Phi_{i,i+1} \circ \Phi_{i+1}^{-1} \circ \exp(-t\partial_\xi).
\end{aligned}$$

After factorization through Π, using (3.3) and Proposition 3.13, we get the expected equivalence relation (4.3) for φ and φ'. Conversely, if $\varphi \approx \varphi'$, then we can trace back the existence of an analytic conjugacy $\Phi : (\mathrm{U}, \mathrm{C}) \to (\mathrm{U}', \mathrm{C})$ by reversing the above implications. □

Remark **4.4.** — We can weaken the notion of analytic equivalence between neighborhoods by considering biholomorphism germs $\Phi : (U, C) \to (U', C)$ inducing translations on C. This means that, in Definition 2.2, we now allow conjugacies $\Phi(z, y) = (cz + O(y), O(y))$ with $c \in \mathbf{C}^*$ in formula (2.2), i.e. translations on the elliptic curve. In that case, the corresponding cocycles are related by

$$\varphi'_{i,i+1} = \phi \circ \varphi_i \circ \varphi_{i,i+1} \circ \varphi_{i+1}^{-1} \circ \phi^{-1}$$

where $\phi(X, Y) = (aX, bY)$ for arbitrary $a, b \in \mathbf{C}^*$.[23] Having in mind the description given in (3.4), one observes that two cocycles are equivalent iff they lie on the same orbit over some action (that the reader will easily explicit) of the fiber product $(\mathcal{O}^* \times \mathcal{O}^*)^{\times^4_{\mathbf{C}^* \times \mathbf{C}^*}}$ of 4 copies of $\mathcal{O}^* \times \mathcal{O}^*$ with respect to the natural morphism $\mathcal{O}^* \times \mathcal{O}^* \ni (f, g) \to (f(0), g(0)) \in \mathbf{C}^* \times \mathbf{C}^*$.

To summarize, we have just associated to each $(U, C) \overset{\text{for}}{\sim} (U_{1,0,0}, C)$ a cocycle

$$\varphi = (\varphi_{i,i+1})_{i \in \mathbf{Z}_4}, \quad \varphi_{i,i+1} \in \mathrm{Diff}^1(V_{i,i+1}, p_{i,i+1}) \tag{4.4}$$

and constructed a map from the moduli space $\mathcal{U}_{1,0,0}$ of such neighborhood up to analytic equivalence $\overset{\text{an}}{\sim}$ to the moduli space $\mathcal{C}$ of cocycles φ like (4.4) up to equivalence (4.3):

$$\mu \ : \ \mathcal{U}_{1,0,0} = \{(U, C) \overset{\text{for}}{\sim} (U_{1,0,0}, C)\}/_{\overset{\text{an}}{\sim}} \longrightarrow \mathcal{C} = \{\varphi\}/_{\approx} \tag{4.5}$$

which is proved to be injective in Proposition 4.3. In Section 5, we prove the surjectivity by constructing an inverse map $\varphi \mapsto U_\varphi$. Before that, we want to reinterpret the cocycle φ as transition maps of an atlas for a neighborhood (V_φ, D).

5. Construction of U_φ

In this section, we construct a large class of non analytically equivalent neighborhoods, all of them formally equivalent to $(U_{1,0,0}, C)$. This is done by sectorial surgery, extending the complex structure along C by means of Newlander-Nirenberg Theorem. In order to do this, we have to work with smooth functions (i.e. of class C^∞).

Definition **5.1.** — *For any open sector Σ_I (Definition 3.1), we denote by $\mathcal{E}^\infty(\Sigma_I)$ the $\mathbf{C}$-algebra of those complex smooth functions $f : \Sigma_I \to \mathbf{C}$ satisfying the following estimates*

$$\forall \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbf{N}^4, \ \forall n \in \mathbf{N}, \ \forall K \subset \mathbf{C}^* \ \textit{compact}, \ \exists C > 0 \ \textit{such that:}$$

$$\forall (z, \xi) \in \Sigma_I, \ z \in K, \quad \textit{we have} \ \left| \frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\alpha_4} f(z, \xi)}{\partial z^{\alpha_1} \partial \xi^{\alpha_2} \partial \bar{z}^{\alpha_3} \partial \bar{\xi}^{\alpha_2}} \right| \leq \frac{C}{|\xi^{n+1}|}.$$

[23] Note that those are precisely the transformation arising from the natural torus action on $\mathbf{P}^1 \times \mathbf{P}^1$ (see also Section 8). For the sake of clarity, we will state our general result (Section 10) modulo analytic isomorphisms inducing translations (and not only the identity) on C.

Passing to inductive limits and sheafification as in Section 3.1, we get a sheaf $\mathcal{E}^\infty$ of differential algebra on the circle $\mathbf{S}^1$. Like in Section 3.3, we can also define the sheaf of groups $\mathcal{D}^\infty$ on the circle, whose sections $\Psi \in \mathcal{D}^\infty(\mathrm{I})$ are smooth sectorial diffeomorphisms asymptotic to the identity, i.e. of the form $\Psi(z,\xi) = (z + h_1, \xi + h_2)$ with $h_1, h_2 \in \mathcal{E}^\infty(\mathrm{I})$. The following property somehow expresses that a cocycle defined by a collection of sectorial biholomorphisms is a coboundary in the C^∞ category.

Lemma **5.2.** — *Let* $(\mathrm{J}_i)_{i\in\mathrm{I}}$ *be a covering of* $\mathbf{S}^1$ *by open intervals threewise disjoints. Assume also that there exist on non empty intersections* $\mathrm{J}_{ij} := \mathrm{J}_i \cap \mathrm{J}_j$ *a family of sectorial biholomorphisms* $\Phi_{ij} \in \mathcal{G}^\infty(\mathrm{J}_{ij})$ *with* $\Phi_{ji} = \Phi_{ij}^{-1}$ *(in particular* $\Phi_{ii} = id$*). Then, there exist smooth sectorial diffeomorphisms flat to identify* $\psi_i \in \mathcal{D}^\infty(\mathrm{J}_i)$ *such that* $\Phi_{ij} = \psi_i \circ \psi_j^{-1}$.

Proof. — One can extract from this covering a finite covering (J_k), $k \in \mathbf{Z}_n$ such that only consecutive sectors J_k and J_{k+1} intersect. It clearly suffices to prove the Lemma for this particular subcovering. Let (θ_k) a partition of the unity subordinate to this covering. Write

$$\Phi_{k,k+1}(z,\xi) = (z + h^1_{k,k+1}, \xi + h^2_{k,k+1}) \quad \text{with } h^1_{k,k+1}, h^2_{k,k+1} \in \mathcal{A}^\infty(\mathrm{J}_{k,k+1}).$$

First define $\tilde\psi_k \in \mathcal{D}^\infty(\mathrm{J}_k)$ for $k \in \mathbf{Z}_n$ by

$$\tilde\psi_k = \begin{cases} \mathrm{Id} & \text{when } \arg(\xi) \in \mathrm{J}_k \setminus \mathrm{J}_{k,k+1}, \\ \mathrm{Id} + \theta_{k+1}(\arg\xi)(h^1_{k,k+1}, h^2_{k,k+1}) & \text{when } \arg(\xi) \in \mathrm{J}_{k,k+1} \end{cases}$$

Next, define $\psi_k \in \mathcal{D}^\infty(\mathrm{J}_k)$ by

$$\psi_k = \begin{cases} \Phi_{k,k-1} \circ \tilde\psi_{k-1} & \text{when } \arg(\xi) \in \mathrm{J}_{k-1,k} \\ \tilde\psi_k & \text{when } \arg(\xi) \in \mathrm{J}_k \setminus \mathrm{J}_{k-1,k} \end{cases}$$

One easily check that ψ_k are smooth, equal to the identity outside intersections, and satisfy $\psi_k = \Phi_{k,k-1} \circ \psi_{k-1}$ on intersections as expected. □

Corollary **5.3.** — *Notations and assumptions like in Lemma 5.2. There exist sectorial biholomorphisms tangent to identity* $\Psi_i \in \mathcal{G}^1(\mathrm{J}_i)$ *such that* $\Phi_{ij} = \Psi_i \circ \Psi_j^{-1}$. *In particular, asymptotic expansions coincide* $\hat\Psi_i = \hat\Psi_j$.

Proof. — Let $\tilde{\mathrm{U}}_i$ be the sectorial domain of definition of ψ_i and $\tilde{\mathrm{U}}$ be their union together with the section $\tilde{\mathrm{C}}$ defined by $\xi = \infty$. Lemma 5.2 allows to write $\Phi_{ij} = \psi_i \circ \psi_j^{-1}$ where $\psi_i \in \mathcal{D}^\infty(\mathrm{I}_i)$. In particular, denoting by I the standard complex structure on $\mathbf{C}^2$, $\mathrm{J} := \psi_i^*\mathrm{I} = \psi_j^*\mathrm{I}$ is a new complex structure on $\tilde{\mathrm{U}} \setminus \tilde{\mathrm{C}}$ which extends to $\tilde{\mathrm{U}}$ as a complex structure by Newlander-Nirenberg's Theorem. In fact, because of flatness of ψ_i to the identity, the almost complex structure J extends at 0 as a C^∞ almost complex structure on $\tilde{\mathrm{U}}$; by construction, it is integrable on $\tilde{\mathrm{U}}_i$'s and therefore Nijenhuis tensor vanishes identically on $\tilde{\mathrm{U}} \setminus \tilde{\mathrm{C}}$, and by continuity on $\tilde{\mathrm{U}}$. Then, Newlander-Nirenberg's Theorem tells us that J is integrable. Note that $\mathrm{I} = \mathrm{J}$ in restriction to $\tilde{\mathrm{C}}$ which is then conformally equivalent to $\mathbf{C}^*$ for both structures. Now, we use the fact that two-dimensional germs of neighborhood of $\mathbf{C}^*$ are analytically equivalent as recalled in Section 2. This can be translated into the existence of a smooth diffeomorphism ψ of $(\tilde{\mathrm{U}}, \tilde{\mathrm{C}})$ such that $\psi_* \mathrm{I} = \mathrm{J}$. Up to making a right composition by a biholomorphism of $(\tilde{\mathrm{U}}, \tilde{\mathrm{C}})$ with respect to I, one can suppose (exploiting that $\mathrm{I} = \mathrm{J}$ on $\mathrm{T}\tilde{\mathrm{U}}_{|\tilde{\mathrm{C}}}$) that ψ is tangent to the identity along $\tilde{\mathrm{C}}$. This implies that for every i, $\Psi_i := \psi_i \circ \psi \in \mathscr{G}^1(\mathrm{U}_i)$ and, because the Φ_{ij}'s are flat to identity, admit an asymptotic expansion $\hat{\Psi}_i$ along $\tilde{\mathrm{C}} = \mathbf{C}^*$ independent of i. By construction, we have $\Phi_{ij} = \Psi_i \circ \Psi_j^{-1}$ as desired. Obviously, all along this proof, we might have shrinked the domain $\tilde{\mathrm{U}}$ of definition without mentioning it. $\square$

Remark **5.4.** — The use of the Newlander-Nirenberg in this context is not new and can be traced back to Malgrange [16] and Martinet-Ramis [17].

We now specialize to our covering of $\mathbf{S}^1$ determined by the intervals I_i defined by (4.1) in Lemma 4.1. Let us show how to construct a neighborhood realizing a given cocycle $\varphi = (\varphi_{i,i+1})$ as in (4.4). We first define $\Phi_{i,i+1} \in \mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_{i,i+1})$ satisfying $\Pi \circ \Phi_{i,i+1} = \varphi_{i,i+1} \circ \Pi$. Then use Corollary 5.3 to obtain $\Psi_i \in \mathscr{G}^1(\mathrm{I}_i)$ such that $\Phi_{i,i+1} = \Psi_i \circ \Psi_{i+1}^{-1}$. As $\Phi_{i,i+1}$ commute to $\mathrm{F}_{1,0,0}$, we have on intersections:

$$(\Psi_i \circ \Psi_{i+1}^{-1}) \circ \mathrm{F}_{1,0,0} = \mathrm{F}_{1,0,0} \circ (\Psi_i \circ \Psi_{i+1}^{-1})$$

which rewrites

$$\Psi_{i+1}^{-1} \circ \mathrm{F}_{1,0,0} \circ \Psi_{i+1} = \Psi_i^{-1} \circ \mathrm{F}_{1,0,0} \circ \Psi_i.$$

Therefore, we can define a global diffeomorphism of $(\tilde{\mathrm{U}}, \tilde{\mathrm{C}})$ by setting

$$\mathrm{F}_\varphi := \Psi_i^{-1} \circ \mathrm{F}_{1,0,0} \circ \Psi_i$$

on U_i's and extending by continuity as the identity mapping on $\tilde{\mathrm{C}}$. By construction, the quotient

$$(\mathrm{U}_\varphi, \mathrm{C}) := (\tilde{\mathrm{U}}, \tilde{\mathrm{C}}) / < \mathrm{F}_\varphi >$$

has cocycle φ and is formally equivalent to $\mathrm{U}_{1,0,0}$. This proves the surjectivity of the map (4.5) whose injectivity has been proved in Proposition 4.3. It remains to prove the Sectorial Normalization Lemma 4.1 (i.e. Lemma A in the introduction), which will be done in Section 9. Modulo this technical but central Lemma, we have achieved the proof of Theorem C.

6. Construction of V_φ

In this section, keeping notations of Section 1.3, we generalize Serre isomorphism $\Pi : \mathrm{U}_{1,0,0} \setminus \mathrm{C} \to \mathbf{C}^*_{\mathrm{X}} \times \mathbf{C}^*_{\mathrm{Y}}$ to the case of a general neighborhood $(\mathrm{U}, \mathrm{C}) \overset{\text{for}}{\sim} (\mathrm{U}_{1,0,0}, \mathrm{C})$.

Theorem **6.1.** — *Given a germ of neighborhood* $(\mathrm{U}_\varphi, \mathrm{C}) \overset{\mathit{for}}{\sim} (\mathrm{U}_{1,0,0}, \mathrm{C})$, *there exists a neighborhood germ* $(\mathrm{V}_\varphi, \mathrm{D})$ *of* D *where each* L_i *has trivial normal bundle, and an isomorphism germ*

$$\Pi_\varphi \ : \ (\mathrm{U}_\varphi \setminus \mathrm{C}, \mathrm{C}) \xrightarrow{\sim} (\mathrm{V}_\varphi \setminus \mathrm{D}, \mathrm{D}) \tag{6.1}$$

canonically attached to the analytic class of $(\mathrm{U}_\varphi, \mathrm{C})$ *in the following sense: if* $(\mathrm{U}_{\varphi'}, \mathrm{C})$ *is another neighborhood germ, then*

$$(\mathrm{U}_\varphi, \mathrm{C}) \overset{an}{\sim} (\mathrm{U}_{\varphi'}, \mathrm{C}) \quad \Leftrightarrow \quad (\mathrm{V}_\varphi, \mathrm{D}) \overset{an}{\sim} (\mathrm{V}_{\varphi'}, \mathrm{D}) \tag{6.2}$$

where the analytic equivalence allows translations[24] *on* C *for the left-hand-side, and preserves the numbering of lines* L_i *on the right-hand-side.*

Proof. — Given a cocycle not necessarily tangent to the identity

$$\varphi = (\varphi_{i,i+1})_{i\in\mathbf{Z}_4}, \quad \varphi_{i,i+1} \in \mathrm{Diff}(\mathrm{V}_{i,i+1}, p_{i,i+1}),$$

we define a new germ of analytic neighborhood of D as follows. We consider the disjoint union of neighborhood germs $(\mathrm{V}_i, \mathrm{L}_i)$, and patch them together through the transition maps

$$\varphi_{i,i+1} : (\mathrm{V}_{i+1}, p_{i,i+1}) \xrightarrow{\sim} (\mathrm{V}_i, p_{i,i+1}).$$

The resulting analytic manifold V_φ contains a copy of D, namely the union of lines L_i identified at points $p_{i,i+1}$, and only the germ of neighborhood V_φ makes sense

$$(\mathrm{V}_\varphi, \mathrm{D}) := \sqcup_i (\mathrm{V}_i, \mathrm{L}_i)/(\varphi_{i,i+1}).$$

This germ of neighborhood comes with embeddings

$$\psi_i : (\mathrm{V}_i, \mathrm{L}_i) \hookrightarrow (\mathrm{V}_\varphi, \mathrm{D}).$$

Conversely, if (V, D) is a germ of neighborhood of D where all lines L_i have zero self-intersection, then there exist trivialization maps (preserving D)

$$\psi_i : (\mathrm{V}_i, \mathrm{L}_i) \xrightarrow{\sim} (\mathrm{V}, \mathrm{L}_i)$$

[24] We emphasize that this is not exactly the equivalence relation defined in Definition 2.2.

(where (V, L_i) denotes the germ of V along L_i) in such a way that, near $p_{i,i+1}$ we have

$$\psi_i = \varphi_{i,i+1} \circ \psi_{i+1} \quad \text{for some } \varphi_{i,i+1} \in \mathrm{Diff}(V_{i,i+1}, p_{i,i+1}).$$

It is clear from above arguments that, for another cocycle φ', we have

$$(V_\varphi, D) \overset{\text{an}}{\sim} (V_{\varphi'}, D) \quad \Leftrightarrow \quad \varphi \sim \varphi'$$

where

$$\varphi \sim \varphi' \quad \overset{\text{def}}{\Leftrightarrow} \quad \exists \varphi_i \in \mathrm{Diff}(V_i, L_i), \quad \varphi_i \circ \varphi'_{i,i+1} = \varphi_{i,i+1} \circ \varphi_{i+1},$$

and in that case, the isomorphism $V_\varphi \xrightarrow{\sim} V_{\varphi'}$ is given by patching

$$(V_\varphi, D) \xleftarrow{\psi_i} (V_i, L_i) \xrightarrow{\varphi_i} (V_i, L_i) \xrightarrow{\psi'_i} (V_{\varphi'}, D).$$

From the linear part of equivalence relation $\varphi \sim \varphi'$, we see that any cocycle φ is equivalent to a cocycle such that

- $\varphi_{i,i+1} \in \mathrm{Diff}^1(V_{i,i+1}, p_{i,i+1})$ (tangent to the identity) for $i = 1, 2, 3$.
- $\varphi_{4,1}(X, Y) = (aX + \cdots, bY + \cdots)$ for $a, b \in \mathbf{C}^*$ independent of the choice.

The pair (a, b) is an invariant of the neighborhood V_φ. Cocycles arising from $(U, C) \overset{\text{for}}{\sim} (U_{1,0,0}, C)$ have invariants $a = b = 1$. In order to prove the equivalence (6.2), we just have to note that, for equivalent cocycles $\varphi \sim \varphi'$ normalized as above (in particular when all $\varphi_{i,i+1}$, $\varphi'_{i,i+1}$ are tangent to the identity) then all four conjugating maps φ_i have the same linear part. Then apply Remark 4.4 to show that it corresponds to analytic equivalence of (U_φ, C) and $(U_{\varphi'}, C)$ up to a translation of the curve.

Finally, we construct the isomorphism (6.1) by patching together the sectorial ones

$$U_i \xrightarrow{\Pi_i} V_i^* \overset{\psi_i}{\hookrightarrow} V_\varphi \setminus D$$

using the identity (4.2) $\Pi_i = \varphi_{i,i+1} \circ \Pi_{i+1}$. □

7. Foliations

Recall that our model $(U_{1,0,0}, C)$ carries a pencil of foliations

$$\mathcal{F}_t \ : \ \{\omega_0 + t\omega_\infty = 0\}, \quad \text{where } \omega_0 = d\xi \text{ and } \omega_\infty = \frac{1}{2i\pi\tau}\frac{dz}{z};$$

moreover, there is no other formal foliation on $(U_{1,0,0}, C)$ either tangent, or transversal to C (see [15, Section 2.3]). Via the isomorphism $\Pi : U_{1,0,0} \setminus C \to V_0 \setminus D$, we get the corresponding pencil

$$\Pi_*\mathcal{F}_t\ :\ (1-t)\tau\frac{d\mathrm{X}}{\mathrm{X}}+t\frac{d\mathrm{Y}}{\mathrm{Y}},$$

where one recalls that $(\mathrm{X}=e^{2i\pi\xi},\mathrm{Y}=ze^{2i\pi\tau\xi})$. The monodromy (or holonomy) of $\mathcal{F}_t$ is given by

$$\pi_1(\mathrm{C})\to\mathrm{Aut}(\mathbf{C})\ ;\ \begin{cases}1\mapsto[\xi\mapsto\xi+\frac{t}{\tau}]\\ \tau\mapsto[\xi\mapsto\xi+t-1]\end{cases}$$

In particular, for $\frac{m}{n}\in\mathbf{Q}\cup\{\infty\}$, are equivalent

- $\mathcal{F}_t$ has trivial monodromy along $m+\tau n\in\Gamma\setminus\{0\}$, viewed as a loop of $\pi_1(\mathrm{C})\simeq\Gamma$;
- $t=\frac{\tau n}{m+\tau n}$, or equivalently $\left(\frac{1}{t}-1\right)\tau=\frac{m}{n}$;
- $\Pi_*\mathcal{F}_t$ admits the rational first integral $\mathrm{X}^m\mathrm{Y}^n$.

We will say that $\mathcal{F}_t$ is of *rational type* if there is $\frac{m}{n}\in\mathbf{Q}\cup\{\infty\}$ with these properties, and of *irrational type* if not. We note that rational type foliations are characterized by the fact that their holonomy group is cyclic (one generator), and also that the space of leaves (after deleting C and regarding them as global foliations on S_0) is somehow “rational”, and not “elliptic”.

If (U, C) is any analytic neighborhood with a formal conjugacy

$$\hat{\Psi}:(\mathrm{U},\mathrm{C})\overset{\sim}{\longrightarrow}(\mathrm{U}_{1,0,0},\mathrm{C}),$$

then it also carries the pencil of formal foliations $\hat{\mathcal{F}}_t:=\hat{\Psi}^*\mathcal{F}_t$. As we shall prove, these foliations are divergent in general. In fact, recall (see [15, Theorem 4])

Theorem **7.1.** — *Let* (U, C) *be an analytic neighborhood formally equivalent to* $(\mathrm{U}_{1,0,0},\mathrm{C})$. *Assume*

- *three elements* $\hat{\mathcal{F}}_{t_1},\hat{\mathcal{F}}_{t_2},\hat{\mathcal{F}}_{t_3}$ *of the pencil are convergent,*
- *or two elements* $\hat{\mathcal{F}}_{t_1},\hat{\mathcal{F}}_{t_2}$ *of the pencil are convergent, both of irrational type:* $\left(\frac{1}{t_i}-1\right)\tau\notin\mathbf{Q}$ *for* $i=1,2$.

Then the full pencil $\hat{\mathcal{F}}_t$ *is convergent, and* (U, C) *is analytically equivalent to* $(\mathrm{U}_{1,0,0},\mathrm{C})$ *(in fact* $\hat{\Psi}$ *is convergent).*

In [15, Theorem 5], the two first authors and O. Thom construct infinite dimensional deformations of neighborhoods with two convergent foliations $\hat{\mathcal{F}}_{t_1}$ and $\hat{\mathcal{F}}_{t_2}$, provided that one or two of them is of rational type. In fact, Écalle-Voronin moduli spaces are shown to embed in moduli spaces of neighborhoods through these bifoliated constructions. Now, we know from our main result Theorem C, that the moduli space of neighborhoods is larger, comparable with $\mathbf{C}\{\mathrm{X},\mathrm{Y}\}$, in contrast with Écalle-Voronin moduli space which is comparable with $\mathbf{C}\{X\}$. The point is that we have missed all neighborhoods with only one, or with no convergent foliation in the aforementioned work.

7.1. *Existence of foliations.* — We now start examining under which condition on the glueing cocycle $\varphi = (\varphi_{i,i+1})$ the neighborhood (U_φ, C) admits a convergent foliation. Here, we follow notations Ψ_i, Π_i, Φ_i,... of Section 4.

Lemma **7.2.** — *Let $t \in \mathbf{P}^1 \setminus \{0, 1, \}$. If the formal foliation $\hat{\mathcal{F}}_t$ of (U, C) is convergent, then the induced foliation $\Pi_{i*}\hat{\mathcal{F}}_t$ on V_i^* extends as a singular foliation on V_i and is defined by a closed logarithmic 1-form*

$$\theta_i = (1-t)\tau\frac{dX}{X} + t\frac{dY}{Y} + \eta_i \tag{7.1}$$

with η_i closed holomorphic on V_i.

Remark **7.3.** — On V_i, $i = 1, 2, 3, 4$, the closed holomorphic 1-form writes

$$\eta_i = df \quad \text{with } f \in \mathbf{C}\{Y\},\ \mathbf{C}\{1/X\},\ \mathbf{C}\{1/Y\},\ \mathbf{C}\{X\} \text{ respectively.}$$

We note that are equivalent:

- θ is a closed logarithmic 1-form on V_i with non trivial poles supported on $D \cap V_i$,
- $\theta = \alpha\frac{dX}{X} + \beta\frac{dY}{Y} + \eta$ with $(\alpha, \beta) \neq (0, 0)$ and η holomorphic and closed on V_i,
- $\theta = \varphi_i^*\left\{\alpha\frac{dX}{X} + \beta\frac{dY}{Y}\right\}$ where $(\alpha, \beta) \neq (0, 0)$ and $\varphi_i \in \mathrm{Diff}^1(V_i, L_i)$.

For instance, on V_1, if $\eta = df$ with $f \in \mathbf{C}\{Y\}, f(0) = 0$ and $\alpha \neq 0$, one can choose $\varphi_1 = (Xe^{\frac{f(Y)}{\alpha}}, Y)$.

Proof. — Let us start with the tangent case $t \neq \infty$. By transversality of $\mathcal{F}_0$ and $\mathcal{F}_\infty$, and the fact that the ω_t's are $F_{1,0,0}$-invariant, one deduces that the sectorial foliation $\Psi_{i*}\hat{\mathcal{F}}_t$ on $(U_{1,0,0}, C)$ is defined by a unique 1-form writing as $\omega = \omega_0 + u \cdot \omega_\infty$ for a function $u \in \mathcal{A}[F_{1,0,0}](I_i)$, obviously satisfying $\hat{u} = t$. If $i = 4$ (resp. $i = 2$), then $u = t + f$ with $f \in \mathbf{C}\{X\}, f(0) = 0$ (resp. $f \in \mathbf{C}\{1/X\}, f(\infty) = 0$) (see Proposition 3.7). Then,

$$\frac{2i\pi\tau t}{u}\omega = t\frac{1-u}{u}\tau\frac{dX}{X} + t\frac{dY}{Y} = (1-t)\tau\frac{dX}{X} + t\frac{dY}{Y} - \frac{f}{t+f}\tau\frac{dX}{X}.$$

If $i = 1$ (resp. $i = 3$), then $u = t + f(Y)$ with $f \in \mathbf{C}\{Y\}, f(0) = 0$ (resp. $f \in \mathbf{C}\{1/Y\}$, $f(\infty) = 0$) and we arrive at a similar situation

$$\frac{2i\pi\tau(1-t)}{1-u}\omega = (1-t)\tau\frac{dX}{X} + t\frac{dY}{Y} + \frac{f}{1-t-f}\tau\frac{dY}{Y}.$$

Of course, we have used $t \neq 0, 1$ in order to get non zero multiple of ω.

Let us end with the case $\hat{\mathcal{F}}_\infty$. The foliation, in that case, can be defined by a closed holomorphic 1-form ω extending the holomorphic 1-form on C. Now, up to a multiplicative constant, we can write $\Psi_{i*}\omega = \omega_\infty + \eta$ for a closed 1-form $\eta = u\omega_0 + v\omega_\infty$ where $u, v \in \mathcal{A}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_i)$ One concludes as above. □

Lemma **7.4.** — *If the formal foliation $\hat{\mathcal{F}}_0$ (resp. $\hat{\mathcal{F}}_1$) of* (U, C) *is convergent, then the induced foliation $\Pi_{i*}\hat{\mathcal{F}}_t$ on V_i^* extends as a singular foliation on V_i and is defined by a 1-form*

$$\theta_i = \begin{cases} \frac{d\mathrm{X}}{\mathrm{X}} + f_i(\mathrm{X})\frac{d\mathrm{Y}}{\mathrm{Y}} & \text{if } i = 2, 4 \\ \frac{d\mathrm{X}}{\mathrm{X}} + \eta_i & \text{if } i = 1, 3 \end{cases}$$

$$\left(\textit{resp.} \quad \theta_i = \begin{cases} \frac{d\mathrm{Y}}{\mathrm{Y}} + \eta_i & \text{if } i = 2, 4 \\ \frac{d\mathrm{Y}}{\mathrm{Y}} + g_i(\mathrm{Y})\frac{d\mathrm{X}}{\mathrm{X}} & \text{if } i = 1, 3 \end{cases} \right)$$

with $f_i \in \mathbf{C}\{\mathrm{Z}\}$ vanishing at $\mathrm{Z} = 0$, where $\mathrm{Z} = \mathrm{Y}, \frac{1}{\mathrm{X}}, \frac{1}{\mathrm{Y}}, \mathrm{X}$ for $i = 1, 2, 3, 4$ respectively, and η_i is a closed holomorphic 1-form on V_i.

Proof. — It is similar to the proof of Lemma 7.2. For the case $t = 0$, once we have defined $\hat{\mathcal{F}}_t$ by $\omega = \omega_0 + u \cdot \omega_\infty$ for a function $u \in \mathcal{A}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_i)$, then

$$\frac{2i\pi\tau}{1-u}\omega = \tau\frac{d\mathrm{X}}{\mathrm{X}} + \frac{\tau u}{1-u}\frac{d\mathrm{Y}}{\mathrm{Y}}.$$

Again, by Proposition 3.7, we see that if $i = 1, 3$, then $u = f_i(\mathrm{Y})$ with f_i as in the statement, and we are done. However, when $i = 2, 4$, then $u = f(\mathrm{X})$, but we cannot divide by $\frac{f}{1-f}$ to get a closed logarithmic 1-form as before: the multiplicity of the polar locus is likely to increase (see remark 7.5). □

Remark **7.5.** — In Lemma 7.4, we can always define the foliation $\Pi_{i*}\hat{\mathcal{F}}_t$ by a closed meromorphic 1-form on V_i provided that we allow non logarithmic poles. For instance, in the case $t = 0$ and $i = 2, 4$, if $f \equiv 0$ (is identically vanishing), there is nothing to do, it is the logarithmic case; if $f \not\equiv 0$, then after division, we get

$$\frac{2i\pi}{f}\omega = \frac{1-f}{f}\frac{d\mathrm{X}}{\mathrm{X}} + \frac{d\mathrm{Y}}{\mathrm{Y}} = \tilde{f}(\mathrm{X})\frac{d\mathrm{X}}{\mathrm{X}^{k+1}} + \frac{d\mathrm{Y}}{\mathrm{Y}}$$

with $\tilde{f} \in \mathbf{C}\{\mathrm{X}\}, \tilde{f}(0) \neq 0$, and $k \in \mathbf{Z}_{>0}$. As it is well-known (see [15, Section 2.2]), we can write

$$2i\pi\tau\varphi_i^*\frac{\omega}{f} = \frac{d\mathrm{X}}{\mathrm{X}^{k+1}} + \alpha\frac{d\mathrm{X}}{\mathrm{X}} + \frac{d\mathrm{Y}}{\mathrm{Y}}$$

for some $\alpha \in \mathbf{C}$ (the residue of $\tilde{f}(\mathrm{X})\frac{d\mathrm{X}}{\mathrm{X}^{k+1}}$) and $\varphi_i \in \mathrm{Diff}^1(\mathrm{V}_i, \mathrm{L}_i)$.

We can now prove Theorem D.

Corollary **7.6.** — *The formal foliation* $\hat{\mathcal{F}}_t$ *of* $(\mathrm{U}_\varphi, \mathrm{C})$ *is convergent if, and only if, there exist* η_i *closed holomorphic* 1*-forms on* $(\mathrm{V}_i, \mathrm{L}_i)$ *such that*

$$({\varphi_{i,i+1}}^*\theta_i) \wedge \theta_{i+1} = 0 \quad \text{where } \theta_i = (1-t)\tau\frac{d\mathrm{X}}{\mathrm{X}} + t\frac{d\mathrm{Y}}{\mathrm{Y}} + \eta_i.$$

Equivalently, there exists an equivalent cocycle $\varphi' \sim \varphi$ *such that*

$$({\varphi'_{i,i+1}}^*\theta^0) \wedge \theta^0 = 0 \quad \text{where } \theta^0 = (1-t)\tau\frac{d\mathrm{X}}{\mathrm{X}} + t\frac{d\mathrm{Y}}{\mathrm{Y}}.$$

Proof. — When $t \neq 0, 1$, the proof easily follows from Lemma 7.2. Indeed, all $\Pi_{i*}\hat{\mathcal{F}}_t$ are defined by $\theta_i = 0$ and have to patch via the glueing maps $\varphi_{i,i+1}$. Conversely, if $\theta_i = 0$ patches via the glueing maps $\varphi_{i,i+1}$, then this means that we get a foliation $\mathcal{F}$ on $\mathrm{U}_\varphi \setminus \mathrm{C}$ which is flat to $\hat{\mathcal{F}}_t$ along C, and therefore extends by Riemann. Using Remark 7.3 and Definition 4.2, one easily derives the second (equivalent) assertion. Finally, in the case $t = 0$ for instance, after applying Lemma 7.4 in a very similar way, we note that θ_i defines a regular foliation on V_i for $i = 1, 3$ (as $\eta_i = df, f \in \mathbf{C}\{\mathrm{Y}^{\pm 1}\}$). On the other hand, on V_i for $i = 2, 4$, θ_i defines a singular foliation as soon as $f_i \not\equiv 0$ (non identically vanishing), i.e. with a saddle-node singular points at the two points $p_{i,i+1}$ and $p_{i-1,i}$; therefore, $f_i \equiv 0$ in the case we have a global foliation and we are back to the logarithmic case. The proof ends-up like before. □

Remark **7.7.** — The statement of Corollary 7.6 can be reformulated as follows. The formal foliation $\hat{\mathcal{F}}_t$ of $(\mathrm{U}_\varphi, \mathrm{C})$ is convergent if, and only if, there exists a foliation $\mathcal{G}_t$ on $(\mathrm{V}_\varphi, \mathrm{D})$ which is locally defined by a closed logarithmic 1-form with poles supported on D and having residues t on L_1 and $(1-t)\tau$ on L_4 (we have automatically opposite residues on opposite sides of D). Indeed, the local foliations $\theta_i = 0$ patch together.

We can precise Corollary 7.6 for generic t as follows.

Proposition **7.8.** — *If* $\hat{\mathcal{F}}_t$ *is not of rational type, i.e.* $\left(\frac{1}{t} - 1\right)\tau \notin \mathbf{Q}$, *then are equivalent*

1. $\hat{\mathcal{F}}_t$ *is convergent,*
2. $\hat{\mathcal{F}}_t$ *is defined by a closed (convergent) meromorphic* 1*-form* ω,
3. $(\varphi_{i,i+1})^*\theta_i = \theta_{i+1}$ *with* θ_i *like in Corollary 7.6,*
4. *there is a closed logarithmic* 1*-form* θ *on* $(\mathrm{V}_\varphi, \mathrm{D})$ *with poles supported on* D *and having residues* t *on* L_1 *and* $(1-t)\tau$ *on* L_4.

Obviously, $\omega = \Pi_\varphi^*\theta$ *up to a constant.*

Proof. — When $\hat{\mathcal{F}}_t$ is not of rational type, then we have

$$(\varphi_{i,i+1}{}^*\theta_i) \wedge \theta_{i+1} = 0 \quad \Leftrightarrow \quad (\varphi_{i,i+1}{}^*\theta_i) = \theta_{i+1}.$$

Indeed, if $\varphi_{i,i+1}{}^*\theta_i$ is colinear to θ_{i+1}, then it is proportional to θ_{i+1}, i.e. it writes $f_i \cdot \theta_i$ with f_i meromorphic on V_i (de Rham-Saito Lemma). But since it is also closed, we have

$$0 = d(f_i \cdot \theta_i) = df_i \wedge \theta_i + f_i \wedge \underbrace{d\theta_i}_{=0}$$

and f_i is a meromorphic first integral for $\theta_i = 0$, which must be constant in the irrational type. This constant must be $= 1$ as the residues are preserved. As a consequence, all θ_i patch together on V_φ. Finally, note that if $\hat{\mathcal{F}}_t$ is convergent, then its holonomy is not cyclic (because not of rational type) and therefore preserves a meromorphic 1-form on the transversal that we can extend as a closed meromorphic 1-form ω defining the foliation. □

Let us now illustrate how different is the situation for foliations of rational type by revisiting the classification [15, Theorem 5] of neighborhoods with 2 convergent foliations, in the particular case of $\hat{\mathcal{F}}_0$ and $\hat{\mathcal{F}}_1$, corresponding respectively to vertical and horizontal foliations on V_φ. The proof is a straightforward application of the above criteria.

Proposition **7.9.** — *The formal foliations* $\hat{\mathcal{F}}_0$ *and* $\hat{\mathcal{F}}_1$ *on* (U, C) *are convergent if, and only if,* (U, C) *can be defined by a cocycle of the form*

$$\begin{cases} \varphi_{i,i+1}(X, Y) = (\alpha_i(X), Y) & \textit{for } i = 1, 3 \\ \varphi_{i,i+1}(X, Y) = (X, \alpha_i(Y)) & \textit{for } i = 2, 4 \end{cases}$$

for 1*-variable diffeomorphisms* α_i *tangent to the identity, and the corresponding foliations on* V_φ *are respectively defined in charts* V_i *by* $dX = 0$ *and* $dY = 0$*. Moreover, this normalization is unique up to conjugacy by* $\phi(X, Y) = (e^t X, e^{\tau t} Y)$.

The space of leaves of $\hat{\mathcal{F}}_0$ on $U \setminus C$ corresponds to the space of orbits for its holonomy map, and therefore to Martinet-Ramis' "*Chapelet de sphères*" (see [18, page 591]). It is given by two copies of $\mathbf{C}^*_X$ patched together by means of diffeomorphism germs $\alpha_1(X)$ at $X = \infty$ and $\alpha_3(X)$ at $X = 0$. A similar description holds for $\hat{\mathcal{F}}_1$ with Martinet-Ramis' cocycle α_2 and α_4. The invariants found by the third author in [36] are related with the corresponding periodic transformations in variable ξ (see Fig. 5).

Remark **7.10.** — It follows from [15], or from the unicity of the formal pencil $\hat{\mathcal{F}}_t$, that for given $t \in \mathbf{P}^1$, one cannot find two different collections $(\theta_i)_i$ and $(\theta'_i)_i$ defining two global logarithmic foliations $\mathcal{G}_t$ and $\mathcal{G}'_t$ on V, like in Corollary 7.6. One way to see

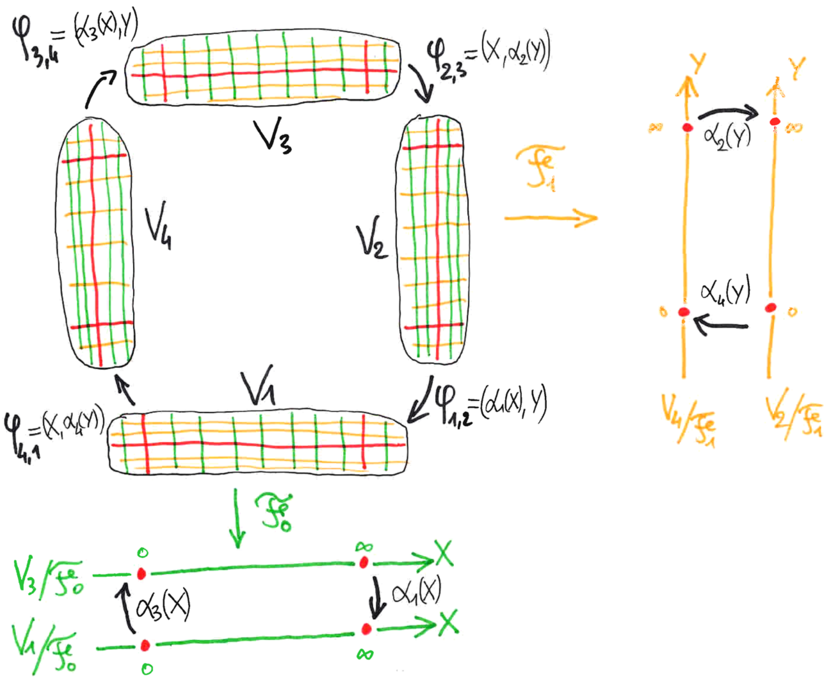

FIG. 5. — Martinet-Ramis moduli

this directly from the point of view of this section is as follows. In the irrational case $\left(\frac{1}{t}-1\right)\tau \notin \mathbf{Q}$, we see from Proposition 7.8 that θ_i's patch as a global closed logarithmic 1-form. But the difference between two closed logarithmic 1-forms with the same residues is a closed holomorphic 1-form η on (V,C). Now, η must be zero, even if we restrict on two consecutive line neighborhoods $(\mathrm{V}_i,\mathrm{L}_i)\cup_{\varphi_{i,i+1}}(\mathrm{V}_{i+1},\mathrm{L}_{i+1})$, as it only depends on X or on Y depending on the sector. In the rational case, $\left(\frac{1}{t}-1\right)\tau \in \mathbf{Q}$, there is also unicity of θ_i's on two consecutive line neighborhoods whose residues have quotient >0; indeed, after blowing-up, we get a rational fibration which must be unique by Blanchard Lemma.

7.2. *Non existence of foliations.* — For a generic neighborhood $(\mathrm{U}_\varphi,\mathrm{C})$, there is no convergent foliation. In order to prove this, it is enough to provide a single example without foliation. Such an example has been given quite recently by Mishustin in [21]. With our Corollary 7.6, it is not too difficult to provide an example without foliations.

Theorem **7.11.** — *Let* $(\mathrm{U}_\varphi,\mathrm{C})$ *be a neighborhood such that*

$$\varphi_{1,4}(\mathrm{X},\mathrm{Y}) = (\mathrm{X}(1+\mathrm{XY}),\mathrm{Y}(1+\mathrm{X}^2\mathrm{Y})).$$

Then all foliations $\hat{\mathcal{F}}_t$ *belonging to the formal pencil* $(\hat{\mathcal{F}}_t)_{t\in\mathbf{P}^1}$ *are divergent.*

Even the transversal fibration $\hat{\mathcal{F}}_\infty$ is divergent in that case.

Proof. — Suppose by contradiction that there exists at least one convergent foliation in the pencil. Then, by Corollary 7.6, there exists on each V_i a non trivial logarithmic 1-form

$$\theta_i = \alpha\frac{dX}{X} + \beta\frac{dY}{Y} + \begin{cases} f_i(X)dX & \text{if } i \text{ even} \\ f_i(Y)dY & \text{if } i \text{ odd}\end{cases}$$

with $f_i : (\mathbf{C}, 0) \to \mathbf{C}$ holomorphic, such that $\theta_{i+1} \wedge (\varphi_{i,i+1})^*\theta_i = 0$. One has

$$(\varphi_{1,4})^*\theta_1 = \alpha\frac{dX}{X} + \alpha\frac{d(XY)}{1+XY} + \beta\frac{dY}{Y} + \beta\frac{d(X^2Y)}{1+X^2Y} + f_1(Y(1+X^2Y))\cdot((1+2X^2Y)dY + 2XY^2dX).$$

The residual parts of the 2-form $\theta_4 \wedge (\varphi_{1,4})^*\theta_1$ at $X = 0$ and $Y = 0$ respectively write

$$\alpha f_1(Y)\frac{dX}{X} \wedge dY \quad \text{and} \quad \beta f_4(X)dX \wedge \frac{dY}{Y}. \tag{7.2}$$

Both expressions must be vanishing identically. If $\beta = 0$, then $\alpha \neq 0$ and we deduce from (7.2) that $f_1 \equiv 0$. This implies that $(\varphi_{1,4})^*\theta_1$ only depends on $X(1+XY)$, while θ_4 only depends on X, contradiction. Assume now $\alpha = 0$ (and $\beta \neq 0$); then, by (7.2), we have $f_4 \equiv 0$. Again, we conclude that $(\varphi_{1,4})^*\theta_1$ only depends on $X(1+X^2Y)$, while θ_4 only depends on Y, contradiction. Finally, assume that $\alpha \neq 0$ and $\beta \neq 0$; then, by (7.2), we have $f_1, f_2 \equiv 0$ and we obtain

$$\theta_4 \wedge (\varphi_{1,4})^*\theta_1 = \frac{\alpha(\alpha-\beta) + \beta(\alpha - 2\beta)X + (\alpha^2 - 2\beta^2)X^2Y}{(1+XY)(1+X^2Y)}dX \wedge dY.$$

Clearly, this expression cannot be zero if α and β are both non zero. □

7.3. *Only one convergent foliation.* — To complete the picture, it is interesting to provide an example of a foliation having only one convergent foliation $\hat{\mathcal{F}}_{t_0}$ in the pencil $(\hat{\mathcal{F}}_t)_{t\in\mathbf{P}^1}$ for an arbitrary t_0.

Theorem **7.12.** — *Let $t_0 = [u_0 : v_0] \in \mathbf{P}^1$, and let (U_φ, C) be the neighborhood such that*

$$\begin{cases} \varphi_{1,4} = (X(1+XYe^Y)^{-\beta_0}, Y(1+XYe^Y)^{\alpha_0}) \\ \quad \textit{where} \quad \alpha_0 = \tau(v_0 - u_0) \textit{ and } \beta_0 = u_0 \end{cases} \quad \textit{and} \quad \begin{cases} \varphi_{i,i+1} = Id \\ \quad \textit{for } i = 1, 2, 3. \end{cases}$$

Then $\hat{\mathcal{F}}_{t_0}$ is the unique convergent foliation in the formal pencil $(\hat{\mathcal{F}}_t)_{t\in\mathbf{P}^1}$.

Proof. — Note that φ preserves the foliation defined by the logarithmic form $\alpha_0\frac{dX}{X} + \beta_0\frac{dY}{Y}$ which then descends on V_φ, i.e. the formal foliation $\hat{\mathcal{F}}_{t_0}$ is indeed convergent on U_φ. Assuming by contradiction that there is another convergent foliation $\hat{\mathcal{F}}_t$

with $t \neq t_0$, let θ_i be the associated logarithmic 1-form on V_i. As in the proof of Theorem 7.11, we get

$$\theta_1 = \alpha\frac{dX}{X} + \beta\frac{dY}{Y} + f_1(Y)dY \quad \text{and} \quad \theta_4 = \alpha\frac{dX}{X} + \beta\frac{dY}{Y} + f_4(X)dX$$

with $f_i : (\mathbf{C}, 0) \to \mathbf{C}$ holomorphic, and $[\alpha : \beta] \neq [\alpha_0 : \beta_0]$. We derive

$$(\varphi_{1,4})^*\theta_1 = \alpha\frac{dX}{X} + \beta\frac{dY}{Y} + (\alpha_0\beta - \alpha\beta_0)\frac{d(XYe^Y)}{1+XYe^Y} \\ + f_1(Y(1+XYe^Y)^{\alpha_0}) \cdot d(Y(1+XYe^Y)^{\alpha_0}).$$

The residual parts of the 2-form $\theta_4 \wedge (\varphi_{1,4})^*\theta_1$ at $X = 0$ and $Y = 0$ respectively write

$$\alpha f_1(Y)\frac{dX}{X} \wedge dY \quad \text{and} \quad \beta f_4(X)dX \wedge \frac{dY}{Y}. \tag{7.3}$$

We are led to a similar discussion as in the proof of Theorem 7.11. When $\beta = 0$, then $f_1 \equiv 0$ and $(\varphi_{1,4})^*\theta_1$ only depends on $X(1 + XY)^{-\beta_0}$, while θ_4 only depends on X; we get a contradiction since $\beta_0 \neq 0$ in this case. When $\alpha = 0$, then $f_4 \equiv 0$ and $(\varphi_{1,4})^*\theta_1$ only depends on $Y(1 + XY)^{\alpha_0}$, while θ_4 only depends on Y; contradiction. Finally, when both $\alpha \neq 0$ and $\beta \neq 0$, then

$$\theta_4 \wedge (\varphi_{1,4})^*\theta_1 = \underbrace{(\alpha\beta_0 - \alpha_0\beta)}_{\neq 0}\frac{d(XYe^Y)}{1+XYe^Y} \wedge \left(\alpha\frac{dX}{X} + \beta\frac{dY}{Y}\right)$$

which cannot be zero, again a contradiction. □

Remark **7.13.** — More generally, given $f \in \mathbf{C}\{X, Y\}$ vanishing along $X = 0$ and $Y = 0$, the same proof shows that the cocycle defined by $\varphi_{1,4} = (Xe^{-\beta_0 f}, Ye^{\alpha_0 f})$ and $\varphi_{i,i+1} = \mathrm{Id}$ otherwise also provides a neighborhood (U_φ, C) with only one convergent foliation, namely $\hat{\mathcal{F}}_{t_0}$, provided that $df \wedge d(X^pY^q) \not\equiv 0$ for all $p, q \in \mathbf{Z}_{>0}$. Moreover, one easily checks that two different such f, says f and f', define non equivalent neighborhoods provided that their difference do not take the form $f' - f = g(X) + h(Y)$.

8. Symmetries

We follow notations of Sections 3 and 4.

Let (U_φ, C) be a neighborhood formally equivalent to $(U_{1,0,0}, C)$. Via formal conjugation, formal symmetries (or automorphisms) of (U_φ, C) which restrict to translations on C are those of $(U_{1,0,0}, C)$. They form the group denoted $\mathrm{Aut}(S_0)$ and are explicited in the left hand side in the correspondence below (see Corollary 3.10):

$$\textbf{(8.1)}\qquad \begin{cases} (z,\xi)\mapsto (cz,\xi+t) \\ c\in\mathbf{C}^*,\ t\in\mathbf{C}\end{cases} \quad\leftrightarrow\quad (a,b):(\mathrm{X},\mathrm{Y})\mapsto (\underbrace{e^{2i\pi t}}_{a}\mathrm{X},\underbrace{ce^{2i\pi\tau t}}_{b}\mathrm{Y})$$

The subgroup $\mathrm{Aut}(\mathrm{U}_\varphi,\mathrm{C})$ of convergent automorphisms (which restrict to translations on C) thus identifies with a subgroup of the two dimensional linear algebraic torus:

$$\mathrm{G}\subset \mathrm{Aut}^0(\mathbf{P}^1\times\mathbf{P}^1,\mathrm{D})\simeq \mathbf{C}^*\times\mathbf{C}^*.$$

Theorem **8.1.** — *Let* $(\mathrm{U}_\varphi,\mathrm{C})$ *and* G *be as above. Then the subgroup* $\mathrm{G}\subset\mathbf{C}^*\times\mathbf{C}^*$ *is algebraic. In particular, we are in one of the following cases:*

- G *is finite and* $\mathrm{G}=\{(a,b)\ ;\ a^pb^q=a^{p'}b^{q'}=1\}$ *for some non proportional* $(p,q),(p',q')\in\mathbf{Z}^2\setminus(0,0)$*;*
- $\mathrm{G}=\{(a,b)\ ;\ a^pb^q=1\}$ *for some* $(p,q)\in\mathbf{Z}^2\setminus(0,0)$*; in particular, a finite index subgroup of* G *is generated by the flow of the rational vector field* $p\mathrm{X}\partial_\mathrm{X}+q\mathrm{Y}\partial_\mathrm{Y}$*;*
- $\mathrm{G}=\mathbf{C}^*\times\mathbf{C}^*$ *and* $(\mathrm{U}_\varphi,\mathrm{C})\overset{an}{\sim}(\mathrm{U}_{1,0,0},\mathrm{C})$.

Moreover, in the first two cases, up to equivalence $\approx$*, the cocycle takes the form*

$$\varphi_{i,i+1}(\mathrm{X},\mathrm{Y})=(\mathrm{X}\cdot u_{i,i+1},\mathrm{Y}\cdot v_{i,i+1})$$

where $u_{i,i+1}$, $v_{i,i+1}$ *are Laurent series in* $\mathrm{X}^p\mathrm{Y}^q$ *and* $\mathrm{X}^{p'}\mathrm{Y}^{q'}$ *in the first case, in* $\mathrm{X}^p\mathrm{Y}^q$ *in the second one and the action of* G *is linear in each chart* $(\mathrm{V}_i,\mathrm{L}_i)$.

Remark **8.2.** — Note that G is not algebraic in general as a subgroup of $\mathrm{Aut}(\mathrm{U}_{1,0,0},\mathrm{C})=\mathrm{Aut}^0(\mathrm{S}_0)$. Actually, the correspondence between $\mathrm{Aut}(\mathrm{U}_{1,0,0},\mathrm{C})$ and $\mathrm{Aut}^0(\mathbf{P}^1\times\mathbf{P}^1,\mathrm{D})$ specified above is only of analytic nature.

Proof. — Let $f\in\mathrm{Aut}(\mathrm{U}_\varphi,\mathrm{C})$. Set $f_i=\Psi_i\circ f\circ\Psi_i^{-1}$. For every i, one can then write $f_i=h\circ\Phi_i$ where $\Phi_i\in\mathscr{G}^\infty[\mathrm{F}_{1,0,0}](\mathrm{I}_i)$ and $h=\hat\Psi\circ f\circ\hat\Psi^{-1}$. According to the correspondence depicted in (8.1) and Proposition 3.13, there exists a collection of automorphisms $g_i\in\mathrm{Diff}(\mathrm{V}_i,\mathrm{L}_i)$ satisfying

$$\Pi\circ f_i=g_i\circ\Pi.$$

Moreover, these $g_i's$ define an automorphism of the neighborhood of four lines $(\mathrm{V}_\varphi,\mathrm{D})$ associated to U_φ. The gluing conditions read

$$\textbf{(8.2)}\qquad g_i\circ\varphi_{i,i+1}=\varphi_{i,i+1}\circ g_{i+1}$$

Note that the corresponding element (a,b) to h in $\mathbf{C}^*\times\mathbf{C}^*$ is the linear part of g_i at the crossing points $p_{i-1,i},p_{i,i+1}$ and is independent of i.

Conversely, the datum of $(g_i) \in \prod_i \mathrm{Diff}(V_i, L_i)$ fulfilling (8.2) determine f (use the necessary and sufficient condition provided by Proposition 3.13).

We claim that g_i can be linearized in each chart (V_i, L_i). Let us consider for instance the transformation g_4 on (V_4, L_4). Let us first prove that g_4 is locally linearizable at intersection point $p_{4,1}$. We note that g_4 preserves the vertical fibration $dX = 0$ and sections $Y = 0, \infty$; it therefore takes the form

$$(X, Y) \mapsto (aX \cdot u(X), bY \cdot v(X)), \quad u(0) = v(0) = 1.$$

The gluing condition involving $\varphi_{4,1}$ shows that the restriction $g_4|_{L_1} : X \mapsto aX \cdot u(X)$ becomes linear in the chart (V_1, L_1). Therefore, after changing the X coordinate on (V_4, L_4), we can assume

$$g_4(X, Y) = (aX, bY \cdot v(X)), \quad v(0) = 1.$$

On the other hand, g_1 preserves the fibration $dY = 0$ of (V_1, L_1), and by compatibility condition, g_4 must also preserve the local foliation with first integral $\bar{Y} = \varphi_{4,1}^* Y$ near $p_{4,1}$. In other words, in new coordinates $(X, \bar{Y})$, the transformation g_4 preserves $dX = d\bar{Y} = 0$ and is linear. Now, in order to prove the claim, we have to show that this linearization can be done globally along (V_4, L_4). For this, consider the change of coordinate $\bar{Y} = \phi(X, Y)$ used to linearize locally and decompose $\phi(X, Y) = \sum_{k \geq 1} \phi_k(X) Y^k$. Expanding the linearization equation in powers of Y shows that the truncated change of coordinate $\tilde{Y} = \phi_1(X) Y$ also linearizes g_4.

As a by-product, G must contain the Zariski closure of $< (a, b) > \subset \mathbf{C}^* \times \mathbf{C}^*$. Indeed, all $\varphi_{i,i+1}$ have to commute with $g(X, Y) = (aX, bY)$. Writing $\varphi_{i,i+1}(X, Y) = (X \cdot u(X, Y), Y \cdot v(X, Y))$, we see that u, v have to be invariant by g, i.e. $u \circ g = u$ for instance; equivalently, all non zero monomials of u and v are g-invariant. These monomials define an algebraic subgroup $H \subset \mathbf{C}^* \times \mathbf{C}^*$ which is the group of linear transformations commuting with $\varphi_{i,i+1}$. We conclude that $(a, b) \in H \subset G$.

If g was Zariski dense in $\mathbf{C}^* \times \mathbf{C}^*$, we are done: the commutation of $\varphi_{i,i+1}$ with all linear transformations shows that $\varphi_{i,i+1}$ is linear (hence trivial) and $(U_\varphi, C) \overset{\mathrm{an}}{\sim} (U_{1,0,0}, C)$. Now, assuming that G is a strict subgroup of $\mathbf{C}^* \times \mathbf{C}^*$, we want to prove that it can be linearized globally.

Assume first that G is finite. Then it can be linearized on each line neighborhood (V_i, L_i). Indeed, for instance on (V_4, L_4), G acts by transformations of the form

$$g(X, Y) = (aX \cdot u(X), bY \cdot v(X)), \quad u(0) = v(0) = 1,$$

and we denote by $\mathrm{lin}(g)$ its linear part (aX, bY). Then the transformation

$$\varphi_4 := \frac{1}{\#G} \sum_{g \in G} \mathrm{lin}(g)^{-1} \circ g$$

is of the form $\varphi_4(X, Y) = (X \cdot u(X), Y \cdot v(X))$, $u(0) = v(0) = 1$ and linearizing the group:

$$\varphi_4 \circ g = \mathrm{lin}(g) \circ \varphi_4, \quad \forall g \in G.$$

We can therefore assume that G acts linearly in each chart (V_i, L_i) and the cocycle φ has to commute with all elements. It is well known that the group G is generated by two elements (a_1, b_1) and (a_2, b_2) of finite order; moreover, by duality, G is defined by 2 independent monomial equations $a^p b^q = a^{p'} b^{q'} = 1$. Gluing conditions with $\varphi_{i,i+1}$ show that $u_{i,i+1}, v_{i,i+1}$ must be G-right-invariant and therefore factor through the two monomial equations.

On the other hand, if G contains an element g of infinite order, then we can first linearize this element. The Zariski closure H of its iterates $< g >$ in $\mathbf{C}^* \times \mathbf{C}^*$ is one dimensional, if strictly smaller than $\mathbf{C}^* \times \mathbf{C}^*$, and defined by a monomial equation $a^p b^q = 0$. If G is larger than H, then the quotient G/H is generated by an element of finite order $g' \in G$, and we can linearize the finite group $< g' >$ like above; since g' and its linear part both commute with H, the linearizing transformations also commute with H and G is linearized. It is therefore algebraic, defined by monomial equations of $\varphi_{i,i+1}$, and they all factor into a single monomial. □

Remark **8.3.** — From the description above, we note that the convergence of a non trivial automorphism g of (U_φ, C) inducing the identity on C implies that $(U_\varphi, C) \overset{\mathrm{an}}{\sim} (U_{1,0,0}, C)$, since the group generated by g must be Zariski dense in $\mathbf{C}^* \times \mathbf{C}^*$.

Remark **8.4.** — In Proposition 7.9, the foliation $\hat{\mathcal{F}}_0$ is defined by a holomorphic vector field if, and only if, $\alpha_2(Y) = \alpha_4(Y) = Y$. Equivalently, the Martinet-Ramis invariant of $\hat{\mathcal{F}}_1$ are trivial, i.e. $\hat{\mathcal{F}}_1$ can be defined by a closed 1-form.

9. Sectorial normalization

We maintain the foregoing notations. Recall that we have set $\varpi = \arg \tau \in]0, \pi[$. Let (U, C) be formally equivalent to $(U_{1,0,0}, C)$. We want to show that (U, C) has the form U_φ. In other word, we want to prove Lemma A, or equivalently Lemma 4.1.

9.1. *Overview of the proofs.* — One can suppose that $(U, C) = (\tilde{U}, \tilde{C})/F$ where

$$F(z, \xi) = \underbrace{F_{1,0,0}(z, \xi)}_{(qz, \xi - 1)} + (\Delta_1, \Delta_2)$$

Here, the terms Δ_i $i = 1, 2$ appearing in the residual part are well defined on a neighborhood of $\{\xi = \infty\}$ in $\mathbf{C}^* \times \bar{\mathbf{C}}\}$. Moreover, one can require that $\Delta_i = O(\xi^{-N})$[25]

[25] It must be understood that for every compact K of $\mathbf{C}^*$, there exists $C_K > 0$ and $M_K > 0$ such that for all (z, ξ) such that $z \in K$ and $|\xi| \geq M_K$, $|\Delta_i(z, \xi)| \leq \frac{C_K}{|\xi^N|}$.

where $N \gg 0$ is an arbitrarily large integer (depending on the order of truncation of the formal conjugacy map), so that there exists a formal diffeomorphism

$$\hat{\Psi}(z,\xi) = (z + \hat{g}, \xi + \hat{h}), \quad \hat{g} = \sum_{n\geq 1} b_n \xi^{-n}, \ \hat{h} = \sum_{n\geq 1} a_n \xi^{-n}$$

where a_n, b_n are entire functions on $\tilde{C} = \mathbf{C}^*$ such that

$$F \circ \hat{\Psi} = \hat{\Psi} \circ F_{1,0,0}.$$

This can be reformulated as

$$\hat{g} \circ F_{1,0,0} - q\hat{g} = \Delta_1 \circ \hat{\Psi} \tag{9.1}$$

$$\hat{h} \circ F_{1,0,0} - \hat{h} = \Delta_2 \circ \hat{\Psi} \tag{9.2}$$

Basically, we will show that there exists a *holomorphic* solution $\Psi = \mathrm{Id} + (g,h)$ of the previous fonctional equations $F \circ \Psi = \Psi \circ F_{1,0,0}$, i.e. with h, g satisfying

$$g \circ F_{1,0,0} - qg = \Delta_1 \circ \Psi \tag{9.3}$$

$$h \circ F_{1,0,0} - h = \Delta_2 \circ \Psi \tag{9.4}$$

defined on "suitable sectorial domains", namely on $U_i := \Pi^{-1}(V_i - L_i)$, $i = 1, 2, 3, 4$, and admitting asymptotic expansion along $\tilde{C}$ compatible with the formal conjugacy map $\hat{\Psi}$. More precisely $g, h \in \mathcal{A}(I_i)$ with the notations of section 3.1 with respective asymptotic expansions $\hat{g}$ and $\hat{h}$. To this end, we will first exhibit solutions in $\mathcal{O}(I_i)$ satisfying some suitable growth behaviour of the following linearized equations:

$$g \circ F_{1,0,0} - qg = \Delta_1 \tag{9.5}$$

$$h \circ F_{1,0,0} - h = \Delta_2 \tag{9.6}$$

Most of Section 9 is devoted to the construction of such sectorial solutions on the sector U_1, and it will be explained in Section 9.6 how to deduce normalization on other sectors.

Remark **9.1.** — The equation (9.5) can be reduced to equation (9.6); indeed, after setting

$$g(z,\xi) = z\tilde{g}(z,\xi) \quad \text{and} \quad \Delta_1(z,\xi) = qz\widetilde{\Delta}_1(z,\xi),$$

we get

$$\tilde{g} \circ F_{1,0,0} - \tilde{g} = \widetilde{\Delta}_1.$$

This will enable us to solve by a fairly standard fixed point method the initial functional equations (9.3) and (9.4). In order to get rid of the coefficient q on the left hand side, note that both equations (9.4) and (9.3) can be reformulated as:

$$\tilde{g}(qz,\xi-1)-\tilde{g}(z,\xi)=(1+\tilde{g}(z,\xi))\widetilde{\Delta}_1\big(z(1+\tilde{g}(z,\xi)),\xi+h(z,\xi)\big) \tag{9.7}$$

$$h(qz,\xi-1)-h(z,\xi)=\Delta_2\big(z(1+\tilde{g}(z,\xi)),\xi+h(z,\xi)\big) \tag{9.8}$$

where the symbol $\sim$ stands for the same modification than in Remark 9.1 for the linear case.

9.2. *The linearized/homological equation.* — Our purpose is to construct some sectorial solution of the linearized functional equations (9.6) (and therefore (9.5) by Remark 9.1) belonging to $\mathcal{O}(\mathrm{I}_1)$. Actually, one will just firstly state some results and use this material to undertake the resolution of the complete (non linear) conjugacy equation (over I_1). We will detail the resolution of the linearized equation (the most technical part) in Section 9.5. The existence of other sectorial conjugacy maps over I_i, $i\neq 1$ can be obtained in a very similar way and we indicate briefly how to proceed in Section 9.6.

Let us first settle some notations. Recall that $q=e^{2i\pi\tau}$ with $\Im\tau>0$, so that $|q|<1$. As one only focuses on transversal sectorial domain determined by $\mathrm{I}_1=]-\varpi,\pi-\varpi[$, $\varpi=\arg(\tau)$, we are going to work in domains S of the following shape. Fix $0<a<b$ such that $a<|q|b$ and consider the annulus

$$\mathcal{C}_{a,b}=\{a\leq|z|\leq b\}.$$

Let $\delta_{(a,b)}>0$ small enough and for $0<\delta\leq\delta_{(a,b)}$, set

$$\mathrm{S}_{a,b,\delta}=\{(z,\xi)\in\mathbf{C}^2 \mid |\mathrm{Y}(z,\xi)|\leq\delta \text{ and } z\in\mathcal{C}_{a,b}\}$$

where $\mathrm{Y}(z,\xi)=ze^{2i\pi\tau\xi}$ (recall that $|\mathrm{Y}|<\delta$ corresponds to a neighborhood V_1 of L_1). Alternatively, this set can be described by the inequation $\Re(2i\pi\tau\xi)<\log(\frac{\delta}{|z|})$, $z\in\mathcal{C}_{a,b}$ so that in particular $\arg\xi\in\mathrm{I}_1=]-\varpi,\pi-\varpi[$ or equivalently $\Im(\tau\xi)>0$. Note that $\mathrm{F}_{1,0,0}(\mathrm{S}_{a,b,\delta})=\mathrm{S}_{|q|a,|q|b,\delta}$. It is thus coherent to investigate the existence of a solution h of (9.6) on the domain $\mathrm{S}_{|q|a,b,\delta}=\mathrm{S}_{a,b,\delta}\cup\mathrm{F}_{1,0,0}(\mathrm{S}_{a,b,\delta})$.

In what follows, we will indeed provide a solution of (9.6) with "good estimates" on a domain of the form $\mathrm{S}_{|q|a,b,\delta}$ using a "leafwise" resolution with respect to the foliation defined by the levels of Y. For the sake of notational simplicity we will omit for a while the subscript a, b by setting $\mathrm{S}_\delta:=\mathrm{S}_{a,b,\delta}$ and $\mathrm{S}'_\delta:=\mathrm{S}_{|q|a,b,\delta}$. If $\delta_1\leq\delta_2$, remark that S_{δ_1} and S'_{δ_1} are respectively subdomains of S_{δ_2} and S'_{δ_2}. To state precisely our result, let us fix some additional notations and definitions. Let $m\geq 3$ a positive integer and consider the subspace H_δ^m of $\mathcal{H}(\mathrm{S}_\delta)$[26] defined by the functions Δ such that

$$\|\Delta\|_m := \sup_{(z,\xi)\in\mathrm{S}_\delta} |\Delta(z,\xi)||\xi|^m < \infty.$$

We will also introduce the space $\mathrm{H}_\delta^{\infty\prime}$ of *bounded* holomorphic functions h on S'_δ equipped with the natural norm

$$\|h\|_\infty := \sup_{(z,\xi)\in\mathrm{S}'_\delta} |h(z,\xi)| < \infty.$$

Theorem **9.2.** — *Fix a, b as above. Then, there are positive constants $\delta_{(a,b)}$, C such that: for every $\delta \leq \delta_{(a,b)}$ and every function $\Delta_\delta \in \mathrm{H}_\delta^m$, $m \geq 3$, there exists a unique function $h_\delta \in \mathrm{H}_\delta^{\infty\prime}$ satisfying*

1. $h_\delta \circ \mathrm{F}_{1,0,0} - h_\delta = \Delta_\delta$.
2. $\|h_\delta\|_\infty \leq \mathrm{C}\|\Delta_\delta\|_m$.
3. *For every $(z,\xi) \in \mathrm{S}'_\delta \cap \{\Im(\xi) \geq 1\}$, we have $|h_\delta(z,\xi)| \leq \frac{\mathrm{C}\|\Delta_\delta\|_m}{\sqrt{\Im(\xi)}}$.*

In addition, there exists a positive constant D_θ only depending on $\theta \in]0, \frac{\pi}{2}]$ such that

$$|h_\delta(z,\xi)| \leq \frac{\mathrm{D}_\theta\|\Delta_\delta\|_m}{|\xi|^{m-2}}$$

for every $(z,\xi) \in \mathrm{S}'_\delta \cap \{\theta - \varpi \leq \arg\xi \leq \pi - \theta - \varpi\}$.

As mentioned before, we will postpone the proof of Theorem 9.2 to Section 9.5. Condition (3) is needed for the unicity, and after to produce a norm with a unique fixed point when solving the functional equation. For the time being, we detail how it provides a section Ψ_1 of $\mathscr{G}^1(\mathrm{I}_1)$ of the form $(z+g, \xi+h)$ such that the pair (g,h) admits $(\hat{g},\hat{h})$ as asymptotic expansion and satisfies in addition the equations (9.8) and (9.7). In other words, we are going to exhibit a transversely sectorial conjugacy map between F and $\mathrm{F}_{1,0,0}$:

$$\mathrm{F} \circ \Psi_1 = \Psi_1 \circ \mathrm{F}_{1,0,0}.$$

9.3. *Solving the functional equation.* — Notations as in Theorem 9.2. It is worth mentioning that the strategy developed here as well as the resolution of the linearized/homological equation in the forthcoming Section 9.5 owes a lot to [38].

As before, a, b are fixed, $\delta_{(a,b)} > 0$ is small enough and may be adjusted from line to line in order to guarantee the validity of the estimates below. We will denote by δ any positive number such that $0 < \delta \leq \delta_{(a,b)}$. We will omit for a while the subscript (a,b). Let us introduce two Banach spaces.

[26] Let $\mathrm{A} \subset \mathbf{C}^\mathrm{N}$, in this paragraph and hereafter, $\mathcal{H}(\mathrm{A})$ will denote the algebra of *holomorphic functions on* A, that is the $\mathbf{C}$-valued continuous functions on A which are holomorphic in the interior in the usual sense.

First, let $m \geq 3$, and equip $\mathrm{H}_\delta^m \times \mathrm{H}_\delta^m$ with the natural norm

$$\mathrm{N}_m(\mathrm{D}_1, \mathrm{D}_2) := \|\mathrm{D}_1\|_m + \|\mathrm{D}_2\|_m$$

(recall that $\|\mathrm{D}\|_m := \sup_{(z,\xi)\in \mathrm{S}_\delta} |\mathrm{D}(z,\xi)||\xi|^m$).

On the other hand, let $\mathrm{H}_\delta^{\infty,\infty\prime}$ be the subspace of those $(h_1, h_2) \in \mathcal{H}(\mathrm{S}'_\delta) \times \mathcal{H}(\mathrm{S}'_\delta)$ defined by $\mathrm{N}'_\infty(h, \tilde{g}) < \infty$ where

$$\mathrm{N}'_\infty(h_1, h_2) := \|h_1\|'_\infty + \|h_2\|'_\infty$$
$$\text{with } \|h\|'_\infty := \underbrace{\sup_{(z,\xi)\in \mathrm{S}'_\delta} |h(z,\xi)|}_{\|h\|_\infty} + \sup_{(z,\xi)\in \mathrm{S}'_\delta \cap \Im(\xi)\geq 1} |h(z,\xi)|\sqrt{|\Im(\xi)|}.$$

Note that both normed spaces are Banach spaces, and from Theorem 9.2, one inherits a continuous linear map between them:

$$\mathcal{L} \,:\, (\mathrm{H}_\delta^m \times \mathrm{H}_\delta^m, \mathrm{N}_m) \to (\mathrm{H}_\delta^{\infty,\infty\prime}, \mathrm{N}'_\infty)\ ;\ \overrightarrow{\mathrm{D}} = (\mathrm{D}_1, \mathrm{D}_2) \mapsto \overrightarrow{h} = (h_1, h_2)$$

defined by solving

$$h_i \circ \mathrm{F}_{1,0,0} - h_i = \mathrm{D}_i \quad \text{for } i = 1, 2.$$

To be more precise, for every δ small enough, and every $\overrightarrow{\mathrm{D}} \in \mathrm{H}_\delta^m \times \mathrm{H}_\delta^m$, one has

$$\mathrm{N}'_\infty(\mathcal{L}(\overrightarrow{\mathrm{D}})) \leq \mathrm{C} \cdot \mathrm{N}_m(\overrightarrow{\mathrm{D}})$$

with the positive constant C given by Theorem 9.2.

We now define a non linear continuous map in the other way. Let us come back to the expression of the transformation $\mathrm{F} = \mathrm{F}_{1,0,0} + (\Delta_1, \Delta_2)$ defining the formally equivalent neighborhood (U, C) as explicited in Section 9.1. One can assume $\Delta_i(z,\xi) = \mathrm{O}(\xi^{-\mathrm{N}})$ for a fixed arbitrary integer $\mathrm{N} \geq 4$. Recall that the Δ_i's are analytic on a neighborhood of $\{\xi = \infty\}$ and consequently are well defined as an element of H_δ^m, $m < \mathrm{N}$, whose $\| \ \|_m$ norm tends to zero when δ goes to zero. For every $\mathrm{M} > 0$, let us denote by

$$\left(\mathrm{H}_\delta^m \times \mathrm{H}_\delta^m\right)(\mathrm{M}) \subset \mathrm{H}_\delta^m \times \mathrm{H}_\delta^m \quad \text{and} \quad \mathrm{H}_\delta^{\infty,\infty\prime}(\mathrm{M}) \subset \mathrm{H}_\delta^{\infty,\infty}$$

the respective closed balls of radius M. Then, for δ small enough, we a have a well defined map

$$\mathcal{R} \,:\, \mathrm{H}_\delta^{\infty,\infty\prime}(1) \to \mathrm{H}_\delta^m \times \mathrm{H}_\delta^m\ ;\ \overrightarrow{h} = (h_1, h_2) \mapsto \overrightarrow{\mathrm{D}} = (\mathrm{D}_1, \mathrm{D}_2)$$
$$\text{where } \begin{cases} \mathrm{D}_1(z,\xi) = (1 + h_2(z,\xi))\widetilde{\Delta}_1\big(z(1 + h_2(z,\xi)), \xi + h_1(z,\xi)\big) \\ \mathrm{D}_2(z,\xi) = \Delta_2\big(z(1 + h_2(z,\xi)), \xi + h_1(z,\xi)\big) \end{cases}$$

Indeed, if $N'_\infty(h_1, h_2) \leq 1$, then in particular $\|h_i\|_\infty \leq 1$, $i = 1, 2$, and this follows from the decreasing properties of Δ_i as precised by Footnote 25. For the same reasons, note also that the image of $H_\delta^{\infty,\infty'}(1)$ by $\mathcal{R}$ lies in $\left(H_\delta^m \times H_\delta^m\right)(R_\delta)$ where $\lim_{\delta\to 0} R_\delta = 0$. The proof of the following is straightforward:

Lemma **9.3.** — *Let* $\varepsilon > 0$. *Then, for* δ *small enough, one has*

$$\forall \overrightarrow{g}, \overrightarrow{h} \in H_\delta^{\infty,\infty'}(1) \quad \Rightarrow \quad N_m(\mathcal{R}(\overrightarrow{h}) - \mathcal{R}(\overrightarrow{g})) \leq \varepsilon \cdot N'_\infty(\overrightarrow{h} - \overrightarrow{g}).$$

In particular, $\mathcal{R}$ *is continuous (Lipschitz).*

Let $\varepsilon > 0$ such that $\varepsilon C < 1$. Then, the composition $\mathcal{L} \circ \mathcal{R}$ induces a (non linear) contracting map of the complete metric space $H_\delta^{\infty,\infty'}(1)$. The unique fixed point is a solution to the functional equations (9.8), (9.7). This provides a solution of the original functional equations (9.6) and (9.5), taking into account the renormalization indicated in Remark 9.1. By uniqueness, the solution $\overrightarrow{h}_\delta$ attached to δ induces by restriction the solution attached to δ' for $\delta' \leq \delta$.

One can complete this picture by taking into account all the properties required in the statement of Theorem 9.2. This leads to the following list of properties of the solution exhibited above as a fixed point of a non linear operator. We reintroduce the subscript (a, b) (with obvious notations) in order to recall that the choice of δ depends on a fixed arbitrary annulus in the z variable:

Proposition **9.4.** — *Notations as above. Let* $\widetilde{\Delta}_1, \Delta_2 = O(\xi^{-N})$ *two germs of holomorphic functions in the neighborhood of* $\tilde{C} \subset \mathbf{C}^2$ *with* $N \geq 4$ *(as defined from the conjugation equation introduced in Section 9.1). Let* $3 \leq m < N$. *Let* $0 < a < b < +\infty$ *such that* $a < |q|b$. *Then there exists* $\delta_{(a,b)} > 0$ *such that for every* $0 < \delta \leq \delta_{(a,b)}$, *the system of equations (9.7), (9.8) admits a unique solution* $(\tilde{g}_{\delta,a,b}, h_{\delta,a,b}) \in H_{\delta,a,b}^{\infty,\infty'}(1) \times H_{\delta,a,b}^{\infty,\infty'}(1)$. *Moreover,*

- $(\tilde{g}_{\delta',a,b}, h_{\delta',a,b})$ *is the restriction of* $(\tilde{g}_{\delta,a,b}, h_{\delta,a,b})$ *if* $0 < \delta' \leq \delta \leq \delta_{(a,b)}$.
- $\lim_{\delta\to 0} N'_\infty(\tilde{g}_{\delta,a,b}, h_{\delta,a,b}) = 0$.
- *there exists a positive number* $D = D(\theta)$ *depending only on* $\theta \in]0, \frac{\pi}{2}]$ *such that for every* $\delta \leq \delta_{(a,b)}$ *and every* $(z, \xi) \in S'_{\delta,a,b} \cap \{\theta - \varpi \leq \arg \xi \leq \pi - \theta - \varpi\}$, *one has*

$$|\tilde{g}_{\delta,a,b}(z, \xi)| \leq \frac{D}{|\xi|^{m-2}} \quad \textit{and} \quad |h_{\delta,a,b}(z, \xi)| \leq \frac{D}{|\xi|^{m-2}}.$$

9.4. *Asymptotic expansion.* — Notations as above. We start by fixing $N \geq 4$ and a, b as before. Let us denote by $(\tilde{g}_{a,b}, h_{a,b})$ the germ of sectorial solution induced by $(\tilde{g}_{\delta,a,b}, h_{\delta,a,b})$ by taking $\delta \to 0$. Let a', b' be positive real numbers such that $a' \leq a < b \leq b'$. By Proposition 9.4, note that the unique solution of (9.7), (9.8) lying in $H_{\delta,a',b'}^{\infty,\infty'}(1) \times H_{\delta,a',b'}^{\infty,\infty'}(1)$ induces by restriction the unique solution of the same functional equation in $H_{\delta,a,b}^{\infty,\infty'}(1) \times H_{\delta,a,b}^{\infty,\infty'}(1)$. In particular, $(\tilde{g}_{\delta,a,b}, h_{\delta,a,b})$ is the restriction of $(\tilde{g}_{\delta,a',b'}, h_{\delta,a',b'})$. Then, if one takes projective limit with respect to $a \to 0$, $b \to +\infty$ and exploits the last asymptotic estimate in the Proposition 9.4, one get a solution $(\tilde{g}, h)$ of (9.7) and (9.8) well defined as a flat element of $\mathcal{A}^{m-3}(\mathrm{I}_1) \times \mathcal{A}^{m-3}(\mathrm{I}_1)$. Set $\Psi = \mathrm{Id} + (g, h)$ with $g = z\tilde{g}$ so that, by virtue of 9.1,

$$\mathrm{F} \circ \Psi = \Psi \circ \mathrm{F}_{1,0,0}$$

where equality takes place in $\mathcal{O}(\mathrm{I}_1)$ componentwise.

Now, consider an integer $p \gg \mathrm{N}$ arbitrarily large. Let $k \geq p$ be a positive integer and consider the truncation (or k-jet) $\mathrm{J}^k\hat{\Psi}$ of $\hat{\Psi}$ at order k:

$$\mathrm{J}^k\hat{\Psi}(z, \xi) = \left(z + \sum_{n=1}^{k} a_n \xi^{-n}\ ,\ \xi + \sum_{n=1}^{k} b_n \xi^{-n}\right).$$

If k is large enough, then one has

$$(\mathrm{J}^k\hat{\Psi})^{-1} \circ \mathrm{F} \circ \mathrm{J}^k\hat{\Psi}\ (\xi, z) = \left(qz + \Delta_1^k(z, \xi), \xi - 1 + \Delta_2^k(z, \xi)\right)$$

where $\Delta_i^k(z, \xi) = \mathrm{O}(\xi^{-p})$. Replacing F by the analytic conjugate $(\mathrm{J}^k\hat{\Psi})^{-1} \circ \mathrm{F} \circ \mathrm{J}^k\hat{\Psi}$, one can apply Proposition 9.4 to get existence and uniqueness of $\tilde{g}_\delta^k, h_\delta^k \in \mathrm{H}_{\delta,a,b}^{\infty,\infty\prime}(1)$ (δ small enough) such that

$$\mathrm{F} \circ \mathrm{J}^k\hat{\Psi} \circ \Psi^k = \mathrm{J}^k\hat{\Psi} \circ \Psi^k \circ \mathrm{F}_{1,0,0}$$

with $\Psi^k = \mathrm{Id} + (g_\delta^k, h_\delta^k)$, where $g_\delta^k(z, \xi) = z\tilde{g}_\delta^k(z, \xi)$. As before, one can invoke uniqueness and restrictions considerations to infer that these solutions are in fact induced by a flat element of $\mathcal{A}^{p-4}(\mathrm{I}_1) \times \mathcal{A}^{p-4}(\mathrm{I}_1)$ and that $\Psi = \mathrm{J}^k\hat{\Psi} \circ \Psi^k$. Thus,

$$\Psi = \left(z + \sum_{n=1}^{p-4} a_n \xi^{-n}\ ,\ \xi + \sum_{n=1}^{p-4} b_n \xi^{-n}\right) + \mathrm{R}_k$$

where $\mathrm{R}_k \in \mathcal{A}^{p-4}(\mathrm{I}_1) \times \mathcal{A}^{p-4}(\mathrm{I}_1)$ is flat componentwise. As p can be chosen arbitrarily large, we eventually get that $\Psi \in \mathcal{G}^1(\mathrm{I}_1)$ and admits $\hat{\Psi}$ as asymptotic expansion. We have thus obtain the sought normalization $\Psi_1 := \Psi$ on the germ of sector of opening I_1.

9.5. *Solving the linearized equation.* — The goal of this section is to prove Theorem 9.2.

Consider the foliation defined by the level sets $\{\mathrm{Y} = c\}$ of $\mathrm{Y} = ze^{2i\pi\tau\xi}$. For every complex number c, $0 < |c| < \delta$, consider

$$\mathrm{S}_{a,b,c} := \{\mathrm{Y} = c\} \cap \mathrm{S}_{a,b,\delta} = \{(z, \xi) = (ce^{-2i\pi\tau\xi}, \xi)\ ;\ \xi \in \Sigma_{a,b,c}\}$$

where

$$\Sigma_{a,b,c} = \{\xi \in \mathbf{C} : (\log|c| - \log b) \leq \Re(2i\pi\tau\xi) \leq (\log|c| - \log a)\}.$$

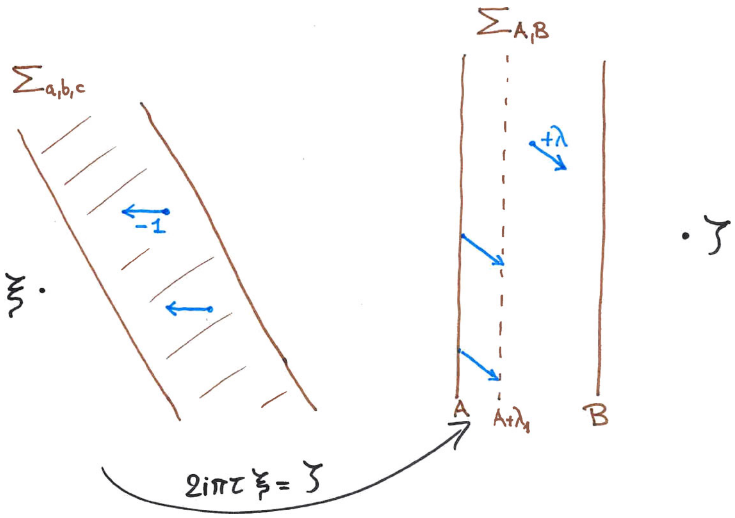

FIG. 6. — Strips Σ from ξ-plane to ζ-plane

Note that $\bigcup_{0<|c|<\delta} \mathrm{S}_{a,b,c} = \mathrm{S}_{a,b,\delta}$ and the linearized equation has a simple form restricted to these slices. To simplify the presentation, we introduce the *notation* (see Fig. 6)

$$\zeta = 2i\pi\tau\xi \quad \text{and} \quad \lambda := -2i\pi\tau \quad \text{with } \lambda = \lambda_1 + i\lambda_2,\ \lambda_1, \lambda_2 \in \mathbf{R}$$

and note that the real part $\lambda_1 > 0$. In particular, the linearized/homological equation

$$h_\delta \circ \mathrm{F}_{1,0,0} - h_\delta = \Delta_\delta$$

can be then rewritten as

$$\varphi_c(\zeta + \lambda) - \varphi_c(\zeta) = \Delta_c(\zeta) \tag{9.9}$$

where $\varphi_c(\zeta) = h_\delta(ce^{-\zeta}, -\frac{\zeta}{\lambda})$, and $\Delta_c(\zeta) = \Delta_\delta(ce^{-\zeta}, -\frac{\zeta}{\lambda})$. We are then led to solve the family of difference equations (9.9) with respect to the parameter c in the vertical strip

$$\Sigma_{a,b,c} = \{\zeta \in \mathbf{C} : \log|c| - \log b \leq \Re(\zeta) \leq \log|c| - \log a\}$$

where we impose φ_c to be holomorphic, defined on the larger strip

$$\Sigma_{|q|a,b,c} = \{\zeta \in \mathbf{C} : \log|c| - \log b \leq \Re(\zeta) \leq \log|c| - \log a + \lambda_1\}$$

and to depend analytically on the parameter c in order to recover a holomorphic solution to (1) in Theorem 9.2.

9.5.1. *Resolution of a difference equation.* — We now proceed to the construction of φ_c. It is essentially a consequence of the following general result (with additional estimates).

Theorem **9.5.** — *Let* $A < A + \lambda_1 < B \leq -1 - \lambda_1$, *and let* Δ *be holomorphic on the strip*

$$\Sigma_{A,B} = \{\zeta \ ; \ A \leq \Re\zeta \leq B\}.$$

Suppose moreover that, for some $m \geq 3$, *we have:*

$$\|\Delta\|_m := sup_{\zeta\in\Sigma_{A,B}}|\Delta(\zeta)||\zeta|^m < \infty.$$

Let $\lambda \in \mathbf{C}$ *such that* $\Re\lambda = \lambda_1$.

Then, there exists a bounded holomorphic function φ *on* $\Sigma_{A,B+\lambda_1}$ *which solves*

$$\varphi(\zeta+\lambda) - \varphi(\zeta) = \Delta(\zeta). \tag{9.10}$$

Moreover φ *is unique modulo an additive constant.*

Proof. — First notice that, if φ_1, φ_2 are two *bounded* holomorphic functions solving (9.10), the difference $\varphi_1 - \varphi_2$ extends as a bounded λ-periodic entire function, hence constant. Uniqueness part of Theorem 9.5 is therefore obvious.

Concerning the existence part, let us first observe, by Cauchy formula, that

$$\Delta(\zeta) = \underbrace{\frac{1}{2i\pi}\int_{L_+}\frac{\Delta(t)}{t-\zeta}dt}_{F_0^+(\zeta)} - \underbrace{\frac{1}{2i\pi}\int_{L_-}\frac{\Delta(t)}{t-\zeta}dt}_{F_0^-(\zeta)}, \quad A < \Re\zeta < B \tag{9.11}$$

where $L_- = \{\Re t = A\}$ and $L_+ = \{\Re t = B\}$ are both oriented from bottom to top. Since $\|\Delta\|_m < \infty$, we see that the two integrals are well defined, and holomorphic in ζ. Observe that F_0^- and F_0^+ are respectively defined on the half-planes $A < \Re\zeta$ and $\Re\zeta < B$, and can be extended to the boundary by continuity, likely as Δ, by using equality (9.11). Then we define, for $n \geq 0$

$$\begin{cases} F_n^-(\zeta) := F_0^-(\zeta + n\lambda) & \text{holomorphic on } A - n\lambda_1 \leq \Re\zeta < \infty \\ F_n^+(\zeta) := F_0^+(\zeta - n\lambda) & \text{holomorphic on } -\infty < \Re\zeta \leq B + n\lambda_1 \end{cases} \tag{9.12}$$

The solution φ to (9.10) is therefore given by the following series

$$\varphi(\zeta) := \sum_{n\geq 1} F_n^+(\zeta) + \sum_{n\geq 0} F_n^-(\zeta) \ = \ F_0^-(\zeta) + \sum_{n\geq 1}\underbrace{F_n^+(\zeta) + F_n^-(\zeta)}_{F_n(\zeta)} \tag{9.13}$$

that will be proved to converge uniformly on the large strip $\Sigma_{A,B+\lambda_1}$ in Lemma 9.6. We can already check that it is indeed a solution:

$$\varphi(\zeta+\lambda) = \sum_{n\geq 0} F_n^+(\zeta) + \sum_{n\geq 1} F_n^-(\zeta) = \varphi(\zeta) + \underbrace{F_0^+(\zeta) - F_0^-(\zeta)}_{\Delta(\zeta)}.$$

Therefore, Theorem 9.5 is an immediate consequence of the following Lemma which actually provides further informations. $\square$

Lemma **9.6.** — *For* Δ *like in Theorem 9.5, the series (9.13) is well defined, holomorphic on* $\Sigma_{A,B+\lambda_1}$ *providing a solution of (9.10). Moreover, there exists a positive number* $C = C(B-A)$ *only depending on* $B-A$[27] *such that*

$$\textbf{(9.14)} \qquad \sup_{\zeta\in\Sigma_{A,B+\lambda_1}} |\varphi(\zeta)| \leq C\|\Delta\|_m \int_{L^+} \frac{|dt|}{|t|^m}$$

Proof. — Set $I_m := \int_{L_+} \frac{|dt|}{|t|^m}$. We will also use repeatedly (and without mentioning it) that, for $a<0$, $\int_{\Re t=a} \frac{|dt|}{|t|^m} = |a|^{1-m} \int_{\Re t=1} \frac{|dt|}{|t|^m}$. In particular, since $A < B \leq -1$, we have

$$\int_{L_-} \frac{|dt|}{|t|^m} < \int_{L_+} \frac{|dt|}{|t|^m} = I_m.$$

If $A + \frac{\lambda_1}{2} \leq \Re u$, one has

$$\begin{aligned}\left|\frac{1}{2i\pi}\int_{L_-}\frac{\Delta(t)}{t-u}dt\right| &\leq \frac{1}{2\pi}\int_{L_-}\frac{|\Delta(t)||t|^m}{|t-u|}\frac{|dt|}{|t|^m} \leq \frac{1}{2\pi}\frac{\|\Delta\|_m}{\lambda_1/2}\int_{L_-}\frac{|dt|}{|t|^m}\\ &\leq \frac{\|\Delta\|_m}{\pi\lambda_1} I_m.\end{aligned}$$

If $A \leq \Re u \leq A + \frac{\lambda_1}{2}$, Cauchy's formula yields the inequality

$$\left|\frac{1}{2i\pi}\int_{L_-}\frac{\Delta(t)}{t-u}dt\right| \leq |\Delta(u)| + \left|\frac{1}{2i\pi}\int_{L_+}\frac{\Delta(t)}{t-u}dt\right|$$

and then by reversing the role of L_- and L_+ in the previous estimate,

$$\left|\frac{1}{2i\pi}\int_{L_-}\frac{\Delta(t)}{t-u}dt\right| \leq |\Delta(u)| + \frac{\|\Delta\|_m I_m}{\pi\lambda_1} \leq \|\Delta\|_m + \frac{\|\Delta\|_m I_m}{\pi\lambda_1} \leq C_1\|\Delta\|_m I_m$$

with

$$C_1 = \left(\frac{1}{I_m} + \frac{1}{\pi\lambda_1}\right).$$

Following the same principle, we get the inequality

$$\left|\frac{1}{2i\pi}\int_{L_+}\frac{\Delta(t)}{t-u}dt\right| \leq C_1\|\Delta\|_m I_m$$

where $A \leq \Re u \leq B$. This eventually leads to the upper bound

$$\textbf{(9.15)} \qquad |F_n(\zeta)| \leq 2C_1\|\Delta\|_m I_m \quad \forall n \geq 0,\ \forall \zeta \in \Sigma_{A,B+\lambda_1}.$$

[27] In order to give an unambiguous statement, τ, m are fixed but A, B, Δ are allowed to vary provided they satisfy assumptions of Theorem 9.5.

Moreover, when $n > \frac{B-A}{\lambda_1}$, Cauchy's formula allows to write

$$\int_{L_-} \frac{\Delta(t)}{t-\zeta-n\lambda}dt = \int_{L_+} \frac{\Delta(t)}{t-\zeta-n\lambda}dt.$$

Consequently, for these values of n one has

$$F_n(\zeta) = \frac{1}{2i\pi}\int_{L_+} \frac{2(t-\zeta)\Delta(t)}{(t-\zeta-n\lambda)(t-\zeta+n\lambda)}dt$$

The key point will be to find a suitable upper bound for

$$A(t,\zeta) = \sum_{n\geq n_0}\left|\frac{1}{(t-\zeta-n\lambda)(t-\zeta+n\lambda)}\right|$$

where $n_0 > 2\left(\frac{B-A}{\lambda_1}+1\right)$, $t \in L_+$, $\zeta \in \Sigma_{A,B+\lambda_1}$.

For $n \geq n_0$, Cauchy-Schwarz inequality gives

$$(\mathbf{9.16}) \qquad A(t,\zeta) \leq \left(\sum_{n\geq n_0}\frac{1}{(v-n\lambda_2)^2+(\frac{n}{2}\lambda_1)^2}\right)^{\frac{1}{2}}\left(\sum_{n\geq n_0}\frac{1}{(v+n\lambda_2)^2+(\frac{n}{2}\lambda_1)^2}\right)^{\frac{1}{2}}$$

where $v = \Im(t-\zeta)$ and $\lambda = \lambda_1 + i\lambda_2$. Here, we have used the fact that

$$|\Re(t-\zeta)| \leq B - A + \lambda_1 \leq \frac{n}{2}\lambda_1 \quad \text{whenever } n \geq n_0.$$

Therefore,

$$(\mathbf{9.17}) \qquad A(t,\zeta) \leq \sum_{n\geq n_0}\frac{1}{(\frac{n}{2}\lambda_1)^2} \leq \left(\frac{2}{\lambda_1}\right)^2\sum_{n\geq n_0}\frac{1}{n^2} \leq \left(\frac{2}{\lambda_1}\right)^2\int_{n_0-1}^{+\infty}\frac{dt}{t^2} \leq \left(\frac{2}{\lambda_1}\right)^2.$$

This, together with the fact that $\|\Delta\|_m < \infty$ for some $m \geq 3$, proves immediately that the serie $\sum F_n$ converges uniformly on every compact of $\Sigma_{A,B+\lambda_1}$. The function φ is then well defined and holomorphic on $\Sigma_{A,B+\lambda_1}$.

For $n \geq n_0$, consider the decomposition $F_n = G_n + H_n$ where

$$G_n(\zeta) = \frac{1}{2i\pi}\int_{\{t\in L_+, |v|\leq 1\}} \frac{2(t-\zeta)\Delta(t)}{(t-\zeta-n\lambda)(t-\zeta+n\lambda)}dt$$

and

$$H_n(\zeta) = \frac{1}{2i\pi}\int_{\{t\in L_+, |v|> 1\}} \frac{2(t-\zeta)\Delta(t)}{(t-\zeta-n\lambda)(t-\zeta+n\lambda)}dt.$$

Consequently, when $|v| \leq 1$, we deduce that $|t-\zeta| \leq \sqrt{1+(\mathrm{B}-\mathrm{A})^2}$ so that, by (9.17), we find $\sum_{n\geq n_0} |\mathrm{G}_n(\zeta)| \leq \mathrm{C}_2 \|\Delta\|_m \mathrm{I}_m$ where

$$\mathrm{C}_2 = \left(\frac{2}{\lambda_1}\right)^2 \frac{\sqrt{1+(\mathrm{B}-\mathrm{A})^2}}{\pi}.$$

In order to achieve the proof of Lemma 9.6, one needs a little bit more analysis to estimate $\sum_{n\geq n_0} |\mathrm{H}_n(\zeta)|$ (and justify a posteriori the choice of $|v|>1$). This also relies on (9.16), firstly noticing that the first factor in the right-hand-side can be rewritten as

$$\left(\sum_{n\geq n_0} \frac{1}{(v-n\lambda_2)^2+(\frac{n}{2}\lambda_1)^2}\right)^{\frac{1}{2}} = \frac{1}{c_1|v|}\left(\sum_{n\geq n_0} \frac{1}{(c_2\frac{n}{v}-c_3)^2+1}\right)^{\frac{1}{2}}$$

with constants $c_1, c_2 \in \mathbf{R}_{>0}$ and $c_3 \in \mathbf{R}$ depending only on λ_1, λ_2. The map $f(t) = \frac{1}{(c_2\frac{t}{v}-c_3)^2+1}$ reaches its maximum at $t_0 = \frac{c_3 v}{c_2}$ and $f(t_0)=1$. Moreover f is increasing on $(-\infty, t_0]$ and decreasing on $[t_0, +\infty)$. Integral comparison yields

$$\frac{1}{(c_2\frac{n}{v}-c_3)^2+1} \leq \int_n^{n+1} f(t)dt \quad \text{if } n \leq t_0 - 1$$

and

$$\frac{1}{(c_2\frac{n}{v}-c_3)^2+1} \leq \int_{n-1}^{n} f(t)dt \quad \text{if } n \geq t_0 + 1.$$

By summation and the explicit calculation

$$\int_{\mathbf{R}} f(t)dt = \frac{\pi|v|}{c_2^2}$$

one gets the upper bound

$$\left(\sum_{n\geq n_0} \frac{1}{(v-n\lambda_2)^2+(\frac{n}{2}\lambda_1)^2}\right)^{\frac{1}{2}} \leq \frac{1}{c_1|v|}\left(\frac{\pi|v|}{c_2^2}+2\right)^{\frac{1}{2}} \leq \frac{\sqrt{\pi+2c_2^2}}{c_1c_2\sqrt{|v|}} \tag{9.18}$$

where the right hand side follows directly from $|v|>1$. By exactly the same computation, we get the same bound for the second factor of $\mathrm{A}(t,\zeta)$:

$$\left(\sum_{n\geq n_0} \frac{1}{(v+n\lambda_2)^2+(\frac{n}{2}\lambda_1)^2}\right)^{\frac{1}{2}} \leq \frac{\sqrt{\pi+2c_2^2}}{c_1c_2\sqrt{|v|}}$$

Therefore, using that $\frac{|t-\zeta|}{|v|} = \sqrt{\frac{(B-A)^2}{|v|^2}+1} \leq \sqrt{(B-A)^2+1}$ for $|v| > 1$, we conclude that

$$\sum_{n\geq n_0} |H_n(\zeta)| \leq \underbrace{\frac{(\pi + 2c_2^2)(\sqrt{(B-A)^2+1}}{\pi c_1^2 c_2^2}}_{C_3} \|\Delta\|_m I_m$$

for a constant C_3 depending only on $B-A$. By putting together the inequalities above involving C_1, C_2, C_3, one obtains that φ satisfies the estimate (9.14) of Lemma 9.6, thus proving Theorem 9.5. □

9.5.2. *Choice of a canonical solution.* — In this section, we are going to replace the solution $\varphi(\zeta) = F_0^-(\zeta) + \sum_{n\geq 1} F_n(\zeta)$ constructed in the proof of Theorem 9.5 by another one $\psi(\zeta) = \varphi(\zeta) + m_0$ where m_0 is a constant, so that $\psi \to 0$ while $\Im(\zeta) \to -\infty$. As we shall see, the right constant is

$$m_0 = \frac{1}{2\lambda}\int_{L_-} \Delta(t)dt = \frac{1}{2\lambda}\int_{L_+} \Delta(t)dt.$$

Indeed, we have:

Lemma **9.7.** — *Keeping notations as in the proof of Theorem 9.5, set*

$$\psi(\zeta) = \underbrace{F_0^-(\zeta) + \sum_{n\geq 1} F_n(\zeta)}_{\varphi(\zeta)} + \underbrace{\frac{1}{2\lambda}\int_{L_+} \Delta(t)dt}_{m_0}.$$

Then, ψ is the unique solution to the difference equation (9.10) such that, for every $\zeta \in \Sigma_{A,B+\lambda_1}$ satisfying $\Im(\frac{\zeta}{\lambda}) \leq -1$ (i.e. $\Im(\xi) \geq 1$), one has

$$|\psi(\zeta)| \leq \frac{C\|\Delta\|_m}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}, \tag{9.19}$$

for some constant C.

Moreover, for every $\theta \in]0, \frac{\pi}{2}]$, there exists a real number $D = D(\theta, B-A) > 0$ such that for $\zeta \in K_\theta := \Sigma_{A,B+\lambda_1} \cap \{\frac{\pi}{2} + \theta \leq \arg\zeta \leq \frac{3\pi}{2} - \theta\}$, we have

$$|\psi(\zeta)| \leq \left(\frac{A}{B}\right)^{m-2} \frac{D\|\Delta\|_m}{|\zeta|^{m-2}}. \tag{9.20}$$

The first condition (9.19) insures that the solution $\psi \to 0$ at least when $\Im(\frac{\zeta}{\lambda}) \to -\infty$ (or equivalently $\Im(\zeta) \to -\infty$ in the strip $S_{A,B+\lambda_1}$). The second condition is used to prove existence of asymptotic expansions of the sectorial normalization.

Proof. — Let $\zeta \in \Sigma_{A,B+\lambda_1}$ such that $\Im(\frac{\zeta}{\lambda}) < 0$, so that $\zeta \notin \lambda\mathbf{Z}$. Write

$$\begin{aligned}
F_n(\zeta) = \frac{1}{2i\pi}\Bigg[\Bigg(\int_{L_+}\frac{\Delta(t)}{t-\zeta+n\lambda}dt + \int_{L_+}\frac{\Delta(t)}{\zeta-n\lambda}dt\Bigg) \\
+\Bigg(\int_{L_-}\frac{\Delta(t)}{t-\zeta-n\lambda}dt + \int_{L_-}\frac{\Delta(t)}{\zeta+n\lambda}dt\Bigg) \\
-\int_{L_+}\frac{\Delta(t)}{\zeta-n\lambda}dt - \int_{L_-}\frac{\Delta(t)}{\zeta+n\lambda}dt\Bigg]
\end{aligned}$$

Now using that $\int_{L_-}\Delta(t)dt = \int_{L_+}\Delta(t)dt$ and

$$\sum_{n\in\mathbf{Z}}\frac{1}{\zeta+n\lambda} = \frac{\pi}{\lambda}\cot\left(\pi\frac{\zeta}{\lambda}\right),$$

and summing-up the previous equalities, one obtains

$$\begin{aligned}
\varphi(\zeta) = \frac{1}{2i\pi}\sum_{n\geq 1}\int_{L_+}\frac{t\Delta(t)}{(\zeta-n\lambda)(t-\zeta+n\lambda)}dt \\
+\frac{1}{2i\pi}\sum_{n\geq 0}\int_{L_-}\frac{t\Delta(t)}{(\zeta+n\lambda)(t-\zeta-n\lambda)}dt \\
-\frac{\pi}{2i\pi\lambda}\cot(\pi\frac{\zeta}{\lambda})\int_{L_+}\Delta(t)dt
\end{aligned}$$

which makes sense whenever $\Im(\frac{\zeta}{\lambda}) \neq 0$.

For $n=0$, one can make use of Cauchy's formula as in the proof of Lemma 9.6 (replacing $\Delta(t)$ by $t\Delta(t)$), which leads to

$$\left|\frac{1}{2i\pi}\int_{L_-}\frac{t\Delta(t)}{\zeta(t-\zeta)}dt\right| \leq C_1\frac{\|\Delta\|_m I_{m-1}}{|\Im(\frac{\zeta}{\lambda})|} \leq C_1\frac{\|\Delta\|_m I_{m-1}}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

with

$$C_1 = \frac{1}{|\lambda|}\left(\frac{1}{I_{m-1}} + \frac{1}{\pi\lambda_1}\right).$$

Similarly, one obtains for $n=1$

$$\left|\frac{1}{2i\pi}\int_{L_+}\frac{t\Delta(t)}{(\zeta-\lambda)(t-\zeta+\lambda)}dt\right| \leq C_1\frac{\|\Delta\|_m I_{m-1}}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

(here the discussion depends on whether $\Re\zeta \leq B + \frac{\lambda_1}{2}$ or not).

Now, we proceed to estimate the remaining terms in the serie expansion giving φ. To this end, define for $(t,\zeta) \in L_{\pm} \times \Sigma_{A,B+\lambda_1}$:

$$B_-(t,\zeta) = \sum_{n\geq 1} \left| \frac{1}{(\zeta + n\lambda)(t - \zeta - n\lambda)} \right|,$$

$$B_+(t,\zeta) = \sum_{n\geq 2} \left| \frac{1}{(\zeta + n\lambda)(t - \zeta + n\lambda)} \right|.$$

Here, proceeding as for proving (9.16) and (9.17), Cauchy-Schwarz inequality gives

$$B_-(t,\zeta) \leq \left(\sum_{n\geq 1} \frac{1}{|\zeta + n\lambda|^2} \right)^{1/2} \left(\sum_{n\geq 1} \frac{1}{|t - \zeta - n\lambda|^2} \right)^{1/2}$$

$$B_+(t,\zeta) \leq \left(\sum_{n\geq 2} \frac{1}{|\zeta - n\lambda|^2} \right)^{1/2} \left(\sum_{n\geq 2} \frac{1}{|t - \zeta + n\lambda|^2} \right)^{1/2}$$

By doing the kind of integral comparison already performed in (9.18),

$$\begin{aligned} \sum_{n\geq 1} \frac{1}{|\zeta \pm n\lambda|^2} &\leq \frac{1}{|\lambda|^2} \left(\int_{\mathbf{R}} \frac{dt}{\left(\pm t + \Re(\frac{\zeta}{\lambda})\right)^2 + |\Im(\frac{\zeta}{\lambda})|^2} + \frac{2}{\Im(\frac{\zeta}{\lambda})^2} \right) \\ &\leq \frac{\pi + 2}{|\lambda|^2 |\Im(\frac{\zeta}{\lambda})|} \end{aligned}$$

On the other hand,

$$\left(\sum_{n\geq 1} \frac{1}{|t - \zeta - n\lambda|^2} \right) \leq \sum_{n\geq 1} \frac{1}{(\lambda_1 n)^2} \quad \text{for } t \in L_-$$

$$\left(\sum_{n\geq 2} \frac{1}{|t - \zeta + n\lambda|^2} \right) \leq \sum_{n\geq 1} \frac{1}{(\lambda_1 (n-1))^2} \quad \text{for } t \in L^+$$

If we combine the previous upper-bounds, one can conclude that there exists a positive constant C_2 such that for every $\zeta \in \Sigma_{A,B+\lambda_1}$ satisfying $\Im(\frac{\zeta}{\lambda}) \leq -1$, we get:

$$\left| \frac{1}{2i\pi} \sum_{n\geq 0} \int_{L_-} \frac{t\Delta(t)}{(\zeta + n\lambda)(t - \zeta - n\lambda)} dt \right| \leq \frac{C_2 \|\Delta\|_m}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

$$\left|\frac{1}{2i\pi}\sum_{n\geq 1}\int_{\mathrm{L}_+}\frac{t\Delta(t)}{(\zeta-n\lambda)(t-\zeta+n\lambda)}dt\right| \leq \frac{\mathrm{C}_2\|\Delta\|_m}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

Finally, for $\Im(\frac{\zeta}{\lambda}) \leq -1$, a straightforward calculation gives

$$\left|\frac{\pi}{2i\pi\lambda}\cot(\pi\frac{\zeta}{\lambda}) - \frac{1}{2\lambda}\right| \leq \frac{e^{2\pi\Im(\frac{\zeta}{\lambda})}}{|\lambda|(1-e^{-2\pi})} \leq \frac{\mathrm{C}_4}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

for some constant $\mathrm{C}_4 > 0$. These different estimates prove (9.19).

Now, let us establish the upper-bound (9.20). For this end, observe that, on K_θ, we have

$$|\mathrm{B}+\lambda_1| \leq |\zeta| \leq \frac{|\mathrm{A}|}{\sin\theta}$$

so that we have in particular:

$$|\frac{t}{\zeta}| \geq \frac{\mathrm{B}\sin\theta}{\mathrm{A}} \quad \text{for every } t\in\mathrm{L}^+.$$

From Lemma 9.6, one deduces that

$$\sup_{\zeta\in\mathrm{K}_\theta}|\varphi(\zeta)| \leq \frac{\mathrm{C}\|\Delta\|_m}{|\zeta|^{m-2}}\left(\frac{\mathrm{A}}{\mathrm{B}\sin\theta}\right)^{m-2}\int_{\mathrm{L}^+}\frac{|dt|}{|t|^2}$$

where $\mathrm{C}=\mathrm{C}(\mathrm{B}-\mathrm{A})$ is the constant appearing in *loc.cit.* Moreover one gets trivially that

$$|m_0| \leq \frac{\|\Delta\|_m}{2|\lambda|}\int_{\mathrm{L}^+}\frac{|dt|}{|t|^m}$$

Consequently $\mathrm{D} := (\frac{\mathrm{C}}{(\sin\theta)^{m-2}} + \frac{1}{2|\lambda|})\int_{\Re t=1}\frac{|dt|}{|t|^2}$ satisfies (9.20). □

9.5.3. *Version with parameters and end of the proof of Theorem 9.2.* — We resume to notations introduced in Section 9.2 and the beginning of Section 9.5. We have fixed τ with $\Im\tau > 0$ and set

$$\lambda = -2i\pi\tau = \underbrace{\lambda_1}_{>0} + i\lambda_2.$$

Consider $0 < a < b$ such that $a < |q|b$ or equivalently $\log a + \lambda_1 < \log b$, and let $\delta_{(a,b)} > 0$ small enough. For every positive number $\delta \leq \delta_{(a,b)}$ and complex number c such that $0 < |c| \leq \delta$, consider the vertical strip defined in the complex line by

$$\Sigma_{a,b,c} = \{\zeta\in\mathbf{C} : \underbrace{\log|c| - \log b}_{\mathrm{A}_c} \leq \Re(\zeta) \leq \underbrace{\log|c| - \log a}_{\mathrm{B}_c}\}.$$

Consider also the subset of $\mathbf{C}^2$

$$\mathscr{S}_{a,b,\delta} = \bigcup_{|c|\leq\delta} \{c\} \times \Sigma_{a,b,c}.$$

It is worth mentioning that $\mathrm{B}_c - \mathrm{A}_c = \log b - \log a > \lambda_1 = \Re\lambda$ does not depend on c. By a suitable choice of $\delta_{(a,b)}$, one can moreover assume that $\mathrm{B}_c \leq -1 - \lambda_1$ and $\frac{\mathrm{A}_c}{\mathrm{B}_c} \leq 2$. Let $\Delta(c, \zeta)$ be a holomorphic function defined on $\mathscr{S}_{a,b,\delta}$ and assume in addition that

$$\|\Delta\|_m := \sup_{\mathscr{S}_{a,b,\delta}} |\Delta(c,\zeta)||\zeta|^m < \infty$$

for some $m \geq 3$. From the results collected in Sections 9.5.1, 9.5.2, one promptly obtains the following statement:

Proposition **9.8.** — *Let δ small enough and $\Delta \in \mathcal{H}(\mathscr{S}_{a,b,\delta})$ with $\|\Delta\|_m < \infty$, $m \geq 3$. Then, there exists a unique function $\varphi \in \mathcal{H}(\mathscr{S}_{|q|a,b,\delta})$ with the following properties*

1. *For every $(c,\zeta) \in \mathscr{S}_{a,b,\delta}$, one has $\varphi(c, \zeta + \lambda) - \varphi(c,\zeta) = \Delta(c,\zeta)$.*
2. *There exists a positive number $\mathrm{C} = \mathrm{C}(a, b)$ such that for all $0 < \delta \leq \delta_{(a,b)}$,*

$$\sup_{\mathscr{S}_{|q|a,b,\delta}} |\varphi(c,\zeta)| \leq \mathrm{C}\|\Delta\|_m$$

$$\sup_{\mathscr{S}_{|q|a,b,\delta} \cap \{\Im(\frac{\zeta}{\lambda}) \leq -1\}} |\varphi(c,\zeta)| \leq \frac{\mathrm{C}\|\Delta\|_m}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}.$$

Proof. — Define φ_δ by the formula (compare Lemma 9.7)

$$\begin{aligned} \varphi_\delta(c,\zeta) := & \underbrace{\frac{1}{2i\pi}\int_{\mathrm{L}^c_-} \frac{\Delta_\delta(c,t)}{t-\zeta}dt}_{\mathrm{F}_0^-(c,\zeta)} \\ & + \sum_{n=1}^{\infty} \underbrace{\frac{1}{2i\pi}\Big(\int_{\mathrm{L}^c_+} \frac{\Delta_\delta(c,t)}{t-\zeta+n\lambda}dt + \int_{\mathrm{L}^c_-} \frac{\Delta_\delta(c,t)}{t-\zeta-n\lambda}dt\Big)}_{\mathrm{F}_n(c,\zeta)} \\ & + \underbrace{\frac{1}{2\lambda}\int_{\mathrm{L}^c_+} \Delta_\delta(c,t)dt}_{m_0(c)} \end{aligned}$$

setting $\mathrm{L}^c_- = \{\Re t = \mathrm{A}_c\}$, $\mathrm{L}^c_+ = \{\Re t = \mathrm{B}_c\}$. From this integral formula, it is clear that F_0^- and F_n are well defined as holomorphic functions on $\mathscr{S}_{|q|a,b,\delta}$ and that m_0 depends analytically on c. Moreover, one can easily verify (as in Section 9.5.1) that the series $\sum \mathrm{F}_n$

converges uniformly on every compact subset of $\mathscr{S}_{|q|a,b,\delta}$. Thus $\varphi \in \mathcal{H}(\mathscr{S}_{|q|a,b,\delta})$ and fulfills the properties stated in the Proposition as a direct application of the construction performed in Sections 9.5.1 and 9.5.2. $\square$

Then, the proof of Theorem 9.2 immediately follows when translating this existence and uniqueness result into the original variable (z,ξ), with $c = ze^{-2i\pi\tau\xi}$ and $\zeta = 2i\pi\tau\xi$ (see Section 9.5) together with the upper bound (9.20) of Lemma 9.7. $\square$

Recall that once we have solutions to the linearized equation as achieved above, we are in position to obtain a solution to the general functional equations (9.8) and (9.7) by a standard fixed point Theorem as developed in Section 9.3. *This eventually finishes (at least on sectors of opening* I_1*) the proof of Lemma A* (and more precisely Lemma 4.1). One can construct others normalizing conjugacy biholomorphisms on the remaining sectors, namely I_2, I_3, I_4 without further real complications. This is explained below.

9.6. *Construction of other sectorial normalizations.*

9.6.1. *On* I_3. — It is just a slight modification of the previous construction for I_1. The starting point consists again in the resolution of the linearized equation (9.6) on the corresponding domain. In this situation, it is relevant to deal with the levels $\mathrm{Y} = c$ of the first integral $\mathrm{Y} = ze^{2i\pi\tau\xi}$ for $|c| \gg 0$. One adapts the notation of Section 9.2 by defining $\mathrm{S}_{a,b,\delta} = \{(z,\xi) \in \mathbf{C}^2 |\ |\mathrm{Y}(z,\xi)| > \delta \text{ and } z \in \mathcal{C}_{a,b}\}$ $(\delta \gg 0)$. Here again, this amounts to solve an analytic family of difference equations where the solutions satisfy some estimates which eventually leads by the same fixed point consideration to a normalizing conjugation map. This can be carried out following *verbatim* the same method.

9.6.2. *On* I_2 *and* I_4. — Here, we can use the $\mathrm{SL}_2(\mathbf{Z})$ action described in Section 1.6 to deduce existence of sectorial normalization on remaining sectors. Indeed, if we consider the cyclic covering determined by $\mathrm{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (see Section 1.6) then we are led to an alternate presentation of (U,C). Our model $(\mathrm{U}_{1,0,0},\mathrm{C})$ is now viewed as the quotient of $(\mathbf{C}^*_{z'} \times \mathbf{C}_{\xi'}, \{\xi' = \infty\})$ by the semi-hyperbolic transformation

$$\mathrm{F}'_{1,0,0}(z',\xi') = (q'z', \xi' - 1) \quad \text{where} \quad \tau' = \frac{-1}{\tau}, \quad q' = e^{-\frac{2i\pi}{\tau}}$$
$$\text{and} \quad z' = z^{\frac{1}{\tau}}, \quad \xi' = \tau\xi + \frac{\log(z)}{2i\pi}.$$

In these coordinates (U,C) is isomorphic to a quotient of the form $(\mathbf{C}^*_{z'} \times \mathbf{C}_{\xi'}\ \{\xi' = \infty\})$ by a biholomorphism F' such that $\mathrm{F}'(z',\infty) = (z',\infty)$. We can then apply the results of previous sections and obtain an analytic sectorial conjugation

$$\mathrm{F}' \circ \Psi' = \Psi' \circ \mathrm{F}'_{1,0,0}$$

defined on sectorial domains of the shape

$$\arg(\xi') \in \mathrm{I}_1' =]\varpi - \pi, \varpi[\quad \text{and} \quad \arg(\xi') \in \mathrm{I}_3' = \mathrm{I}_1' + \pi,$$

$\varpi' = \arg(\tau') = \pi - \varpi$. Moreover, Ψ' admits an asymptotic expansion $\hat{\Psi}'$ solving the same conjugacy equation. Finally, going back the (z, ξ) coordinates, we get that Ψ' provides the sought sectorial conjugation on

$$\arg(\xi) \in \mathrm{I}_1' - \varpi =] - \pi, 0[= \mathrm{I}_2 \quad \text{and} \quad \arg(\xi) \in \mathrm{I}_3' - \varpi = \mathrm{I}_4$$

taking into account that $\arg(\xi') \sim \varpi + \arg(\xi)$ asymptotically. Maybe composing Ψ' by an automorphism of our model $(\mathrm{U}_{1,0,0}, \mathrm{C})$ (see Section 3.3), we can assume that the asymptotic expansion fits with that of previous sectorial normalizations: $\hat{\Psi}' = \hat{\Psi}$.

10. Generalization to the case of trivial normal bundle

10.1. *General formal classification in the torsion case.* — We first recall here the results obtained in [15] under a slightly more synthesized form including the case of torsion normal bundle and finite Ueda type. As already mentioned in Section 1.2, we do not address the case where C fits into a formal fibration (corresponding to infinite Ueda type). According to a result due to Ueda, this fibration is indeed analytic and we fall into the case which satisfies the formal principal: there is no differences between analytic and formal classification and this latter is very easy to describe, see for instance [15, Section 5.1].

Let m be the torsion order of the normal bundle $\mathrm{N_C}$ and let $k > 0$ be the Ueda type. Recall that this latter is necessary a multiple of m. The linear monodromy of the corresponding unitary connection along the loops 1 and τ is respectively determined by two roots of unity a_1, a_τ of respective orders m_1 and m_τ such that $lcm(m_1, m_\tau) = m$. The triple (a_1, a_τ, k) is obviously a formal invariant of the neighborhood. A complete set of invariants is provided in [15, Section 5.2], from which we borrow and adapt the notation. In particular, there exists on each normal form described below a pencil of regular foliations $(\mathcal{F}_t)_{t\in\mathbf{P}^1}$ fulfilling one of the following properties:

1. C is $\mathcal{F}_t$ invariant for $t \in \mathbf{C}$,
2. $\mathcal{F}_\infty$ is either tangent to C, either totally transverse to C .

Conversely every formal foliation satisfying one of these two properties fits into this pencil and is in particular convergent. Moreover, this family of foliations can be defined by a pencil of closed meromorphic forms $\omega_t = \omega_0 + t\omega_\infty$ whose expression is recalled below according to the value of the parameters characterizing the normal forms that we proceed to describe now.

Set $\varphi_{k,\nu} = \exp\left(\frac{y^{k+1}}{1+\nu y^k}\partial_y\right)$, and given $\mathrm{P}(z) = \sum_{i=0}^{k'-1} \lambda_i z^i$ a polynomial, $k = mk'$, define

$$\omega_{\mathrm{P}} := \mathrm{P}\left(\frac{1}{y^m}\right)\frac{dy}{y}, \tag{10.1}$$

$$\text{and} \quad g_{k,\nu,\mathrm{P}}(y) := \tau \int_0^y [(a_\tau\varphi_{k,\nu})^*\omega_{\mathrm{P}} - \omega_{\mathrm{P}}].$$

The group $\mathbf{Z}_{k'}$ of k'^{th} roots of unity acts on the set of polynomials P as follows:

$$(\mu, \mathrm{P}(\mathrm{X})) \mapsto \mathrm{P}(\mu\mathrm{X}) \tag{10.2}$$

Theorem **10.1.** — *[15]Notations as above. There exist $\nu \in \mathbf{C}$ and $\mathrm{P} \in \mathbf{C}[z]$ of degree at most $k'-1$, unique up to the $\mathbf{Z}_{k'}$-action (10.2), such that* (U, C) *is formally equivalent*[28] *to the quotient* $\mathrm{U}_{k,\nu,\mathrm{P},a_1,a_\tau}$ *of* $(\mathbf{C}_x \times \mathbf{C}_y, \{y=0\})$ *by the group generated by*

$$\begin{cases} \phi_1(x,y) = (x+1\ ,\ a_1 y) \\ \phi_\tau(x,y) = (x+\tau+g_{k,\nu,\mathrm{P}}(y)\ ,\ a_\tau\varphi_{k,\nu}(y)) \end{cases} \tag{10.3}$$

The pencil $\omega_t = \omega_0 + t\omega_\infty$ of closed 1-forms (and therefore foliation $\mathcal{F}_t$) is generated by

$$-\omega_0 = \frac{dy}{y^{k+1}} + \nu\frac{dy}{y} \quad \textit{and} \quad \omega_\infty = \frac{dx}{\tau} - \omega_{\mathrm{P}}. \tag{10.4}$$

By duality, we get a 2-parameter family of holomorphic vector fields spanned by

$$v_0 = \tau\partial_x \quad \text{and} \quad v_\infty = \frac{y^{k+1}}{1+\nu y^k}\partial_y + \tau\mathrm{P}\left(\frac{1}{y}\right)\frac{y^k}{1+\nu y^k}\partial_x.$$

They provide a 2-parameter group of automorphisms of the neighborhood, preserving the curve C, and the isotropy group of C is generated by v_∞.

Actually (see [15]), the case $\mathrm{P}=0$ (i.e $\omega_{\mathrm{P}}=0$) corresponds exactly to the existence of a (necessarily unique) transverse fibration (Property (2) above), given at the level of the formal normal forms described above by $\omega_\infty = \frac{dx}{\tau}$. If in addition, $k=1$, $\nu=0$ (and necessarily $m=1$) one recovers Serre's example. The case where $\mathrm{P}=\mu$ is the constant polynomial, is covered by [15, Theorem 5.3] whereas $\deg(\mathrm{P}) > 0$ is covered by [15, Theorem 5.4].

When $m=1$, meaning that $\mathrm{N_C}$ analytically trivial, one will denote the corresponding neighborhood simply by $\mathrm{U}_{k,\nu,\mathrm{P}}$. In that case, one easily check that ϕ_τ is the flow at time 1 of the holomorphic vector field

$$v_{k,\nu,\mathrm{P}} = v_0 + v_\infty$$

[28] As noticed in [15], this result is unchanged if one allows formal conjugacy maps inducing translation on C.

where v_0 and v_∞ are as above. Indeed, in the (x,y) coordinates, this diffeomorphism reads

$$\phi_\tau(x,y) = (x(1), y(1))$$

where $t \mapsto x(t)$, $t \mapsto y(t)$ are solutions of the autonomous system

$$\dot{x} = \tau \left(\frac{y^k \mathrm{P}(\frac{1}{y})}{1+\nu y^k} + 1 \right), \quad \dot{y} = \frac{y^{k+1}}{1+\nu y^k}.$$

This yields $y(t) = \exp t v_{k,\nu}(y(0))$ where $v_{k,\nu} = \frac{y^{k+1}}{1+\nu y^k}\partial_y$ and consequently

$$x(1) = x(0) + \tau \left(1 + \int_0^1 \mathrm{P}\left(\frac{1}{y(t)}\right) \frac{\dot{y}(t)}{y(t)} dt \right).$$

By change of variable $t \to y$, the integrand in the expression above also reads

$$\tau \int_0^1 \mathrm{P}\left(\frac{1}{y(t)}\right) \frac{\dot{y}(t)}{y(t)} dt = g_{k,\nu,\mathrm{P}}(y(0))$$

where

$$g_{k,\nu,\mathrm{P}}(y) := \tau \int_0^y \left(\varphi_{k,\nu}^* \omega_{\mathrm{P}} - \omega_{\mathrm{P}}\right)$$

and $\omega_{\mathrm{P}} := \mathrm{P}(\frac{1}{y})\frac{dy}{y}$. This enables to give the explicit form

$$\phi_\tau(x,y) = \left(x + \tau + g_{k,\nu,\mathrm{P}}(y), \varphi_{k,\nu}(y)\right)$$

where $\varphi_{k,\nu} = \exp v_{k,\nu}$. For general torsion neighborhood models $\mathrm{U}_{k,\nu,\mathrm{P},a_1,a_\tau}$, one can notice that $\phi_\tau = \mathrm{A}_\tau \circ \exp v_{k,\nu,\mathrm{P}}$ where $\mathrm{A}_\tau(x,y) = (x, a_\tau y)$.

10.2. *Formal symmetries.* — In order to generalize our approach to the general case with trivial normal bundle, i.e. $m = 1$, it is convenient to view C as a quotient of $\mathbf{C}^*$ by a contraction:

$$\mathrm{C} = \mathbf{C}_z^* / < z \mapsto qz > \quad \text{with } z = e^{2i\pi x} \text{ and } q = e^{2i\pi\tau}, \quad (|q| < 1)$$

and then work in the coordinates $z = e^{2i\pi x}$ and $\xi = 1/y$.

The formal models described before write

$$(\mathrm{U}_{k,\nu,\mathrm{P}}, \mathrm{C}) := (\mathbf{C}_z^* \times \overline{\mathbf{C}}_\xi, \{\xi = \infty\}) / \langle \mathrm{F}_{k,\nu,\mathrm{P}} \rangle$$

where $\mathrm{F}_{k,\nu,\mathrm{P}} = \exp v_{k,\nu,\mathrm{P}}$ and $v_{k,\nu,\mathrm{P}} = v_0 + v_\infty$ with $v_0 = \frac{1}{\xi^k + \nu}(-\xi\partial_\xi + 2i\pi\tau z \mathrm{P}(\xi)\partial_z)$, $v_\infty = 2i\pi\tau z\partial_z$. This can be directly borrowed from the presentation given in the (x,y) variable.

Concerning the structure of the automorphism group $\mathrm{Aut}(\mathrm{U}_{k,\nu,\mathrm{P}}, \mathrm{C})$ of formal automorphisms inducing translations on C, we have the following characterization which generalizes Lemma 3.9.

Lemma **10.2.** — *Any formal automorphism in* $\mathrm{Aut}(\mathrm{U}_{k,\nu,\mathrm{P}}, \mathrm{C})$ *is actually convergent and there exists a positive integer* $n = n(k, \mathrm{P})$ *dividing* k *such that* $\mathrm{Aut}(\mathrm{U}_{k,\nu,\mathrm{P}}, \mathrm{C})$ *is isomorphic to* $\mathbf{Z}/n \times (\mathbf{C}^* \times \mathbf{C}^*)$. *Moreover the Lie algebra associated to the infinitesimal action of the second factor* $\mathbf{C}^* \times \mathbf{C}^*$ *is spanned by* v_0 *and* v_∞.

Proof. — Let E be the **C**-vector space generated by ω_0 and ω_∞. Set $\mathrm{G} = \mathrm{Aut}(\mathrm{U}_{k,\nu,\mathrm{P}}, \mathrm{C})$. This group acts naturally on the pencil of foliations $(\mathcal{F}_t)$ and because there is no constant meromorphic function constant on C (finiteness of the Ueda type), also on $\mathrm{E} - \{0\}$. Moreover, any $g \in \mathrm{G}$ acts trivially on $\mathrm{H}^1(\mathrm{C})$, then for any $\omega \in \mathrm{E} - \{0\}$, $\omega \neq \omega_\infty$, $g^*\omega$ and ω must have the same periods (one refers to [15, Section 2.4] for the notion of periods involved). As the period mapping is injective on $\mathrm{E} - \{0\}$ (see [15, Corollary 2.7]), one concludes that g^* is the identity on E. In particular, according to the writing of ω_0 and ω_∞ and up to composition with the flow of v_∞, one can assume that g restricts to identity on C and then deduce that it takes the form

$$g(x,y) = \left(x + yh(y), a\exp\left(\frac{ty^{k+1}}{1+\nu y^k}\partial_y\right)\right)$$

where

- $a^k = 1$,
- ω_P is invariant by the rotation $y \to ay$ and
- $yh(y) = \tau \int_0^y [\exp(\frac{ty^{k+1}}{1+\nu y^k}\partial_y)^* \omega_\mathrm{P} - \omega_\mathrm{P}]$.

Indeed, from $g_{|\mathrm{C}} = \mathrm{Id}_\mathrm{C}$ and $g^*\omega_0 = \omega_0$, one gets $g(x,y) = \big(x + yg_1(x,y), g_2(y)\big)$ where $g_2 \in \mathrm{Diff}(\mathbf{C}, 0)$ is an element of the centralizer of the flow $\exp tv_0$ at any time $t \in \mathbf{C}$. This gives the sought expression of g_2 (cf. for instance [14][Prop. 2.17]). Now, we can exploit that $g^*\omega_\infty = \omega_\infty$, which forces $g_1(x,y) = h(y)$ to depend only on the y variable and then to satisfy the identity $d(yh(y)) = \tau(g_2^*\omega_\mathrm{P} - \omega_\mathrm{P})$. In particular, the right hand side has to be holomorphic in the neighborhood of $\{y = 0\}$. This is the case only if ω_P is invariant by the rotation $y \to ay$.

Note that, when $a = 1$, g is nothing but $\exp tv_0$. Conversely, the transformation defined by $(x,y) \to (x, ay)$ where a satisfies the conditions above and $\exp tv_0$ are elements of G. Set $n = k$ if P is constant. When P has positive degree, let d_P be the supremum of the positive integers d such that P is invariant under the action of the group of d^{th} roots of unity over the polynomials

$$(\mu, \mathrm{P}(\mathrm{X})) \to \mathrm{P}(\mu\mathrm{X})$$

Set $n = \gcd(k, d_{\mathrm{P}})$. From the remarks above, we observe that for every $g \in \mathrm{G}$, there exists $(l, t) \in \mathbf{Z}/n \times \mathbf{C}$ uniquely defined such that $g = h_n{}^l \circ \exp tv_0$ and $h_n(z, \xi) = (z, e^{\frac{2i\pi}{n}}\xi)$, whence the sought isomorphism. Note that $n = 1$ in general. □

We are now ready to undertake the analytic classification.

10.3. *Trivial normal bundle, finite Ueda type: the fundamental isomorphism.* — Still using the formal classification of [15] as recalled in Section 10.2, one can investigate, without any further fundamental change, the analytic classification of neighborhood of elliptic curves with arbitrary finite Ueda type. For the sake of simplicity, we will first focus on the case where the normal bundle $\mathrm{N_C}$ of C is analytically trivial. The formal normal forms $(\mathrm{U}_{k,\nu,\mathrm{P}}, \mathrm{C})$ are parametrized by the triple (ν, k, P) where ν is a complex number, $k \in \mathbf{N}_{>0}$ is the Ueda type and P is a polynomial map of one indeterminate of degree $< k$; P is uniquely defined modulo a certain action of the k^{th} roots of unity on the coefficients of P as described in Section 10.2.

Following Section 10.2, one has

$$\mathrm{F}_{k,\nu,\mathrm{P}} = \Omega^{-1} \circ \mathrm{F}_{1,0,0} \circ \Omega \quad \text{where } \Omega(z, \xi) = (ze^{\int \frac{2i\pi\tau \mathrm{P}(\xi)d\xi}{\xi}}, \frac{\xi^k}{k} + \nu \log \xi)$$

and consequently

$$\Omega^{-1}(z, \xi) = (ze^{-\mathrm{Q}(\varphi^{-1}(\xi))}, \varphi^{-1}(\xi)) \quad \text{where } \begin{cases} \varphi(\xi) = \frac{\xi^k}{k} + \nu \log \xi, \text{ and} \\ \mathrm{Q}(\xi) = \int \frac{2i\pi\tau \mathrm{P}(\xi)d\xi}{\xi} \end{cases}$$

Note that Ω and Ω^{-1} make sense as univalued functions on sectors (with respect to the ξ variable) of opening $< 2\pi$ and that their expressions depend on the choice of the determination of $\log \xi$. Moreover $\varphi^{-1}(\xi) = k^{\frac{1}{k}}\xi^{\frac{1}{k}} + \mathrm{O}(|\xi|^{\frac{1}{k}-1+\varepsilon})$ for all $\varepsilon > 0$. Note that Ω provides a conjugation between the pencils of foliations respectively attached to $(\mathrm{U}_{k,\nu,\mathrm{P}}, \mathrm{C})$ and $(\mathrm{U}_{1,0,0}, \mathrm{C})$.

Let us choose 4 intervals I_i^l, $i = 1, \ldots, 4 \in \mathbf{Z}_4$, $l = 0, \ldots, k-1 \in \mathbf{Z}_k$ as follows:

$$\mathrm{I}_1^0 =]-\frac{\varpi}{k}, \frac{\pi - \varpi}{k}[, \quad \mathrm{I}_2^0 =]0, \frac{\pi}{k}[, \quad \mathrm{I}_3^0 = \mathrm{I}_1^0 + \frac{\pi}{k}, \quad \mathrm{I}_4^0 = \mathrm{I}_2^0 + \frac{\pi}{k}$$

so that $k\mathrm{I}_i^0 = \mathrm{I}_i$ and $\mathrm{I}_i^0 \cap \mathrm{I}_{i+1}^0 \neq \emptyset$, and for $l \neq 0$, set

$$\mathrm{I}_i^l = \mathrm{I}_i^0 + \frac{2l\pi}{k}.$$

Define (see Fig. 7)

$$\Pi_{k,\nu,\mathrm{P}}(z, \xi) := \Pi \circ \Omega(z, \xi) = \left(e^{\frac{2i\pi\xi^k}{k}}\xi^{2i\pi\nu}, ze^{\mathrm{P}_1(\xi) + 2i\pi\tau\frac{\xi^k}{k}}\xi^{2i\pi\tau(\nu + a_0)}\right)$$

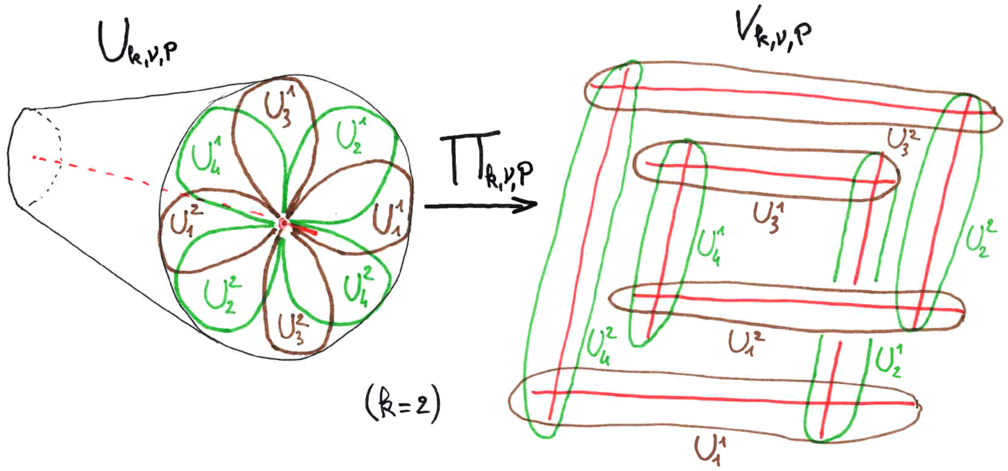

FIG. 7. — The isomorphism $\Pi_{k,\nu,P}$ for $k = 2$

where $a_0 = P(0)$ is the constant term of P and P_1 is a polynomial of degree $< k$ whose exact expression is not relevant in what follows. This transformation $\Pi_{k,\nu,P}$ maps the germ of transversely sectorial domains U_i^l of opening I_i^l (which is $F_{k,\nu,P}$ invariant) onto the germ of deleted neighborhood of L_i. The full picture is thus obtained by taking the successive family of sectors with consecutive overlaps $(I_1^0, \dots, I_4^0, I_1^1, \dots\dots\dots, I_1^{k-1}, \dots., I_4^{k-1})$ and their corresponding system of transversal sectorial neighborhoods. This defines a cover of the deleted neighborhood $U_{k,\nu,P} \setminus C$ equipped with a system of semi-local charts defined by $\Pi_{k,\nu,P}$. If one follows the cyclic order defined by the succession of overlapping sectors, one concludes that this deleted neighborhood can be represented by the union of deleted neighborhoods $V_i^l - L_i^l$ of the components of a cycle $D = \bigcup L_i^l$ of $4k$ rational curves of zero type, namely the image under $\Pi_{k,\nu,P}$ of these sectors. The neighborhood V_i^l of L_i^l is equipped with the standard coordinate system of V_i (neighborhood of L_i in $\mathbf{P}^1 \times \mathbf{P}^1$).

We will also set $V_{i,i+1}^l := V_i^l \cap V_{i+1}^l$ if $i \neq 0 \mod 4$, $V_{4,1}^l := V_4^{l-1} \cap V_1^l$ and consider $p_{i,i+1}^l \in V_{i,i+1}^l$ the crossing points of D.

To summarize we have thus constructed via $\Pi_{k,\nu,P}$ a two dimensional germ of neighborhood $V_{k,\nu,P}$ of D such that each component of D is embedded with zero self intersection and such that

$$V_{k,\nu,P} \setminus D \simeq U_{k,\nu,P} \setminus C.$$

The complex structure of $V_{k,\nu,P}$ is determined by trivial glueings (with respect to the (X, Y) coordinate) along the intersections $V_{i,i+1}^l$ except on $V_{4,1}^0$ where identification is made by the monodromy of $\Pi_{k,\nu,P}$ which is given by the linear diagonal map

$$d_{\nu,P}: \begin{array}{ccc} (V_1^0, p_{4,1}^0) & \to & (V_4^{k-1}, p_{4,1}^0) \\ (X, Y) & \to & (e^{-4\pi^2\nu}X, e^{-4\pi^2\tau(\nu+a_0)}Y) \end{array}$$

Let us denote by $V_{k,\nu,P}$ the neighborhood of four lines obtained by this process. If one specializes this series of observations to $k=1$ and then $P=\mu$ automatically constant, one obtains an isomorphism between the deleted neighborhoods

$$V_{1,\nu,\mu} \setminus D \simeq U_{1,\nu,\mu} \setminus C.$$

In this context, D is a cycle of 4 rational curve, and one recovers the description of Serre's example for the value $(0,0)$ of the parameter (ν,μ).

Remark **10.3.** — On $V_{k,\nu,P}$, the vector fields induced respectively by v_∞ and v_0 simply read as $X_\infty = 2i\pi\tau Y\partial_Y$ and $X_0 = -2i\pi X\partial_X - 2i\pi\tau Y\partial_Y$. In particular the $\mathbf{C}^* \times \mathbf{C}^*$ part of

$$\mathrm{Aut}(U_{k,\nu,P}, C)$$

corresponds at the level of $V_{k,\nu,P}$ to transformations of type $(X,Y) \to (aX, bY)$, $a,b \in \mathbf{C}^*$. According to the description given in (the proof of) Lemma 10.2 it remains to determine the action corresponding to $g = h_n : (z,\xi) \to (z, e^{\frac{2i\pi}{n}}\xi)$. The latter is given in restriction to V_i^l, $0 \le l < k$, by the following transformation

1. $\Theta_n : (X,Y) \in V_i^l \to \left(e^{\frac{-4\pi^2\nu}{n}}X, e^{\frac{-4\pi^2\tau(\nu+a_0)}{n}}Y\right) \in V_i^{l+\frac{k}{n}}$, when $l \le k-1-\frac{k}{n}$,
2. $\Theta_n : (X,Y) \in V_i^l \to \left(e^{4\pi^2\nu(1-\frac{1}{n})}X, e^{-4\pi^2\tau(\nu+a_0)(1-\frac{1}{n})}Y\right) \in V_i^{l+\frac{k}{n}}$ otherwise.

This is done by a straightforward computation.

10.4. *The analytic moduli space.* — Notations of Section 10.3. Our purpose is to give a simple characterization of neighborhoods (U,C) formally conjugated to $(U_{k,\nu,P}, C)$ up to analytic equivalence. The way to proceed is actually suggested in the previous Section where things are basically "modeled" on Serre's example.

Let us settle some notations in accordance with Section 3. For this, denote by $\mathrm{Diff}^1(V_i^l, L_i^l)$ the group of germs of biholomorphisms of (V_i^l, L_i^l) which preserves the germ of divisor $(D \cap V_i^l)$ and tangent to the identity along L_i^l and by $\mathrm{Diff}^1(V_{i,i+1}^l, p_{i,i+1}^l)$ the group germs of diffeomorphisms of $(V_{i,i+1}^l, p_{i,i+1}^l)$ which preserves the germ of divisor $(D \cap V_{i,i+1})$ and tangent to the identity at $p_{i,i+1}^l$.

Definition **10.4.** — *By definition, a cocycle consists in the datum of a family of* $4k$ *germs of diffeomorphisms* $\varphi = (\varphi_{i,i+1}^l)$ *where* $\varphi_{i,i+1}^l \in \mathrm{Diff}^1(V_{i,i+1}^l, p_{i,i+1}^l)$.

Analogously to the case of Serre's example, we investigate in terms of cocycle the structure of the moduli space $\mathcal{U}_{k,\nu,P}$ of neighborhoods (U,C) formally equivalent to $(U_{k,\nu,P}, C)$ up to analytic equivalence.[29]

[29] Recall that we allow for this statement analytic isomorphisms inducing translations on C.

10.4.1. *Normalizing sectorial maps.* — Concerning the construction of normalizing maps, let us explicit as before the modifications needed: let (U, C) be formally equivalent to $(\mathrm{U}_{k,\mathrm{P},\nu}, \mathrm{C})$.

One can suppose that $(\mathrm{U}, \mathrm{C}) = (\tilde{\mathrm{U}}, \tilde{\mathrm{C}})/\mathrm{F}$ where $\mathrm{F}(z, \xi) = \mathrm{F}_{k,\nu,\mathrm{P}}(z, \xi) + (\Delta_1, \Delta_2)$ with $\Delta_i = \mathrm{O}(\xi^{-\mathrm{N}})$ where $\mathrm{N} \gg 0$ is a positive integer arbitrarily large and that there exists a formal diffeomorphism of $(\tilde{\mathrm{U}}, \tilde{\mathrm{C}})$

$$\hat{\Psi}(z, \xi) = (z + \hat{g}, \xi + \hat{h}) = \left(z + \sum_{n\geq 1} b_n \xi^{-n}, \xi + \sum_{n\geq 1} a_n \xi^{-n}\right)$$

where a_n, b_n are entire functions on $\tilde{\mathrm{C}} = \mathbf{C}^*$, such that

$$\mathrm{F} \circ \hat{\Psi} = \hat{\Psi} \circ \mathrm{F}_{k,\nu,\mathrm{P}}.$$

One wants to generalize the construction performed in Section 9 and replace $\hat{\Psi}$ by a collection of transversely sectorial biholomorphisms $\Psi_i^l \in \mathscr{G}^1(\mathrm{I}_i^l)$, $\widehat{\Psi_i^l} = \hat{\Psi}$,[30] that is

$$\mathrm{F} \circ \Psi_i^l = \Psi_i^l \circ \mathrm{F}_{k,\nu,\mathrm{P}} \tag{10.5}$$

One adopts the notation of loc.cit without systematically mentioning the parameters (k, ν, P).

As before, we will just detail the construction of a normalizing conjugation map on the germ of transversal sectorial domain determined by $\mathrm{I}_1 := \mathrm{I}_1^{l_0}$ where $l_0 \in \mathbf{Z}_k$ has been fixed. Consider the annulus $\mathcal{C}_{a,b} = \{a \leq |z| \leq b\}$, $a, b > 0$, $b > |q|^{-1}a$. Consider the locus where the foliation $\mathcal{F}_1$ is defined by the level sets $\{\mathrm{Y}_{k,\nu,\mathrm{P}} = c\} \cap \{\arg \xi \in \mathrm{I}_1\}$, c "small" of

$$\mathrm{Y}_{k,\nu,\mathrm{P}} = \mathrm{Y} \circ \Omega = z e^{\mathrm{P}_1(\xi) + 2i\pi\tau \frac{\xi^k}{k}} \xi^{2i\pi\tau(\nu + a_0)} = z e^{2i\pi\tau\left(\frac{\xi^k}{k} + \mathrm{R}(\xi)\right)}$$

where $\mathrm{R}(\xi) = o(\xi^k)$. It corresponds to the fibration $d\mathrm{Y} = 0$ on a neighborhood V_1^l of L_1^l. Let $\delta(a, b) > 0$ sufficiently small. For every $0 < \delta \leq \delta(a, b)$, set

$$\mathrm{S}_{a,b,\delta} = \{(z, \xi) \in \mathbf{C}^2 | \ |\mathrm{Y}_{k,\nu,\mathrm{P}}(z, \xi)| \leq \delta, \ z \in \mathcal{C}_{a,b}, \ \arg \xi \in \mathrm{I}_1\}.$$

For every complex number c, $0 < |c| \leq \delta$, consider $\mathrm{S}_{a,b,c} = \{\mathrm{Y}_{k,\nu,\mathrm{P}} = c\} \cap \mathrm{S}_{a,b,\delta}$. Note that $\bigcup_{0<|c|\leq\delta} \mathrm{S}_{a,b,c} = \mathrm{S}_{a,b,\delta}$ and that $\mathrm{F}_{k,\nu,\mathrm{P}}(\mathrm{S}_{a,b,\delta}) \approx \mathrm{S}_{|q|a,|q|b,\delta}$. It is thus coherent to investigate the existence of a solution $\Psi = \Psi_1 \in \mathscr{G}^1(\mathrm{I}_1)$ of (10.5) on the domain $\mathrm{S}'_{a,b,\delta} = \mathrm{S}_{a,b,\delta} \cup \mathrm{F}_{k,\nu,\mathrm{P}}(\mathrm{S}_{a,b,\delta})$.

Remark now that the conjugacy equation (10.5) can be equivalently rewritten

$$\mathrm{F}_\Omega \circ \Psi_\Omega = \Psi_\Omega \circ \mathrm{F}_{1,0,0}$$

[30] Notations and definitions of Section 3.3.

where $F_\Omega = \Omega \circ F \circ \Omega^{-1}$, $\Psi_\Omega = \Omega \circ \Psi \circ \Omega^{-1}$. This amounts to determine Ψ_Ω on the domain $\Omega(S'_{a,b,\delta})$. To do this, note that in view of the asymptotic behavior of φ and φ^{-1} described in Section 10.3, one has $F_\Omega = F_{1,0,0} + (\Delta_{1,\Omega}, \Delta_{2,\Omega})$ where one can verify that $(\widetilde{\Delta_{1,\Omega}}, \Delta_{2,\Omega}) = O\left(\frac{1}{|\xi|^{\frac{N}{k}}}\right)$ setting $z\widetilde{\Delta_{1,\Omega}} = \Delta_{1,\Omega}$.

We are looking for solution of the form $\Psi_\Omega = \mathrm{Id} + (g_\Omega, h_\Omega)$. Here again, this can be reformulated as

$$h_\Omega \circ F_{1,0,0} - h_\Omega = \Delta_{2,\Omega} \circ \Psi_\Omega \tag{10.6}$$

$$g_\Omega \circ F_{1,0,0} - qg_\Omega = \Delta_{1,\Omega} \circ \Psi_\Omega \tag{10.7}$$

First, we will still deal with the linearized equations This can be reformulated as

$$h \circ F_{1,0,0} - h = \Delta_2 \tag{10.8}$$

$$g \circ F_{1,0,0} - qg = \Delta_1 \tag{10.9}$$

where we have omit the subscript Ω for notational convenience.

In what follows, we will indeed provide a solution of (10.8) (hence also for (10.9) by the usual transform) with "good estimates" on a domain of the form $\Omega(S'_{a,b,\delta})$ using a "leafwise" resolution with respect to the foliation defined by the levels of Y. This consists as before in solving a family of difference equations parametrized by the leaves space in the variable ξ where the domains are slightly modified as explained now. For every complex number c, $0 < |c| \le \delta$, consider $S_{a,b,c} = \{Y_{k,\nu,P} = c\} \cap S_{a,b,\delta}$ and $\Omega(S_{a,b,c}) = \{(z,\xi) = (ce^{-2i\pi\tau\xi}, \xi)\}$ where ξ belongs to

$$\begin{aligned}\Sigma_{c,a,b} = \Big\{\xi \in \mathbf{C} : (\log|c| - \log b) \le \Re\Big(2i\pi\tau\xi + Q\big(\varphi^{-1}(\xi)\big)\Big)\\ \le (\log|c| - \log a)\Big\}\end{aligned}$$

where a determination of the logarithm has been chosen in the sector in which we are working, namely $\arg\xi \in I_1$. Note also that $Q(\varphi^{-1}(\xi)) = o(\xi)$ and consequently the middle term in the above inequation "behaves" like $\Re(2i\pi\tau\xi)$. The remaining part of the proof then follows *mutatis mutandis* the same line by noticing that the equation

$$h \circ F_{1,0,0} - h = \Delta_2$$

can be then rewritten as

$$\varphi_c(\zeta + \lambda) - \varphi_c(\zeta) = \Delta_c(\zeta) \tag{10.10}$$

where $\lambda = -2i\pi\tau$, $\zeta = 2i\pi\tau\xi$, $\varphi_c(\zeta) = h(ce^{-\zeta}, \frac{\zeta}{2i\pi\tau})$, and $\Delta_c(\zeta) = \Delta_\delta(ce^{-\zeta}, \frac{\zeta}{2i\pi\tau})$ with $0 < |c| \le \delta$.

We are then reduced to solve a family difference equations in the "quasi" vertical strip

$$\Sigma_c = \{\zeta \in \mathbf{C} : \log|c| - \log b \leq \Re(\zeta + Q(\varphi^{-1}(\frac{\zeta}{2i\pi\tau}))) \leq \log|c| - \log a\}$$

depending analytically on the parameter c and we investigate the existence of a solution φ_c on the domain $\Sigma_c \cup (\Sigma_c + \lambda)$. This can be carried out by resolving equation (9.10) using the same method (that is, essentially Cauchy formula) replacing accordingly in the expression (9.13) the integration along the vertical lines L^-, L^+ by $l(\zeta) = A, B$ where $l(\zeta) = \Re(\zeta - R(\varphi^{-1}(\frac{\zeta}{2i\pi\tau})))$. One also obtain in a similar way the same kind of estimates imposing the uniqueness of the solution. This allows by the fixed point method detailed in the previous section to solve the conjugation equation under the forms (10.6) and (10.7) on the relevant domains and finally exhibit a conjugacy sectorial transformation

$$\Psi = \Omega^{-1} \circ \Psi_\Omega \circ \Omega,$$

well defined as a section of $\mathcal{G}^1$ over I_1 and having $\hat{\Psi}$ as asymptotic expansion. One can analogously construct conjugacy maps on other sectors of opening I_i^l, $i = 1, 3$, $l \in \mathbf{Z}_k$ and also on I_i^l, $i = 2, 4$ by exchanging the roles of the foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ following the process described in Section 9.6.

10.4.2. *Cocycles versus analytic class and statement of the main result.* — One borrows notation from Section 3 and defines $\mathcal{G}^\infty[F_{k,\nu,P}]$ to be the subsheaf of $\mathcal{G}^\infty$ formed by germs of sectorial holomorphic transformations flat to identity along $\tilde{C}$ and commuting to $F_{k,\nu,P}$. By exploiting the Proposition 3.13 and the fact that $(U_{k,\nu,P}, C)$ is "sectorially modeled" on (U_0, C) as described in Section 10.3, one obtains the following Proposition:

Proposition **10.5.** — *We have the following characterizations:*

– $\Phi \in \mathcal{G}^\infty[F_{k,\nu,P}](I_i^l)$ *if and only if* $\Pi_{k,\nu,P} \circ \Phi = \varphi \circ \Pi_{k,\nu,P}$ *where* $\varphi \in \mathrm{Diff}^1(V_i^l, L_i^l)$;
– $\Phi \in \mathcal{G}^\infty[F_{k,\nu,P}](I_{i,i+1}^l)$ *if and only if* $\Pi_{k,\nu,P} \circ \Phi = \varphi \circ \Pi_{k,\nu,P}$ *where* $\varphi \in \mathrm{Diff}^1(V_{i,i+1}^l, p_{i,i+1}^l)$ *except for* $i = 4$ *and* $l = k-1$ *for which one has*

$$\Pi_{k,\nu,P} \circ \Phi = d_{\nu,P} \circ \varphi \circ \Pi_{k,\nu,P}.$$

Definition **10.6.** — *We say that two cocycles* φ *and* φ' *are equivalent if*

$$\exists t, t' \in \mathbf{C},\ \exists \varphi_i^l \in \mathrm{Diff}^1(V_i^l, L_i^l), \exists m \in \mathbf{Z}_n$$

$$\text{such that}\quad \varphi'^{(l+m\frac{k}{n})}_{i,i+1} = \Theta_n^m \circ \phi \circ \varphi_i^l \circ \varphi_{i,i+1}^l \circ \left(\varphi_{i+1}^{l+\varepsilon(i)}\right)^{-1} \circ \phi^{-1} \circ \Theta_n^{-m} \tag{10.11}$$

where $\phi = \exp(tX_0 + t'X_\infty)$ *has thus the form* $\phi(X, Y) = (aX, bY)$ *(cf. Remark 10.3). We will denote this equivalence relation by* $\approx$.

Firstly, let us explain as in Section 3 how to associate a cocycle to a neighborhood $(\mathrm{U},\mathrm{C}) = (\tilde{\mathrm{U}},\tilde{\mathrm{C}})/<\mathrm{F}>$ formally conjugated to $(\mathrm{U}_{k,\nu,\mathrm{P}},\mathrm{C})$. Recall that this means that there is a formal diffeomorphism (that can be assumed to be tangent to the identity along $\tilde{\mathrm{C}}$) $\hat{\Psi}$ conjugating F to $\mathrm{F}_{k,\nu,\mathrm{P}}$ i.e. $\mathrm{F}\circ\hat{\Psi} = \hat{\Psi}\circ\mathrm{F}_{k,\nu,\mathrm{P}}$.

Recall (cf. 10.3) that for every pair (i,l), there exists a section Ψ_i^l of $\mathscr{G}^1(\mathrm{I}_i^l)$ such that

$$\Psi_i^l\circ\mathrm{F} = \mathrm{F}_{k,\nu,\mathrm{P}}\circ\Psi_i^l.$$

According to the description of $\mathrm{Aut}(\mathrm{U}_{k,\nu,\mathrm{P}},\mathrm{C})$, namely the fact that it contains only *convergent* automorphisms, one can assume that the asymptotic expansion of the $\hat{\Psi}_i^l$ is constant equal to $\hat{\Psi}$. The Ψ_i^l are unique up to post composition by a sectorial diffeomorphism $g_i^l\in\mathscr{G}^1[\mathrm{F}_{k,\nu,\mathrm{P}}](\mathrm{I}_i^l)$ having the form

$$g_i^l = \exp(tv_0)\circ h_i^l \tag{10.12}$$

where $h_{i,l}\in\mathscr{G}^\infty[\mathrm{F}_{k,\nu,\mathrm{P}}](\mathrm{I}_i^l)$. Actually, the flow of v_0 contains exactly all the automorphisms tangent to identity along C.

It follows that, on intersections $\mathrm{I}_{i,i+1}^l = \mathrm{I}_i^l\cap\mathrm{I}_{i+1}^{l+\varepsilon(i)}$, $\varepsilon(i)=\delta_{i4}$, one obtains $4k$ germs of sectorial glueing biholomorphisms

$$\Phi_{i,i+1}^l := \Psi_i^l\circ\left(\Psi_{i+1}^{l+\varepsilon(i)}\right)^{-1}\in\mathscr{G}^\infty[\mathrm{F}_{k,\nu,\mathrm{P}}](\mathrm{I}_{i,i+1}^l).$$

One can now invoke Proposition 10.5: setting $\Pi_i^l := \Pi_{k,\nu,\mathrm{P}}\circ\Psi_i^l$ (and taking into account the determination of $\Pi_{k,\nu,\mathrm{P}}$ on the corresponding sectors), one can associate $4k$ corresponding cocycles:

$$\Pi_i^l = \Pi\circ\Psi_i = \Pi\circ\Phi_{i,i+1}\circ\Psi_{i+1} = \varphi_{i,i+1}^l\circ\Pi\circ\Psi_{i+1}^l = \varphi_{i,i+1}^l\circ\Pi_{i+1}^l \tag{10.13}$$

except for $i=4$ and $l=k-1$ for which one has

$$\Pi_4^{k-1} = d_{\nu,\mathrm{P}}\circ\varphi_{1,4}^0\circ\Pi_1^0.$$

We have therefore associated to each neighborhood (U,C) formally equivalent to $(\mathrm{U}_{k,\nu,\mathrm{P}},\mathrm{C})$ a cocycle $\varphi = (\varphi_{i,i+1}^l)_{i\in\mathbf{Z}_{4k}}$ which is unique up to the freedom for the choice of Ψ_i's. We have now all the ingredients to state and prove our main Theorem

Theorem **E.** — *Two neighborhood* (U,C) *and* (U',C) *formally equivalent to* $(\mathrm{U}_{k,\nu,\mathrm{P}},\mathrm{C})$ *are analytically equivalent*[31] *if, and only if, the corresponding cocycles are equivalent*

$$(\mathrm{U},\mathrm{C})\overset{an}{\sim}(\mathrm{U}',\mathrm{C}) \quad\Leftrightarrow\quad \varphi\approx\varphi'. \tag{10.14}$$

Moreover, every cocycle can be realized by the process described above.

[31] Recall that formal/analytic conjugations are allowed to induce translations on C.

Proof. — Now, consider a biholomorphism between two neighborhood $(\mathrm{U},\mathrm{C}) \to (\mathrm{U}',\mathrm{C})$ formally conjugated to $(\mathrm{U}_{k,\nu,\mathrm{P}},\mathrm{C})$. According to the description of $\mathrm{Aut}(\mathrm{U}_{k,\nu,\mathrm{P}},\mathrm{C})$, it is represented by a bihomorphism of $(\tilde{\mathrm{U}},\tilde{\mathrm{C}})$ taking the form $h_n^m \circ \exp(t' v_\infty) \circ \Psi$ where $\Psi \in \mathscr{G}^1(\mathbf{S}^1)$ and satisfying $h_n^m \circ \exp(t' v_\infty) \circ \Psi \circ \mathrm{F} = \mathrm{F}' \circ h_n^m \circ \exp(t' v_\infty) \circ \Psi$. Let (Ψ_i^l) and $(\Psi_i'^l)$ be the sectorial normalizations used to compute the invariants φ and φ'. Clearly, $(h_n^m \circ \exp(t' v_\infty))^{-1} \circ \Psi_i'^{l+m\frac{k}{n}} \circ (h_n^m \circ \exp(t' v_\infty)) \circ \Psi$ provides for (U,C) a new collection of sectorial trivializations defined on I_i^l tangent to identity.

By virtue of 10.12, we can then write

$$\Psi_i'^{l+m\frac{k}{n}} \circ h_n^m \circ \exp(t' v_\infty) \circ \Psi = (h_n^m \circ \exp(t' v_\infty)) \circ \exp(t v_0) \circ \Phi_i^l \circ \Psi_i^l$$

with $\Phi_i^l \in \mathscr{G}^\infty[\mathrm{F}_{k,\nu,\mathrm{P}}](\mathrm{I}_i^l)$. Therefore, we have

$$\begin{aligned}\Phi'^{l+m\frac{k}{n}}_{i,i+1} &= (\Psi_i'^{l+m\frac{k}{n}} \circ h_n^m \circ \exp(t' v_\infty) \circ \Psi) \circ (\Psi'^{l+m\frac{k}{n}+\varepsilon(i)}_{i+1} \circ h_n^m \\ &\qquad \circ \exp(t' v_\infty) \circ \Psi)^{-1} \\ &= h_n^m \circ \varphi \circ \Phi_i^l \circ \Phi_{i,i+1}^l \circ \left(\Phi_{i+1}^{l+\varepsilon(i)}\right)^{-1} \circ \varphi^{-1} \circ h_n^{-m}.\end{aligned}$$

where $\varphi := \exp(t v_0) \circ \exp(t' v_\infty) = \exp(t' v_\infty) \circ \exp(t v_0)$. After factorization through $\Pi_{k,\nu,\mathrm{P}}$, using Proposition 10.5, we get the expected equivalence relation (10.11) for φ and φ'. Conversely, if $\varphi \approx \varphi'$, then we can trace back the existence of an analytic conjugacy $\Phi : (\mathrm{U},\mathrm{C}) \to (\mathrm{U}',\mathrm{C})$ by reversing the above implications. Finally, it suffices to mimic the construction performed in Section 5 in order to realize every cocycle. □

Remark **10.7.** — One thus observes, as in the case $k = 1$, that two cocycles are equivalent iff they lie on the same orbit over some action (that the reader will easily explicit) of $\mathbf{Z}/n \times (\mathcal{O}^* \times \mathcal{O}^*)^{\times 4k}_{\mathbf{C}^* \times \mathbf{C}^*}$. One can strengthen the analytic equivalence by demanding that conjugations induce the identity on C. In that case, we have to replace ϕ by $\exp(t\mathrm{X}_0)$ in the statement of definition 10.6.

11. Torsion normal bundle

Recall that the elliptic curve is regarded as the quotient $\mathrm{C} = \mathbf{C}/\mathbf{Z} \oplus \tau\mathbf{Z}$. We maintain notations of Theorem 10.1 and we also set $d := \gcd(m_1, m_\tau)$. Let $(\mathrm{U}_m, \mathrm{C}_m)$ be an m-cyclic cover trivializing the normal bundle having the following form:

$$(\mathrm{U}_m, \mathrm{C}_m) = (\mathbf{C}_x \times \mathbf{C}_y, \{y = 0\})/\langle \phi_1^{m_1}, \phi_1^l \circ \phi_\tau^{\frac{m_\tau}{d}} \rangle$$

where l is an integer such that $a_1^l a_\tau^{\frac{m_\tau}{d}} = 1$. The deck transformation group $\mathrm{G} = \mathbf{Z}/m$ is then generated by $\phi_1^v \circ \phi_\tau^w$ where (v, w) is any pair of integers fulfilling $a_1^v a_\tau^w = e^{\frac{-2i\pi}{m}}$.

Note that, as an effect of this cover, the modulus of the elliptic curve changes accordingly and more precisely C_m is determined by the lattice $\langle m_1, l + \frac{m_\tau}{d}\tau\rangle$.

An easy calculus yields

$$\textbf{(11.1)}\qquad \begin{cases} \phi_1^{m_1}(x,y) = (x+m_1\ ,y) \\ \phi_1^l\circ\phi_\tau^{\frac{m_\tau}{d}}(x,y) = (x+l+\frac{m_\tau}{d}\tau+\tau\int_0^y[(\varphi_{k,\nu}{}^{\frac{m_\tau}{d}})^*\omega_{\mathrm{P}}-\omega_{\mathrm{P}}]\ ,\ \varphi_{k,\nu}{}^{\frac{m_\tau}{d}}(y)) \end{cases}$$

and conjugating by the transformation $\alpha : (x,y) \to (\frac{x}{m_1}, ay)$, where $a = (\frac{m_\tau}{d})^{\frac{1}{k}}$, one can reduce to the simplest and usual normal formal form

$$(\mathrm{U}_m, \mathrm{C}_m) \simeq (\mathbf{C}_x \times \mathbf{C}_y, \{y=0\})/(\mathrm{F}_1, \mathrm{F}_{\tau'})$$

where

$$\mathrm{F}_1(x,y) = (x+1,y), \quad \mathrm{F}_{\tau'}(x,y) = (x+\tau'+h_{k,\nu',\mathrm{P}'}(y), \varphi_{k,\nu'}(y)),$$

and

$$\tau' = \frac{1}{m_1}(l+\frac{m_\tau}{d}\tau), \quad \nu' = \frac{d\nu}{m_\tau}, \quad \mathrm{P}'(\mathrm{X}) = \frac{\tau\mathrm{P}(\mathrm{X}^m)}{\tau' m_1},$$
$$h_{k,\nu',\mathrm{P}'}(y) = \tau\int_0^y[\varphi_{k,\nu'}^*\omega_{\mathrm{P}'}-\omega_{\mathrm{P}'}].$$

If one adopts the presentation in the variable (z,ξ), this corresponds to the neighborhood labeled identically as in Section 10.2 (except that C is replaced by its cover C_m or equivalently τ par τ'): its presentation as a quotient is given by $(\mathbf{C}_z^*\times\overline{\mathbf{C}}_\xi, \{\xi=\infty\})/\langle \mathrm{F}_{k,\nu',\mathrm{P}'}\rangle$ where $\mathrm{F}_{k,\nu',\mathrm{P}'} = \exp v_{k,\nu',\mathrm{P}'}$ and $v_{k,\nu',\mathrm{P}'} = \frac{1}{\xi^k+\nu'}(-\xi\partial_\xi + 2i\pi\tau' z\mathrm{P}'(\xi)\partial_z) + 2i\pi\tau' z\partial_z$. The foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ are respectively defined by the levels of ξ and $\mathrm{Y}_{k,\nu',\mathrm{P}'} = ze^{2i\pi\tau'(\frac{\xi^k}{k}+\mathrm{V}(\xi^m))}\xi^{2i\pi\tau'(\nu'+a_0')}$ where V is a polynomial of degree $< k' = \frac{k}{m}$ that we do not need to explicit and $a_0' = \mathrm{P}'(0)$. From Section 10.3, one knows that $(\mathrm{U}_m, \mathrm{C}_m)$ is parametrized by a neighborhood $\mathrm{V}_{k,\nu',\mathrm{P}'} = \bigcup \mathrm{V}_i^l$ of $4k$ lines via the map

$$\Pi_{k,\nu',\mathrm{P}'} = (e^{\frac{2i\pi\xi^k}{k}}\xi^{2i\pi\nu'}, \mathrm{Y}_{k,\nu',\mathrm{P}'}).$$

For the sake of clarity and coherence, it will be natural to denote this neighborhood by $(\mathrm{U}_{k,\nu,\mathrm{P}'}, \mathrm{C}')$ where $\mathrm{C}' = \mathrm{C}_m$.

In order to recover the structure of the original neighborhood, we have to determine the identifications induced by the deck transformation group G which is cyclic of order m and generated by a transformation

$$g_{v,w} = \Phi_1^v\circ\Phi_\tau^w = \mathcal{A}\circ\exp(-\beta_1 v_{k,\nu',\mathrm{P}'}),$$
$$\Phi_1 := \alpha\circ\phi_1\circ\alpha^{-1},\ \Phi_\tau := \alpha\circ\phi_\tau\circ\alpha^{-1}$$

where

- (v, w) is the fixed pair of integers such that $a_1^v a_\tau^w = e^{-\frac{2i\pi}{m}}$,
- $\beta_1 = -\frac{wd}{m_\tau}$,
- $\mathcal{A}$ is the affine map $\mathcal{A}(x,y) = (x+\beta_2, e^{\frac{-2i\pi}{m}}y)$ with $\beta_2 = \frac{v}{m_1} - \frac{wld}{m}$.

In the (z,ξ) coordinates, $\mathcal{A}$ corresponds to the linear map $d(z,\xi) = (e^{2i\pi\beta_2}z, e^{\frac{2i\pi}{m}}\xi)$. In particular, one has $d^m = \mathrm{id}$. Note that the vector field $\mathrm{X}_{k,\nu',\mathrm{P}'} = -2i\pi \mathrm{X}\partial_{\mathrm{X}}$ on $\mathrm{V}_{k,\nu',\mathrm{P}'}$ corresponds via $\Pi_{k,\nu',\mathrm{P}'}$ to $v_{k',\nu',\mathrm{P}'}$. Its flow at time $-\beta_1$ preserves each individual neighborhood V_i^l of the component L_i^l of the cycle and reads as

$$(\mathrm{X},\mathrm{Y}) \to (e^{2i\pi\beta_1}\mathrm{X},\mathrm{Y}).$$

The action of $g_{v,w}$ by "sectorial permutation" is then explicitly given in the (X,Y) coordinates (determining the neighborhood of $4k$ lines as described in Section 10.3) by

$$\mathscr{D} : (\mathrm{X},\mathrm{Y}) \in \mathrm{V}_i^l \to (e^{\frac{-4\pi^2\nu'}{m}+2i\pi\beta_1}\mathrm{X}, e^{\frac{-4\pi^2\tau'(\nu'+a_0')}{m}+2i\pi\beta_2}\mathrm{Y}) \in \mathrm{V}_i^{l+\frac{k}{m}} \tag{11.2}$$

when $l \le k-1-\frac{k}{m}$, and by

$$\mathscr{D} : (\mathrm{X},\mathrm{Y}) \in \mathrm{V}_i^l \to (e^{4\pi^2\nu'(1-\frac{1}{m})+2i\pi\beta_1}\mathrm{X}, e^{4\pi^2\tau'(\nu'+a_0')(1-\frac{1}{m})+2i\pi\beta_2}\mathrm{Y}) \in \mathrm{V}_i^{l+\frac{k}{m}} \tag{11.3}$$

otherwise. We will denote by

$$(\mathrm{U}^{\mathrm{G}}_{k,\nu',\mathrm{P}'}, \mathrm{C}) := (\mathrm{U}_{k,\nu'}, \mathrm{C}')/\mathrm{G}$$

this description of the original neighborhood as a finite quotient.

At the level of the neighborhood $\mathrm{V}_{k,\nu',\mathrm{P}'}$ of rational curves, these identifications are given by the action of the order m cyclic permutation $\mathscr{D}$ and we eventually end up with a neighborhood of $4k'$ rational curves of zero type, namely the quotient

$$\mathrm{V}^{\mathrm{G}}_{k,\nu',\mathrm{P}'} := \mathrm{V}_{k,\nu',\mathrm{P}'}/\mathscr{D}.$$

Indeed, set $u = \xi^m$ and consider the function

$$\Pi^{\mathrm{G}}_{k,\nu',\mathrm{P}'}(z,\xi) = (\xi^{-m\beta_1}e^{\frac{2i\pi u^{k'}}{k}}u^{\frac{2i\pi\nu'}{m}+\beta_1}, \xi^{-m\beta_2}ze^{2i\pi\tau'(\frac{u^{k'}}{k}+\mathrm{V}(u))}u^{2i\pi\tau'(\frac{\nu'+a_0'}{m})+\beta_2})$$

as defined and univaluate on each of the $4k'$ "multisectorial domains"

$$\tilde{\mathrm{U}}_i^{l'} := \bigsqcup_j \mathrm{U}_i^{l+\frac{jk}{m}},$$

$l' = l[\frac{k}{m}]$ by the choice of a determination of the logarithm of u. One can notice that G acts transitively on the set of sectors of $\tilde{\mathrm{U}}_i^{l'}$ and that $\Pi^{\mathrm{G}}_{k,\nu',\mathrm{P}'}$ is G invariant. On the other hand, $\Pi^{\mathrm{G}}_{k,\nu',\mathrm{P}'}$ coincides with $\Pi_{k,\nu',\mathrm{P}'}$ (up to left composition with a diagonal linear map) on each sector V_i^l, hence is constant and separating along the orbits of $F_{k,\nu',P'}$. This shows, as claimed previously, that the (deleted) neighborhood $(U^G_{k,\nu',P'}, C)$ can be represented as the (deleted) neighborhood $V^G_{k,\nu',P'}$ of a cycle $\tilde{D} = \bigcup \tilde{L}_i^{l'}$ of $4k'$ rational curves of zero type. More precisely, each individual neighborhood $\tilde{V}_i^{l'}$, of $\tilde{L}_i^{l'}$ is the image under $\Pi^G_{k,\nu',P'}$ of $\tilde{U}_i^{l'}$ and is thus equipped as before by coordinates (X, Y) determined by each component of $\Pi^G_{k,\nu',P'}$. These coordinates glue together trivially on overlaps except on $\tilde{V}_1^0$ (defined as in Section 10.3) where it is given by the linear diagonal map

$$d^G : (X, Y) \to (e^{(-4\pi^2 \frac{\nu'}{m} + 2i\pi\beta_1)}X, e^{(-4\pi^2\tau'(\frac{\nu u' + a_0'}{m}) + 2i\pi\beta_2)}Y).$$

The reader should compare with the case of resonant diffeomorphisms of one variable [18, Section 10.3, p. 592].

Let (U^1, C) be a neighborhood formally conjugated to $(U^G_{k,\nu',P'}, C)$. Again by making use of [29] (See proof of Lemma 2.1), one can suppose that (U^1, C) can be represented as the quotient

$$(\mathbf{C}_x \times \mathbf{C}_y, \{y = 0\})/\langle \phi_1, \phi_\tau^1 \rangle$$

so that there exists a formal diffeomorphism of $(\mathbf{C}_x \times \mathbf{C}_y, \{y = 0\})$ commuting with ϕ_1 that can be assumed to be tangent to the identity along C and conjugating ϕ_τ^1 to ϕ_τ.

By taking the same triple of integer (l, v, w) than before, one can argue passing through the m-cyclic (U^1_m, C_m) cover of (U^1, C) trivializing the normal bundle. It can be represented as the quotient $(\mathbf{C}^*_z \times \bar{\mathbf{C}}_\xi, \{\xi = \infty\})/F$. Let G^1 be the attached deck transformation group with generator $g^1_{v,w} = \Phi_1^v \circ {\Phi_\tau^1}^w$ where one sets as before $\Phi_1 := \alpha \circ \phi_1 \circ \alpha^{-1}$, $\Phi_\tau^1 := \alpha \circ \phi_\tau^1 \circ \alpha^{-1}$. Let $\rho : G \to G^1$ the isomorphism of cyclic group mapping $g_{v,w}$ to $g^1_{v,w}$. The formal conjugation between the original neighborhoods, can be translated into the existence of a formal transformation map $\hat{\Psi}$ conjugating F to $F_{k,\nu,P}$, i.e. $F \circ \hat{\Psi} = \hat{\Psi} \circ F_{k,\nu,P}$ and which is in addition *equivariant* with respect to ρ:

$$\forall g \in G, \hat{\Psi} \circ g = \rho(g) \circ \hat{\Psi}.$$

Recall (cf. 10.3) that for every pair (i, l), there exists a section Ψ_i^l of $\mathscr{G}^1(I_i^l)$ such that

$$\Psi_i^l \circ F_{k,\nu,P} = F \circ \Psi_i^l,$$

such that $\hat{\Psi}_i^l = \hat{\Psi}$. Moreover, the collection of these sectorial normalisations can be chosen equivariantly, ie:

$$\Psi_i^{l + j\frac{jk}{m}} \circ g^j_{v,w} = \rho(g^j_{v,w}) \circ \Psi_i^l.$$

This last point is justified by the fact that G and G^1 lie respectively in the centralizer of $F_{k,\nu',P'}$ and F.

This thus provides a “multisectorial conjugation” on each $\tilde{U}_i^{l'}$. One can mimic the arguments presented in Section 10 and thus obtain the description of the analytic moduli space which is thus determined by a cocycle φ with $4k'$ components modulo the identifications given in Theorem E.

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