RIEMANNIAN HYPERBOLIZATION

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ABSTRACT

The strict hyperbolization process of Charney and Davis produces a large and rich class of negatively curved spaces (in the geodesic sense). This process is based on an earlier version introduced by Gromov and later studied by Davis and Januszkiewicz. If M is a manifold its Charney-Davis strict hyperbolization is also a manifold, but the negatively curved metric obtained is very far from being Riemannian because it has a large and complicated set of singularities. We show that these singularities can be removed (provided the hyperolization piece is large). Hence the strict hyperbolization process can be done in the Riemannian setting.

Classical flat geometry is characterized by the condition that the sum of the internal angles of a triangle $\triangle$ is equal to π. We write $\Sigma(\triangle) = \pi$. In other fundamental geometries the equality $\Sigma(\triangle) = \pi$ is replaced by inequalities: in positively curved and negatively curved geometries we have the inequalities $\Sigma(\triangle) > \pi$ and $\Sigma(\triangle) < \pi$, respectively, where $\triangle$ runs over all small non-degenerate triangles in a space. It is natural then to try to find spaces that admit such geometries, and this task has been a driving force in Riemannian Geometry for many decades. But surprisingly there are not too many examples of smooth closed manifolds that support either a positively curved or a negatively curved metric. For instance, besides spheres, in dimensions ≥ 17 (and $\neq 24$) the only positively curved simply connected known examples are complex and quaternionic projective spaces. In negative curvature the situation is arguably more striking because negative curvature has been studied extensively in many different areas in mathematics. Indeed, from the ergodicity of their geodesic flow in Dynamical Systems to their topological rigidity in Geometric Topology; from the existence of harmonic maps in Geometric Analysis to the well-studied and greatly generalized algebraic properties of their fundamental groups, negatively curved Riemannian manifolds are the main object in many important and well-known results in mathematics. Yet the fact remains that very few examples of closed negatively curved Riemannian manifolds are known. Besides the hyperbolic ones ($\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, $\mathbf{O}$), the other known examples are the Mostow-Siu examples (complex dimension 2) which are local branched covers of complex hyperbolic space (1980, [23], see also [39]), the Gromov-Thurston examples (1987, [18]) which are branched covers of real hyperbolic ones, the exotic Farrell-Jones examples (1989, [12]) which are homeomorphic but not diffeomorphic to real hyperbolic manifolds (and there are other examples of exotic type), and the three examples of Deraux (2005, [9]) which are of the Mostow-Siu type in complex dimension 3. Hence, excluding the Mostow-Siu and Deraux examples (in dimensions 4 and 6, respectively), all known examples of closed negatively curved Riemannian manifolds are homeomorphic to either a hyperbolic one or a branched cover of a hyperbolic one.

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This lack of examples in negative curvature changes dramatically if we allow singularities, and a very rich and abundant class of negatively curved spaces (in the geodesic sense) exists due to the strict hyperbolization process of Charney and Davis [5]. The hyperbolization process was originally introduced by Gromov [16], and later studied by Davis and Januszkiewicz [8], and Charney-Davis strict hyperbolization is built on these previous versions. The hyperbolization process is conceptually (but not technically) quite simple since it has a lego type flavor: in the same way as simplicial complexes and cubical complexes are built from a basic set of pieces, basic "hyperbolization pieces" are chosen, and anything that can be built or assembled with these pieces will be negatively curved. This conceptual simplicity could be in some sense a bit deceptive because hyperbolization produces an enormous class of examples with a very fertile set of properties. But the richness and complexity of the hyperbolized objects are matched by the richness and complexity of the singularities obtained, and hyperbolized smooth manifolds are very far from being Riemannian. Interestingly one can relax and lose even more regularity and consider negative curvature from the algebraic point of view, that is consider Gromov's hyperbolic groups, and it can be argued [26] that "almost every group" is hyperbolic. So, negative curvature is in some weak sense generic, but Riemannian negative curvature seems very scarce. It is natural then to inquire about the difference between the class of manifolds with negatively curved metrics with singularities and its subclass of more regular Riemannian counterparts. More specifically we can ask whether the strict hyperbolization process can be brought into the Riemannian universe. In this paper we give a positive answer to this question, and we do this by proving that all singularities of the Charney-Davis strict hyperbolization of a closed smooth manifold can be smoothed, provided the "hyperbolization piece" is large enough (which can always be done). Moreover we prove that we can do this process in a ε-pinched way. Here is the statement of our Main Theorem.

Main Theorem. — Let M^n *be a closed smooth manifold and let* $\varepsilon > 0$. *Then there is a closed Riemannian manifold* N^n *and a smooth map* $f : \mathrm{N} \to \mathrm{M}$ *such that*

(i) *The Riemannian manifold* N *has sectional curvatures in the interval* $[-1-\varepsilon, -1]$.
(ii) *The induced map* $f_* : \mathrm{H}_*(\mathrm{N}, \mathrm{A}) \to \mathrm{H}_*(\mathrm{M}, \mathrm{A})$ *is surjective, for every abelian group* A.
(iii) *If* R *is a commutative ring with identity and* M *is* R*-orientable then* N *is* R*-orientable,* f *has degree one, and* $f^* : \mathrm{H}^*(\mathrm{M}, \mathrm{R}) \to \mathrm{H}^*(\mathrm{N}, \mathrm{R})$ *is injective. This follows from* (ii) *and the naturality of the Poincare Duality Isomorphism.*
(iv) *The map* f^* *sends the rational Pontryagin classes of* M *to the rational Pontryagin classes of* N.

Addendum to Main Theorem. — The manifold N *is the Charney-Davis strict hyperbolization of* M *but with a possibly different smooth structure. The hyperbolization is done with a sufficiently "large" hyperbolization piece* X.

By "large" above we mean that the normal neighborhoods of every face of X has large width. These large pieces always exist (see Proposition 9.1). Also, the underlying cube complex of M is assumed to have the usual intersection property: any two cubes intersect in at most one common subcube. This condition does not seem to be essential in the proof, but is technically useful.

Corollaries 1, 2 and 3 below are the ε-pinched Riemannian versions of classical applications of hyperbolization.

Corollary **1.** — *Every closed smooth manifold is smoothly cobordant to a closed Riemannian manifold with sectional curvatures in the interval* $[-1-\varepsilon, -1]$, *for every* $\varepsilon > 0$.

Corollary **2.** — *The cohomology ring of any finite* CW*-complex embeds in the cohomology ring of a closed Riemannian manifold with sectional curvatures in the interval* $[-1-\varepsilon, -1]$, *for every* $\varepsilon > 0$.

Proof. — Let X be a finite CW-complex. Embed X in some $\mathbf{R}^n$ and let P be a compact neighborhood of X that retracts to X. Let M be the double of P. Then there is a retraction $M \to X$, and Corollary 2 follows from (iii) in the Main Theorem. □

Since degree one maps between closed orientable manifolds are π_1-surjective we obtain the following result.

Corollary **3.** — *For every finite CW-complex* X *there is a closed Riemannian manifold* N *and a map* $f : N \to X$ *such that:* (i) N *has sectional curvatures in the interval* $[-1-\varepsilon, -1]$, (ii) f *is* π_1*-surjective,* (iii) f *is homology surjective.*

All previously known examples of closed negatively curved Riemannian manifolds with less than $\frac{1}{4}$-pinched curvature have zero rational Pontryagin classes (for the Gromov-Thurston branched cover examples this was proved by Ardanza [1]). The next corollary gives examples of such manifolds with nonzero rational Pontryagin classes.

Corollary **4.** — *For every* $\varepsilon > 0$ *and* $n \geq 4$ *there is a closed Riemannian n-manifold with sectional curvatures in the interval* $[-1-\varepsilon, -1]$ *and nonzero rational Pontryagin classes.*

Proof. — Take M in the Main Theorem orientable with nonzero Pontryagin classes. □

All manifolds given in Corollary 4 are new examples of closed negatively curved manifolds, provided $\varepsilon < 3$. We state this in the next corollary.

Corollary **5.** — *For any* $\varepsilon > 0$ *and* $n \geq 4$ *there are closed Riemannian n-manifolds with sectional curvatures in the interval* $[-1-\varepsilon, -1]$ *that are not homeomorphic to a hyperbolic manifold (*$\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, $\mathbf{O}$*), or the Gromov-Thurston branched cover of a real hyperbolic manifold, or one of the Mostow-Siu or Deraux examples.*

Proof. — Let N be as in Corollary 4, with $\varepsilon < 3$. So, N is less that quarter-pinched negatively curved. Then N is not homeomorphic to a real hyperbolic manifold or the Gromov-Thurston branched cover of a real hyperbolic manifold. This follows from Novikov's topological invariance of the rational Pontryagin classes [25], and Ardanza's result in [1] mentioned above. Also the quarter-pinched rigidity results given in (or implied by) the work of Corlette [6], Gromov [17], Hernández [20], and Mok-Siu-Yeung [22] imply that N is not homeomorphic to a quaternionic or Cayley hyperbolic manifold (specifically, one can use Theorem 1 in [22] together with Theorem 2.5(b) in [20]). Finally since a closed Kähler manifold of dimension ≥ 4 cannot be homeomorphic to a less than $\frac{1}{4}$-pinched negatively curved manifold (see remark below), N cannot be homeomorphic to a complex hyperbolic manifold, or any of the Mostow-Siu or Deraux examples. This is because Mostow-Siu and Deraux examples are all Kähler. □

Remark. — In the proof of Corollary 5 we are using the result: *a closed Kähler manifold of dimension ≥ 4 is not homeomorphic to a less than quarter-pinched negatively curved manifold.* One can obtain this result using the work of Hernández (see Theorems 1.1 and 2.5(b) in [20]), though the homeomorphic part is not stated explicitly in Hernández paper. For completeness, here is a sketch of the proof of the homeomorphic part. Let $X \to N$ be a homeomorphism, where X is closed Kähler of dimension $n \geq 4$ and N is less that quarter-pinched negatively curved. By a result of Eells and Sampson [10] f is homotopic to a harmonic h. Since h has degree one, it is onto, so Sard's Theorem implies that there is $x_0 \in X$ such that dh_{x_0} has rank n. On the other hand, since N is less that quarter-pinched, all of its complex sectional curvatures are negative (see 2.5(b) in [20]). We can now apply a result of Sampson (see [35], or 3.1 in [20]) that says that in this situation the rank of dh_x is at most 2, for every $x \in X$, which is a contradiction because $n \geq 4$.

The next application was suggested to us by Stratos Prassidis and deals with cusps of negatively curved manifolds. Recall that if M is a complete finite volume noncompact real hyperbolic manifold then there is a bounded set $B \subset M$ such that $M \setminus B$ is isometric to a manifold of the form $Q \times [b, \infty)$ with the metric $e^{-2t}h + dt^2$, where (Q, h) is a closed flat manifold and $b \in \mathbf{R}$. In this case we say that *the manifold* Q *bounds geometrically a hyperbolic manifold*. More generally, in 1978 Gromov defined almost flat manifolds in [15] and similar facts hold for them replacing hyperbolic manifolds by pinched negatively curved manifolds. That is, let M be a complete finite volume noncompact manifold with pinched negative curvature (i.e. all sectional curvatures lie in a fixed interval $[-a, -b]$, $0 < b \leq a < \infty$). Then there is a bounded $B \subset M$ such that $M \setminus B$ is diffeomorphic to a manifold of the form $Q \times [b, \infty)$, where Q is an almost flat manifold. In this case we say that *the manifold* Q *bounds geometrically a negatively curved manifold*. Of course a necessary condition for Q to bound geometrically as above is to smoothly bound a compact manifold.

Remark. — Here we do not assume Q to be connected.

It was proved by Hamrick and Royster [19] that every closed flat manifold bounds smoothly. This together with the work of Gromov in [14], [15] motivated Farrell and Zdravkovska to make the following well-known conjectures in [13].

Conjecture **1.** — *Every closed almost flat manifold bounds smoothly. This conjecture was also proposed, independently, by Yau in* [38].

Conjecture **2.** — *Every closed flat manifold bounds geometrically a hyperbolic manifold.*

Conjecture **3.** — *Every closed almost flat manifold bounds geometrically a negatively curved manifold.*

It was showed by Long and Reid [21] that Conjecture 2 is false by giving examples of three dimensional flat manifolds that do not bound. The following result says Conjecture 1 implies Conjecture 3.

Theorem **A.** — *Let* Q *be a closed almost flat manifold. Assume that* Q *bounds smoothly. Then* Q *bounds geometrically a negatively curved manifold* M.

Conjecture 1 has generated a lot of research in the last 30 years and it is known to be true for an almost flat manifold in many cases, depending on the holonomy of the manifold. Recall that a nilmanifold is the quotient of a simply connected nilpotent Lie group L by a lattice. Gromov-Ruh [34] proved that every almost flat manifold Q has a finitely-sheeted affine cover that is diffeomorphic to a nilmanifold, and the deck group G of the affine covering is called the *holonomy group of* Q. Let Q be an almost flat manifold and G its holonomy. Conjecture 1 is known to be true in the following cases.

(a) The manifold Q is a nilmanifold.
(b) The holonomy G has order k or $2k$, where k is odd, due to Farrell-Zdravkovska [13].
(c) The holonomy G of Q acts effectively on the center of L, also due to Farrell-Zdravkovska [13].
(d) The holonomy G is cyclic or quaternionic, due to Davis and Fang [7]. Also Upadhyay [37] had proved that Conjecture 1 is true when the following conditions hold: G is cyclic, G acts trivially on the center of L, and L is 2-step nilpotent.

Hence in all of the above cases Q bounds geometrically a pinched negatively curved manifold. Note that for any closed Q we have $\partial(\mathrm{Q}\times \mathrm{I}) = \mathrm{Q}\coprod \mathrm{Q}$. Thus we get the following corollary of Theorem A.

Corollary **6.** — *Let* Q *be a closed almost flat manifold. Then* $\mathrm{Q}\coprod\mathrm{Q}$ *bounds geometrically a pinched negatively curved manifold.*

In other words, for every closed connected almost flat manifold there is a complete finite volume pinched negatively curved manifold with exactly **two** connected cusps, each diffeomorphic to $Q \times [b, \infty)$.

A complete pinched negatively curved metric g on $Q \times \mathbf{R}$ is called a *(pinched negatively curved) cusp metric* if the g-volume of $Q \times [0, \infty)$ is finite. And we say that a cusp metric g on $Q \times \mathbf{R}$ *is an eventually warped cusp metric* if $g = e^{-2t}h + dt^2$, for $t < c$, for some $c \in \mathbf{R}$ and a metric h on Q. Belegradek and Kapovitch [2] showed, based on earlier work by Shen [36], that if Q is almost flat then $Q \times \mathbf{R}$ admits an eventually warped cusp metric.

Addendum to Theorem A. — Let g be an eventually warped cusp metric on $Q \times \mathbf{R}$. If the sectional curvatures of g lie in (a, b), with $a < -1 < b$, then we can take M *in Theorem* A *with sectional curvatures also in (a, b). Moreover the sectional curvatures of* M *away from a cusp can be taken in $[-\varepsilon - 1, -1]$, for any $\varepsilon > 0$.*

Even though a flat manifold may not necessarily bound geometrically a hyperbolic manifold the next corollary says it does bound geometrically an ε-pinched to -1 manifold, for any $\varepsilon > 0$. It follows from the Hamrick and Royster result [19], Theorem A and its addendum.

Corollary **7.** — *Every closed flat manifold bounds geometrically a manifold with sectional curvatures in $[-\varepsilon - 1, -1]$, for any $\varepsilon > 0$.*

We next give a rough idea of some of the methods used in smoothing the singularities of a Charney-Davis hyperbolized smooth manifold. We do this first in dimension two and then in dimension three where we can visualize some of these methods.

A Charney-Davis hyperbolization piece X^n of dimension n is essentially a compact hyperbolic manifold with corners that has the symmetries of an n-cube, and all "faces" intersect perpendicularly. We shall assume throughout this introduction that X is as "large" as we need it to be (see Proposition 9.1).

(i) *Dimension two.*

Fix an X^2 and let K be a cubical 2-complex. Replace each cube by a copy of X^2 to obtain a piecewise hyperbolic space K_X. This is essentially the Charney-Davis hyperbolization of K. We shall identify the vertices of K with the vertices of K_X. Note that the edges (1-faces) match nicely and the piecewise hyperbolic metric σ is smooth away from the vertices. Near a vertex o the metric is a warped product of the form $\sigma = \sigma_L = \sinh^2(t)\frac{L}{4}\sigma_{\mathbf{S}^1} + dt^2$, where:

(1) we are identifying a punctured neighborhood of o with $\mathbf{S}^1 \times (0, r+2)$, for some $r > 0$,

(2) the metric $\sigma_{\mathbf{S}^1}$ is the canonical metric of the circle $\mathbf{S}^1$,
(3) L is the number of 2-cubes (or equivalently, the number of copies of X) containing o.

Of course if $L = 4$ the metric σ_L is already hyperbolic and smooth near o. The problem arises when $L \neq 4$. In this case the solution is given by the *Gromov-Thurston trick.* Choose $d > 0$ with $d < r$ and subdivide $(0, r+2)$ in three pieces $I_1 = (0, r-d]$, $I_2 = [r-d, r]$, $I_3 = [r, r+2)$ and let $\rho = \rho_{L,r,d}$ be a smooth function on $(0, r+2)$ such that $\rho \equiv 1$ on I_1, $\rho \equiv \frac{L}{4}$ on I_3. Consider now $h_L = h_{L,r,d} = \sinh^2(t)\rho(t)\sigma_{\mathbf{S}^1} + dt^2$. Then h_L and σ_L coincide on $\mathbf{S}^1 \times I_3$, hence we can define the *smoothed metric* $\mathcal{G}_L = \mathcal{G}(L, r, d)$ near o to be equal to σ_L outside $\mathbf{S}^1 \times (I_1 \cup I_2)$ and equal to h on $\mathbf{S}^1 \times (0, r)$. Using the Bishop-O'Neill formula in [3] it can be shown that by choosing r and d large enough (depending on L) the metric $\mathcal{G}_L$ will have curvature very close to -1. Furthermore, since $\mathcal{G}_L$ is canonically hyperbolic on $\mathbf{S}^1 \times I_1$ we can extend the metric $\mathcal{G}_L$ to a smooth metric on the whole $(r+2)$-ball centered at o which is hyperbolic on the $(r-d)$-ball. We do this for every vertex and we are done. Note that for the above construction to work the injectivity radius of the vertices of X^2 must be very large.

(ii) *Codimension two and the Gromov-Thurston trick.*

If N is a closed codimension two totally geodesic submanifold of a hyperbolic manifold (M, g), with trivial normal bundle, then N has a neighborhood $\mathcal{N}_{r+2}$ isometric to $N \times \mathbf{B}_{r+2}$ (where $\mathbf{B}_{r+2} \subset \mathbf{H}^2$ is the ball, centered at $0 \in \mathbf{H}^2$, of radius $r+2$) with metric $\cosh^2(t)h + \sigma_{\mathbf{H}^2}$, where $h = g|_N$, $\sigma_{\mathbf{H}^2}$ is the canonical metric on $\mathbf{H}^2$ and t is the distance to $0 \in \mathbf{H}^2$. We call this metric a *hyperbolic extension of* $\sigma_{\mathbf{H}^2}$. Suppose now that we have a singular metric on M, which is smooth outside N, and on $\mathcal{N}_r - N$ is isometric to $N \times (\mathbf{B}_{r+2} - \{0\})$, with metric $\cosh^2(t)h + \sigma_L$. Then we can smooth the metric g to obtain a smooth metric $\mathcal{G}_N = \mathcal{G}(N, L, r, d)$ by changing g using the smooth metric $\cosh^2(t)h + \mathcal{G}_L$ (where $\mathcal{G}_L$ is as in (i)) instead of the singular metric $\cosh^2(t)h + \sigma_L$. This method was used by Gromov and Thurston [18] to smooth singular metrics obtained using branched covers. The smoothed metric $\cosh^2(t)h + \mathcal{G}_L$ is a *hyperbolic extension of* $\mathcal{G}_L$. Note also that $\mathcal{G}_N$ is hyperbolic on $N \times \mathbf{B}_{r-d}$ and equal to g outside $\mathcal{N}_r$.

(iii) *The Farrell-Jones warping trick.*

Before we deal with the dimension three case we have to discuss the *Farrell-Jones warping trick* which in some sense is a generalization of the Gromov-Thurston trick in dimension 2.

Suppose we have a metric h on the sphere $\mathbf{S}^n$. Consider the warp metric $g = \sinh^2(t)h + dt^2$ on $\mathbf{R}^{n+1} - \{0\} = \mathbf{S}^n \times (0, \infty)$. If $h = \sigma_{\mathbf{S}^n}$, the canonical metric on $\mathbf{S}^n$, then g is hyperbolic and, in particular, smooth everywhere. But for general h the metric g is singular at 0. Before we continue here is an important observation that can easily be deduced from Bishop-O'Neill curvature formula in [3].

$$(\mathbf{0.1}) \qquad \text{Given } \varepsilon > 0 \text{ there is } t_0 \text{ such that the sectional curvatures of } g \text{ at } (x,t) \text{ are within } \varepsilon \text{ of } -1, \text{ provided } t > t_0.$$

To smooth the metric g consider the family of metrics (see [12]):

$$g_\alpha(x,t) = \sinh^2(t)\Big(\big(1-\rho_\alpha(t)\big)\sigma_{\mathbf{S}^n}(x) + \rho_\alpha(t)h(x)\Big) + dt^2$$

where $\rho_\alpha(t) = \rho(\frac{t}{\alpha})$, and $\rho : \mathbf{R} \to [0,1]$ is a smooth function with $\rho(t) = 0$ for $t \leq 1$ and $\rho(t) = 1$ for $t \geq 2$. Hence, for $t \leq \alpha$, the metric g_α is hyperbolic, for $t \geq 2\alpha$ we have $g_\alpha = g$ and in between $t = \alpha$ and $t = 2\alpha$ the metric $\sigma_{\mathbf{S}^n}$ deforms to h. The metrics g_α have two important properties:

(1) they are all hyperbolic for $t \leq \alpha$, hence smooth everywhere,
(2) given $\varepsilon > 0$ there is α_0 such that all sectional curvatures of g_α lie within ε of -1, provided $\alpha > \alpha_0$.

Here is an idea why (2) holds. If α is very large the deformation between $\sigma_{\mathbf{S}^n}$ and h happens very slowly (on the "stretched interval" $[\alpha, 2\alpha]$), so g_α is "almost warped", hence the Bishop-O'Neill formula should give a good approximation of the curvatures of g_α. Therefore, by (0.1) the curvatures of g_α should be close to -1, provided we are far away from 0. But since we are assuming α large, we are in fact far away from 0. Interestingly, the actual proof of (2) given in [12] does not follow exactly this intuitive explanation because there is a more direct proof.

In this paper we need a more elaborate version of the Farrell-Jones trick, which we call *the two variable warping deformation*. We need to know to what extent the "stretching" (which in the Farrell-Jones trick [12] is given by a variable α) to be independent of the "far-away constant" (given in the Farrell-Jones trick also by α). Moreover, we need a more quantitative version also. Here is an important remark.

(**0.2**) Given $\varepsilon > 0$, the stretching and the far-away constants needed in the two variable warping deformation (to obtain an ε-pinched to -1 metric) do depend on the metric h.

(iv) *Dimension three.*

Suppose we have a cubical complex K of dimension 3. As in (i) choose X^3 and construct K_X. Call the piecewise hyperbolic metric on K_X by $\sigma = \sigma_{K_X}$. Again as in (i) the codimension one faces (the 2-faces) of X match nicely and there are singularities only on the "1-skeleton" of K_X, that is, along the edges (i.e. 1-faces) and vertices (i.e. 0-faces). The singularities along the 1-faces can be smoothed using the Gromov-Thurston trick as in (i) and (ii), i.e. using smoothing in dimension two plus hyperbolic extension. In this way we obtain a metric σ' which is smooth near (part of) the edges. Let $\mathcal{N}_{r+2}(e)$ be the normal neighborhood of width $r+2$ of the edge e. Then $\sigma = \sigma'$ outside the union $\bigcup_{e\,\text{edge}} \mathcal{N}_{r+2}(e)$. Notice that there is some ambiguity in the definition of the metric σ' because for different edges e, e' with a common vertex the neighborhoods $\mathcal{N}_{r+2}(e)$, $\mathcal{N}_{r+2}(e')$ have nonempty intersection. So σ' is only well-defined outside the s-neighborhoods (i.e. s-balls) $\mathcal{N}_s(o)$ of the vertices o, where s is large enough. Let $\mathrm{L}(e)$ be the number of copies of X that contain the edge e, and write $\mathcal{G}_e = \mathcal{G}_{\mathrm{L}(e)}$ and $\sigma_e = \sigma_{\mathrm{L}(e)}$. Therefore on each $\mathcal{N}_{r+2}(e)$, and outside $\bigcup_{o\,\text{vertex}} \mathcal{N}_s(o)$, the metric σ' is equal to the metric $\cosh^2(t)\sigma_{\mathbf{R}} + \mathcal{G}_e$ which is the hyperbolic extension of the metric $\mathcal{G}_e$.

We are left to smooth the metric near the vertices. Fix a vertex o. Let $\{e_i\}$ be the edges containing o and write $e = e_1$. Let P be the link of o. Then P is a PL-sphere of dimension 2 and it has a natural all-right spherical metric σ_{P}, that is, σ_{P} is the piecewise spherical metric with all edges in P having length $\pi/2$. Note that the metric σ near o is the warped piecewise hyperbolic metric $\sinh^2(s)\sigma_{\mathrm{P}} + ds^2$, where s is the distance to o. Write $\mathrm{L}_i = \mathrm{L}(e_i)$ and $\mathrm{L} = \mathrm{L}_1$. Then the metric σ on $\mathcal{N}_{r+2}(e_i)$ is equal to the hyperbolic extension metric $\cosh^2(t_i)\sigma_{\mathbf{R}} + \sigma_{e_i}$, where t_i is the distance to e_i. Hence σ is sinh-warped from o and cosh-warped from each e_i (near e_i).

What about the metric σ'? It is also cosh-warped from e_i because, by definition, the metric σ' is equal to $\cosh^2(t_i)\sigma_{\mathbf{R}} + \mathcal{G}_{e_i}$ near e_i (i.e. on $\mathcal{N}_{r+2}(e_i) - \bigcup_{o'\,\text{vertex}} \mathcal{N}_s(o')$). But, and this is a key observation, the metric σ' is **not**, in general, warped from o. (Even though σ' is undefined on a neighborhood of o it could still be warped from o away from that neighborhood.) Here is a heuristic idea why this is so. Write $\mathcal{E}_i = \cosh^2(t_i)\sigma_{\mathbf{R}} + \mathcal{G}_{e_i}$ and $\mathcal{E} = \mathcal{E}_1$. Note that $\mathcal{E}$ has rotational symmetry, that is it is invariant by rotations fixing $e = e_1$. Let $d = d_1$ as in (i) and (ii), corresponding to $e = e_1$. Let H be a plane containing e. Since $\mathcal{E}$ has rotational symmetry H is totally geodesic. Then the boundaries of $\mathcal{N}_r(e)$, $\mathcal{N}_{r-d}(e)$ intersect H in two lines each. Let $p \in \mathrm{H} \cap \mathcal{N}_{r+2}(e)$, $p \notin e$, and $x \in e \subset \mathrm{H}$ be the closest point in e to p. Also let v a vector at p perpendicular to H, and denote the circle centered at x, perpendicular to H and passing through p (hence tangent to v) by $\mathrm{S}(p)$. Let $t = t(p)$ be the distance from p to x and $s = s(p)$ the distance from o to p. Now, it can be checked from the definitions that the metrics σ and σ' coincide on vectors tangent to H. They differ on their values on the vectors v as above. These values are directly proportional to the lengths $\ell(\mathrm{S}(p))$, $\ell'(\mathrm{S}(p))$ of the circle $\mathrm{S}(p)$ with respect to the metrics σ and σ', respectively. Hence these metrics can be understood by looking at these lengths. We have $\ell(\mathrm{S}(p)) = 2\pi \frac{\mathrm{L}}{4} \sinh(t)$ and $\ell'(\mathrm{S}(p)) = 2\pi\rho(t)\sinh(t)$ (see (i)). Let θ be the angle at o between e and the geodesic segment $\beta = [o, p]$ (which lies on H). Let $p(s)$ be the point in β at distance s from o. Now if σ' were sinh-warped from o the lengths of the circles $\mathrm{S}(s) = \mathrm{S}(p(s))$ would have the form $c \sinh(s)$ for some constant c. But from the hyperbolic law of sines we have $\sin\theta = \frac{\sinh t}{\sinh s}$, hence $t(s) = \sinh^{-1}(\sin\theta \sinh s)$ and we get

$$\ell'\big(\mathrm{S}(s)\big) = 2\pi\rho\big(t(s)\big)\sinh\big(t(s)\big) = 2\pi\rho\big(t(s)\big)\sin\theta\sinh s \qquad (\mathbf{0.3})$$

Note that if $L = 4$, then $\rho \equiv 1$ and the formula above shows why hyperbolic three space $\mathbf{H}^3$ is at the same time sinh-warped from a point o and cosh-warped from a line containing o. But in general $\rho(t(s))$ is not a constant, hence σ' is not, in general, sinh-warped from o, as we wanted to show. Note that $\rho(t)$ is constant for $t \notin [r-d, r]$.

Why do we want σ' to be sinh-warped from o? Because in this case we could apply two variable warping deformation (see (0.3)) and force/extend the metric σ' to be hyperbolic near o, hence smooth near o. (Even in this case there would be a problem in using two variable warping deformation because of (0.2), but more on this in a moment.) Now, even though σ' is not warped from o it is "very close to being warped", provided r and d are large. Here is an idea why this is true. Since $\sin\theta = \frac{\sinh t}{\sinh s}$, if t is large, so is s and we get $s \approx t - \ln\sin\theta$, and if r and d are large then the function $\rho(t(s))$ in (0.3) even though is not constant, it does in this case change very slowly, hence behaves (locally) almost like a constant. Therefore σ' is "almost warped" in this case, and we can "deform" σ' to a sinh-warped from o metric. We call this process *warp forcing*. Therefore the idea is to first warp force the metric σ' near o and then use the two variable warping deformation to make it hyperbolic near o, hence smooth. In our particular case the sinh-warped metric to which we deform σ' is $\sinh^2(s)\hat{h}_{s_0} + ds^2$, where $\hat{h}_{s_0} = \frac{1}{\sinh s_0} h_{s_0}$, $h_{s_0} = \sigma'|_{\mathbf{S}_{s_0}}$, and $\mathbf{S}_{s_0} = \mathbf{S}_{s_0}(o)$ is the sphere of radius s_0 in K_X centered at o (recall X is as large as needed).

Now we deal with the problem mentioned above. Suppose we succeeded in warp forcing the metric σ' and obtained the sinh-warped from o metric $\sinh^2(s)\hat{h}_{s_0} + ds^2$. Recall that we needed to assume r and d large. By (0.2) the constants needed for two variable warping deformation (call them α_1, α_2) depend on $\hat{h}_{s_0}$, which in turn depend on r, d. It may happen that the $\alpha_i = \alpha_i(r, d)$ are too big for s_0 and we have no space to use the two variable warping deformation. And in fact this may happen if we do not do things in a precise way. To solve this problem we proceed in the following way. Fix an angle $\theta_0 > 0$ and d large as needed but fixed. Consider the plane H as before. Let q be a point in H at distance r from e such that the geodesic (in H, recall H is totally geodesic) $[o, q]$ makes an angle θ_0 at o. Let $s = s(r)$ be the distance from o to q. Now let $p = p(r)$ be the point in H such that the distance from p to o is also s, and the distance to e is $r - d$. Let $\theta_1(r)$ be the angle at o between e and $[o, p]$.

It can be shown with a straightforward calculation that in this particular case the corresponding metrics $\hat{h}_{s(r)}$ C^2-converge to a smooth metric $\hat{h}$. The angles $\theta_1(r)$ also converge. (Here θ_0 is fixed.) And it can be shown that in this case (0.2) does not pose a problem any more because all metrics obtained are in fact very close, so the corresponding constants α_i are close. In particular they do not grow indefinitely. And this is what we needed. Consequently, we first take d very large (for curvature and other considerations) and then take r large so that everything else works, and during all this process we have to make sure to be far enough away from e, and this is given by the constant $\theta_0 > 0$.

Here is a brief description of the paper. In Section 1 we introduce some notation and basic concepts, including the definition of ε-close to hyperbolic metrics. This is a slightly technical but important concept. The idea is to try to measure how close a metric is to being hyperbolic; we do this in a chart by chart fashion. In Section 2 we define and study the "hyperbolic extension" of a metric (or space), which is a key geometric construction. In this section there are no proofs and we essentially collect the main results of [32]. In Section 3 we describe another key geometric construction, hyperbolic forcing; it is the composition of two deformations: warp forcing and the two-variable deformation, which are studied with more detail in [29] and [27], respectively. The results of these two papers are put together in [33]. Section 4 is a family version of Section 3. Again, in Section 4 there are essentially no proofs and we mostly collect the main results of [29], [27], and [33]. In Section 5 we study neighborhoods of simplices of all-right spherical complexes. In this section we introduce a technical device that we called *sequence of widths*. These are sets of positive real numbers that are used as widths for normal neighborhoods of simplices of all-right spherical complexes. We prove that there are sets of widths, independent of the complex, that satisfy very useful properties. These are fundamental objects that make all matching processes work. Section 6 is a sort of a "cone version" of Section 5; in it we study (all-right) piecewise hyperbolic cone complexes, which are just cones over all-right spherical complexes with the metric warped by sinh. In Section 7 we deal with the smoothing issue for cubical and all-right spherical complexes; here we collect the main concepts and results of [28]. We put everything together in Section 8 to smooth hyperbolic cones. Section 9 is dedicated to the Charney-Davis strict hyperbolization process; in this section we collect the results in [31], in particular we mention that strictly hyperbolized smooth manifolds have "normal differentiable structures". Finally we prove the Main Theorem in Section 10 and Theorem A in Section 11. Subsections at the end of Sections 7, 8, 9 deal with generalizations to the case of manifolds with codimension zero singularities. These subsections are used in Section 11.

1. Some notation, definitions, and metrics ε-close to hyperbolic

In this paper ρ will denote a fixed smooth function $\rho : \mathbf{R} \to [0, 1]$ such that: (i) $\rho|_{(-\infty,0+\delta]} \equiv 0$, and (ii) $\rho|_{[1-\delta,\infty)} \equiv 1$, where $\delta > 0$ is small.

Let $A \subset \mathbf{R}^n$ be an open set. Let $|.|_{C^2(A)}$ denote the uniform C^2-norm of $\mathbf{R}^l$-valued functions on A, i.e. if $f = (f_1, \ldots, f_l) : A \to \mathbf{R}^l$, then $|f|_{C^2(A)} = \sup_{z\in A, 1\leq i\leq l, 1\leq j,k\leq n}\{|f_i(z)|, |\partial_j f_i(z)|, |\partial_{j,k} f_i(z)|\}$. Sometimes we will write $|.|_{C^2} = |.|_{C^2(A)}$ when the context is clear. Given a Riemannian metric g on A, the number $|g|_{C^2(A)}$ is computed considering g as the $\mathbf{R}^{n^2}$-valued function $z \mapsto (g_{ij}(z))$ where, as usual, $g_{ij} = g(e_i, e_j)$, and the e_i's are the canonical vectors in $\mathbf{R}^n$.

Let M^n be a complete Riemannian manifold with metric h. We say that a point $o \in M$ is a *center of* M if the exponential map $\exp_o : T_oM \to M$ is a diffeomorphism. In particular M is diffeomorphic to $\mathbf{R}^n$. For instance if M is Hadamard manifold every point is a center. In this paper we will always use the same symbol "o" to denote a chosen center of a Riemannian manifold, unless it is necessary to specify the manifold M, in which case we will write o_{M}. Using the diffeomorphism $\exp_o$ onto M and an identification of $T_o M$ with $\mathbf{R}^n$ via some fixed choice of an orthonormal basis in $T_o M$, we can identify M with $\mathbf{R}^n$ and $M - \{o\}$ with $\mathbf{S}^{n-1} \times \mathbf{R}^+$. By the Gauss lemma the metric h of M, restricted to $M - \{o\}$, can be written as $h_t + dt^2$ on $\mathbf{S}^{n-1} \times \mathbf{R}^+$, where $\{h_t\}_{t>0}$ is a one-parameter family of metrics on $\mathbf{S}^{n-1}$. Moreover, the set of curves $t \mapsto (x, t) \in \mathbf{S}^{n-1} \times \mathbf{R}^+$ are speed one h-geodesics on $M - \{o\} = \mathbf{S}^{n-1} \times \mathbf{R}^+$, and we call this set *the set of rays of* M *with respect to* o, or simply *the set of rays of* M, if the center is understood.

Remarks **1.1.**

1. We will need a partial version of the concept of sets of rays. Let $U \subset M$ be an open set and let f be a metric on U. We say that f is *ray compatible with* (M, o) *over* U if the following two conditions hold. First, the restriction of every speed one geodesic of M to U is a speed one f-geodesic. Second, each of these restrictions is f perpendicular to the spheres of M centered at o.
2. If $U = M$, that is f is globally defined, then f is ray compatible with (M, o) if and only if $M = (M, h)$ and (M, f) have the same set of rays.
3. The metric f defined on U is ray compatible with (M, o) if and only if we can write $f = f_r + dr^2$ on $U \cap (M \setminus \{o\}) = U \cap (\mathbf{S}^{n-1} \times \mathbf{R}^+)$.
4. Trivially, for any U, $h|_{\mathrm{U}}$ is ray compatible with (M, o).
5. If f is ray compatible with (M, o) over U and V, then f is ray compatible with (M, o) over $U \cup V$.

The standard flat metric on $\mathbf{R}^l$ will be denoted by $\sigma_{\mathbf{R}^l}$. Similarly, $\sigma_{\mathbf{H}^l}$ and $\sigma_{\mathbf{S}^{l-1}}$ will denote the standard hyperbolic and round metrics on $\mathbf{H}^l$ and $\mathbf{S}^{l-1}$, respectively.

Let $\mathbf{B} = \mathbf{B}^{l-1} \subset \mathbf{R}^{l-1}$ be the unit ball, with the metric $\sigma_{\mathbf{R}^{l-1}}$. Write $I_\xi = (-1 - \xi, 1 + \xi) \subset \mathbf{R}$, $\xi > 0$. Our basic models are $\mathbf{T}^l_\xi = \mathbf{T}_\xi = \mathbf{B} \times I_\xi \subset \mathbf{R}^l$, with hyperbolic metric $\sigma = e^{2t}\sigma_{\mathbf{R}^{l-1}} + dt^2$. In what follows we may sometimes suppress the subindex ξ, if the context is clear. The number ξ is the *excess* of $\mathbf{T}_\xi$.

Remarks.

1. One of the reasons to introduce the excess is that the process of hyperbolic extension (see Section 2) decreases the excess of the charts, as shown in the statement of Theorem 2.7.
2. In the applications we may actually need warp product metrics with warping functions that are multiples of hyperbolic functions. All these functions are close to the exponential e^t (for t large), so instead of introducing one model for each hyperbolic function we introduced only the exponential model.

Let $\varepsilon > 0$. A Riemannian manifold (M^l, g) is ε*-close to hyperbolic* if there is $\xi > 0$ such that for every $p \in M$ there is an ε*-close to hyperbolic chart with center* p *and excess* ξ, that is, there is a chart $\phi : \mathbf{T}_\xi \to M$, $\phi(0, 0) = p$, with $|\phi^* g - \sigma|_{C^2(\mathbf{T}_\xi)} < \varepsilon$. Note that all charts are defined on the same model space $\mathbf{T}_\xi$. More generally, a subset $S \subset M$ is ε-close to hyperbolic if every $p \in S$ is the center of an ε-close to hyperbolic chart in M with fixed excess ξ.

If N^l has center o we say that $S \subset N$ is *radially ε-close to hyperbolic (with respect to o)* if there is ξ such that for every $p \in S$ there is a *radially ε-close to hyperbolic chart ϕ with center p and excess ξ*, where the latter means that there is an $a \in \mathbf{R}$ and an ε-close to hyperbolic chart ϕ with center p and excess ξ such that for every t the projection of $\phi(.,t)$ on the $\mathbf{R}^+$-factor of $N - \{o\} = \mathbf{S}^{l-1} \times \mathbf{R}^+$ is $t + a$. Here the "radial" directions are $(-1-\xi, 1+\xi)$ and $\mathbf{R}^+$ in $\mathbf{T}_\xi$ and $N - \{o\} = \mathbf{S}^{l-1} \times \mathbf{R}^+$, respectively.

Remarks **1.2.**

1. The definition of radially ε-close to hyperbolic metrics is well suited to studying metrics of the form $g_t + dt^2$ for t large, but for small t this definition is not useful because we need some space to fit the charts. An undesired consequence is that even punctured hyperbolic space $\mathbf{H}^n - \{o\} = \mathbf{S}^{n-1} \times \mathbf{R}^+$ (with warp product metric $\sinh^2 t\sigma_{\mathbf{S}^{n-1}} + dt^2$) is not radially ε-close to hyperbolic for t small. In fact there is $\mathsf{a} = \mathsf{a}(n,\varepsilon)$ such that hyperbolic n-space is ε-close to hyperbolic for $t > \mathsf{a}$ (and not for all $t \le \mathsf{a}$), see Corollary 4.14 [27]. This is not essential for what follows.
2. For every n there is a function $\varepsilon' = \varepsilon'(\varepsilon, \xi, n)$ such that: if a Riemannian metric g on a manifold M^n is ε'-close to hyperbolic, with charts of excess ξ, then the sectional curvatures of g all lie ε-close to -1. This choice is possible, and depends only on n and ξ, because the curvature depends only of the derivatives up to order 2 of $\phi^* g$ on $\mathbf{T}_\xi$, where ϕ is an ε-close to hyperbolic chart with excess ξ.

Lemma **1.3.** — *Let $\phi : \mathbf{T}_\xi \to M$ be a radially ε-close to hyperbolic chart with center $p \in M$. Then*

$$d_M\big(\phi(q), p\big) \le 2 + \xi + n^2\varepsilon$$

for every $q \in \mathbf{T}_\xi$.

Proof. — Write $q = (x_0, t_0) \in \mathbf{B} \times I_\xi$. Consider the path $\alpha(t) = (tx_0, 0)$, $t \in [0,1]$, $\beta(t) = (x_0, tt_0)$, $t \in [0,1]$, and $\gamma = \alpha * \beta$. Write $g' = \phi^* g$ and we have $g' = \sigma + h$, with $|h|_{C^2(\mathbf{T}_\xi)} < \varepsilon$. Then the g-length $\ell_g(\phi \circ \gamma)$ of $\phi \circ \gamma$ is $\ell_{g'}(\gamma) = \ell_{g'}(\alpha) + \ell_{g'}(\beta) \le \ell_\sigma(\alpha) + \ell_h(\alpha) + (1+\xi) \le 1 + \varepsilon n^2 + (1+\xi)$. Hence $d_M(\phi(q), p) \le \ell_g(\phi \circ \gamma) \le 2 + \xi + n^2\varepsilon$. □

Next we deal with a natural and useful class of metrics. These are metrics on $\mathbf{R}^n$ (or on a manifold with center) that are already hyperbolic on the closed ball $\bar{B}_a = \bar{B}_a(0)$ of radius a centered at 0, and are radially ε-close to hyperbolic outside $\bar{B}_{a'}$ (here a' is slightly less than a). Here is the detailed definition. Let M^n have center o and let $B_a = B_a(o)$, $\bar{B}_a = \bar{B}_a(o)$ be the open and closed balls in M of radius a centered at o, respectively. We say that a metric h on M is (B_a, ε)-*close to hyperbolic, with charts of excess ξ*, if

(1) On $\bar{B}_a - \{o\} = \mathbf{S}^{n-1} \times (0, a)$ we have $h = \sinh^2 t\sigma_{\mathbf{S}^{n-1}} + dt^2$. Hence h is hyperbolic on $\bar{B}_a$.
(2) The metric h is radially ε-close to hyperbolic outside $\bar{B}_{a-1-\xi}$, with charts of excess ξ.

Remarks **1.4.**

1. We have dropped the word "radially" to simplify the notation. But it does appear in condition (2), where "radially" refers to the center of B_a.
2. We will always assume $a > \mathsf{a} + 1$, where a is as in Remark 1.2(1).
3. Let ε' be as in Remark 1.2(2). Then the following is also true: if a Riemannian metric g on a manifold M^n is (B_a, ε')-*close to hyperbolic, with charts of excess* ξ, then the sectional curvatures of g all lie ε-close to -1.
4. If a metric is (B_a, ε)-close to hyperbolic with charts of excess ξ then it is (B_a, ε)-close to hyperbolic with charts of excess ξ', with $0 < \xi' \leq \xi$.

Let $\mathsf{c} > 1$. A metric g on a compact manifold M is c-*bounded* if $|g|_{C^2(M)} < \mathsf{c}$ and $|\det g|_{C^0(M)} > 1/\mathsf{c}$. A set of metrics $\{g_\lambda\}$ on the compact manifold M is c-*bounded* if every g_λ is c-bounded.

Remarks **1.5.**

1. Here the uniform C^k-norm $|.|_{C^k}$ is taken with respect to a fixed finite atlas $\mathcal{A}$.
2. We will assume that the finite atlas $\mathcal{A}$ is "nice", that is, it has"extendable" charts, i.e. charts that can be extended to the (compact) closure of their domains.

2. Hyperbolic extensions

Recall that hyperbolic n-space $\mathbf{H}^n$ is isometric to $\mathbf{H}^k \times \mathbf{H}^{n-k}$ with warp product metric $(\cosh^2 r)\sigma_{\mathbf{H}^k} + \sigma_{\mathbf{H}^{n-k}}$, where $\sigma_{\mathbf{H}^l}$ denotes the hyperbolic metric of $\mathbf{H}^l$, and $r : \mathbf{H}^{n-k} \to [0, \infty)$ is the distance to a fixed point in $\mathbf{H}^{n-k}$. For instance, in the case $n = 2$, since $\mathbf{H}^1 = \mathbf{R}^1$ we have that $\mathbf{H}^2$ is isometric to $\mathbf{R}^2 = \{(u, v)\}$ with warp product metric $\cosh^2 v\,du^2 + dv^2$. In the following paragraph we give a generalization of this construction.

Let (M^n, h) be a complete Riemannian manifold with *center* $o = o_M \in M$. The warp product metric

$$g = \big(\cosh^2 r\big)\sigma_{\mathbf{H}^k} + h$$

on $\mathbf{H}^k \times M$ is the *hyperbolic extension (of dimension k)* of the metric h. Here r is the distance-to-o function on M. We write $\mathcal{E}_k(M, h) = (\mathbf{H}^k \times M, g)$, and $g = \mathcal{E}_k(h)$. We also say that $\mathcal{E}_k(M) = \mathcal{E}_k(M, h)$ is the *hyperbolic extension (of dimension k) of* (M, h) (or just of M). Hence, for instance, we have $\mathcal{E}_k(\mathbf{H}^l) = \mathbf{H}^{k+l}$. For $S \subset M$ and $A \subset \mathbf{H}^k$ we define the *partial hyperbolic* *extension* $\mathcal{E}_A(S) = A \times S \subset \mathcal{E}_k(M)$. Also write $\mathbf{H}^k = \mathbf{H}^k \times \{o_M\} \subset \mathcal{E}_k(M)$; any $p \in \mathbf{H}^k$ is a center of $\mathcal{E}_k(M)$ (see Remark 3.3(1) in [32] or 2.3 below).

In this paper *convex subset* means specifically the following. A subset S of a length metric space X (e.g. a Riemannian manifold) is convex in X if every two points in S can be joined by a minimizing geodesic contained in S.

Let η be a complete geodesic line in M passing though o and let η^+ be one of its two geodesic rays (beginning at o). Then η is a totally geodesic subspace of M and η^+ is convex (see (ii) of Section 3 in [32]). Also, let γ be a complete geodesic line in $\mathbf{H}^k$. The following two results are proved in [32] (Lemma 3.1 and Corollary 3.2 in [32], respectively).

Lemma **2.1.** — *The subset $\gamma \times \eta^+$ is a convex subset of $\mathcal{E}_k(M)$ and $\gamma \times \eta$ is totally geodesic in $\mathcal{E}_k(M)$.*

Corollary **2.2.** — *The subsets $\mathbf{H}^k \times \eta^+$ and $\gamma \times M$ are convex in $\mathcal{E}_k(M)$. Also $\mathbf{H}^k \times \eta$ is totally geodesic in $\mathcal{E}_k(M)$.*

We also have that $\mathbf{H}^k$ and every $\{y\} \times M$ are convex in $\mathcal{E}_k(M)$ (see Section 3 in [32]).

Remarks **2.3.**

1. Note that $\mathbf{H}^k \times \eta$ (with the metric induced by $\mathcal{E}_k(M)$) is isometric to $\mathbf{H}^k \times \mathbf{R}$ with warp product metric $\cosh^2 v\sigma_{\mathbf{H}^k} + dv^2$, which is just hyperbolic $(k+1)$-space $\mathbf{H}^{k+1}$. Also $\gamma \times \eta$ is isometric to $\mathbf{R} \times \mathbf{R}$ with warp product metric $\cosh^2 v du^2 + dv^2$, which is just hyperbolic 2-space $\mathbf{H}^2$. In particular every point in $\mathbf{H}^k = \mathbf{H}^k \times \{o\} \subset \mathcal{E}_k(M)$ is a center.
2. Recall that the concept of *sets of rays* was introduced in Section 1. It follows from Lemma 2.1 and Remark 2.3(1) that the set of rays of $\mathcal{E}_k(h)$ with respect to any center $o_{\mathbf{H}^k} \in \mathbf{H}^k \subset \mathcal{E}_k(M)$ only depends on the set of rays of M and the center $o_{\mathbf{H}^k}$. That is, if h and h', defined on M, have the same sets of rays with respect to o, then $\mathcal{E}_k(h)$, $\mathcal{E}_k(h')$ have the same sets of rays with respect to any $o \in \mathbf{H}^k \subset \mathcal{E}_k(M)$.
3. Denote by $\mathbf{B}_r(M)$ the ball of radius r of M centered at o. Note that if h and h' on M have the same sets of rays then the balls $\mathbf{B}_r(M)$ coincide.
4. Recall that $\mathbf{H}^k$ is convex in $\mathcal{E}_k(M)$. Moreover, for $l \leq k$, let H be a convex subset of $\mathbf{H}^k$ isometric to $\mathbf{H}^l$. If h and h' on M have the same sets of rays then the r-neighborhoods (with respect to h and h') of the convex subset H in $\mathcal{E}_k(M)$ coincide.

As before (see Section 1) we use h to identify $M - \{o\}$ with $\mathbf{S}^{n-1} \times \mathbf{R}^+$. Sometimes we will denote a point $v = (u, r) \in \mathbf{S}^{n-1} \times \mathbf{R}^+ = M - \{o\}$ by $v = ru$. Fix a center $o \in \mathbf{H}^k \in \mathcal{E}_k(M)$. Since $\mathbf{H}^k$ is convex in $\mathcal{E}_k(M)$ we can write $\mathbf{H}^k - \{o\} = \mathbf{S}^{k-1} \times \mathbf{R}^+ \subset \mathbf{S}^{k+n-1} \times \mathbf{R}^+$

and $\mathbf{S}^{k-1} \subset \mathbf{S}^{k+n-1}$. Then, for $y \in \mathbf{H}^k - \{o\}$ we can also write $y = tw$, $(w, t) \in \mathbf{S}^{k-1} \times \mathbf{R}^+$. Similarly, using the exponential map we can identify $\mathcal{E}_k(\mathrm{M}) - \{o\}$ with $\mathbf{S}^{k+n-1} \times \mathbf{R}^+$, and for $p \in \mathcal{E}_k(\mathrm{M}) - \{o\}$ we can write $p = sx$, $(x, s) \in \mathbf{S}^{k+n-1} \times \mathbf{R}^+$.

A point $p \in \mathcal{E}_k(\mathrm{M}) - \mathbf{H}^k$ has two sets of coordinates: the *polar coordinates* $(x, s) = (x(p), s(p)) \in \mathbf{S}^{k+n-1} \times \mathbf{R}^+$ and the *hyperbolic extension coordinates* $(y, v) = (y(p), v(p)) \in \mathbf{H}^k \times \mathrm{M}$. Write $\mathrm{M}_o = \{o\} \times \mathrm{M}$. Therefore we have the following functions:

the distance to o function:	$s : \mathcal{E}_k(\mathrm{M}) \to [0, \infty)$,	$s(p) = d_{\mathcal{E}_k(\mathrm{M})}(p, o)$
the direction of p function:	$x : \mathcal{E}_k(\mathrm{M}) - \{o\} \to \mathbf{S}^{n+k-1}$	$p = s(p)x(p)$
the distance to $\mathbf{H}^k$ function:	$r : \mathcal{E}_k(\mathrm{M}) \to [0, \infty)$,	$r(p) = d_{\mathcal{E}_k(\mathrm{M})}(p, \mathbf{H}^k)$
the projection on $\mathbf{H}^k$ function:	$y : \mathcal{E}_k(\mathrm{M}) \to \mathbf{H}^k$,	
the projection on M function:	$v : \mathcal{E}_k(\mathrm{M}) \to \mathrm{M}$,	
the projection on $\mathbf{S}^{n-1}$ function:	$u : \mathcal{E}_k(\mathrm{M}) - \mathbf{H}^k \to \mathbf{S}^{n-1}$	$v(p) = r(p)u(p)$
the length of y function:	$t : \mathcal{E}_k(\mathrm{M}) \to [0, \infty)$,	$t(p) = d_{\mathbf{H}^k}(y(p), o)$
the direction of y function:	$w : \mathcal{E}_k(\mathrm{M}) - \mathrm{M}_o \to \mathbf{S}^{k-1}$	$y(p) = t(p)w(p)$

Note that $r = d_{\mathrm{M}}(v, o)$. Note also that, by Lemma 2.1, the functions w and u are constant on geodesics emanating from $o \in \mathcal{E}_k(\mathrm{M})$, that is $w(sx) = w(x)$ and $u(sx) = u(x)$.

Let ∂_r and ∂_s be the gradient vector fields of r and s, respectively. Since the M-fibers $\mathrm{M}_y = \{y\} \times \mathrm{M}$ are convex the vectors ∂_r are the velocity vectors of the speed one geodesics of the form $a \mapsto (y, au)$, $u \in \mathbf{S}^{n-1} \subset \mathrm{M}$. These geodesics emanate from (and orthogonally to) $\mathbf{H}^k \subset \mathcal{E}_k(\mathrm{M})$. Also the vectors ∂_s are the velocity vectors of the speed one geodesics emanating from $o \in \mathcal{E}_k(\mathrm{M})$. For $p \in \mathcal{E}_k(\mathrm{M})$, denote by $\triangle = \triangle(p)$ the right triangle with vertices $o, y = y(p), p$ and sides the geodesic segments $[o, p] \in \mathcal{E}_k(\mathrm{M})$, $[o, y] \in \mathbf{H}^k$, $[p, y] \in \{y\} \times \mathrm{M} \subset \mathcal{E}_k(\mathrm{M})$. (These geodesic segments are unique and well-defined because: (1) $\mathbf{H}^k$ is convex in $\mathcal{E}_k(\mathrm{M})$, (2) $(y, o) = o_{\{y\}\times\mathrm{M}}$ and o are centers in $\{y\} \times \mathrm{M}$ and $\mathbf{H}^k \subset \mathcal{E}_k(\mathrm{M})$, respectively.)

Let $\alpha : \mathcal{E}_k(\mathrm{M}) - \mathbf{H}^k \to [0, \pi]$ be the angle between ∂_s and ∂_r, thus $\cos\alpha = g(\partial_r, \partial_s)$. Then $\alpha = \alpha(p)$ is the interior angle, at $p = (y, v)$, of the right triangle $\triangle = \triangle(p)$. We call $\beta(p)$ the interior angle of this triangle at o, that is $\beta(p) = \beta(x)$ is the spherical distance between $x \in \mathbf{S}^{k+n-1}$ and the totally geodesic sub-sphere $\mathbf{S}^{k-1}$. Alternatively, β is the angle between the geodesic segment $[o, p] \subset \mathcal{E}_k(\mathrm{M})$ and the convex submanifold $\mathbf{H}^k$. Therefore β is constant on geodesics emanating from $o \in \mathcal{E}_k(\mathrm{M})$, that is $\beta(sx) = \beta(x)$. The following corollary follows from Lemma 2.1 (or see Lemma 4.1 in [32]).

Corollary **2.4.** — *Let η^+ (or η) be a geodesic ray (line) in* M *through o containing $v = v(p)$ and γ a geodesic line in $\mathbf{H}^k$ through o containing $y = y(p)$. Then $\triangle(p) \subset \gamma \times \eta^+ \subset \gamma \times \eta$.*

Note that the right geodesic triangle $\triangle(p)$ has sides of length $r = r(p)$, $t = t(p)$ and $s = s(p)$. By Lemma 2.1 and Remark 2.3 we can consider $\triangle$ as contained in hyperbolic 2-space. Hence using hyperbolic trigonometric identities we can find relations between r, t, s, α and β. For instance, using the hyperbolic law of cosines we get: $\cosh(s) = \cosh(r)\cosh(t)$. Note that this implies $t \le s$. Here is an application of this equation.

Proposition **2.5** *(Iterated hyperbolic extensions). — The following identity holds*

$$\mathcal{E}_l\big(\mathcal{E}_k(\mathrm{M})\big) = \mathcal{E}_{l+k}(\mathrm{M}),$$

where we are identifying $\mathbf{H}^{l+k}$ *with* $\mathbf{H}^l \times \mathbf{H}^k$ *with warp product metric* $(\cosh^2 t)\sigma_{\mathbf{H}^l} + \sigma_{\mathbf{H}^k}$.

This proposition is proved in [32] (it is Proposition 4.1 in [32]).

Remarks **2.6.**

1. Note that the identification of $\mathbf{H}^{l+k}$ with $\mathbf{H}^l \times \mathbf{H}^k$ (as a warp product) depends on the order of l and k, that is, on the order in which the hyperbolic extensions are taken.
2. As before, here the function $t : \mathbf{H}^k \to [0, \infty)$ is the distance in $\mathbf{H}^k$ to the point $o \in \mathbf{H}^k$.

We next explore the relationship between hyperbolic extensions and metrics ε-close to hyperbolic. Since $\mathcal{E}_k(\mathbf{H}^l) = \mathbf{H}^{k+l}$ one would expect that if M is "close" to $\mathbf{H}^l$, then $\mathcal{E}_k(\mathrm{M})$ would be close to $\mathbf{H}^{k+l}$. This motivates the following question.

Question. — What can we say about the hyperbolic extension of a $(\mathrm{B}_a, \varepsilon)$*-close to hyperbolic metric?*

The next result answers this question; it is Theorem B in [32].

Theorem **2.7.** — *Let* M^n *have center* o. *Assume* M *is* $(\mathrm{B}_a, \varepsilon)$*-close to hyperbolic, with charts of excess* $\xi > 0$. *Then* $\mathcal{E}_k(\mathrm{M})$ *is* $(\mathrm{B}_a, \mathrm{C}\varepsilon)$*-close to hyperbolic, with charts of excess* ξ', *provided* a *is sufficiently large. Explicitly we want*

$$a \ge \mathrm{R} = \mathrm{R}(\varepsilon, k, \xi).$$

Here $\mathrm{C} = \mathrm{C}(n, k, \xi)$, *and* $\xi' = \xi - e^{-a/2} > 0$.

Explicit formulas for C and R are given in [32] (the constant C here is called C_2 in [32]). Note that the excess of the charts decreases. This is one of the main reasons to introduce the excess. In Section 3 (see also [29]) we describe another geometric process, warp forcing, which also reduces the excess of the charts.

3. Deformations of metrics

The goal of this section is to describe the "hyperbolic forcing" method. It has as input a metric on $\mathbf{R}^n$ of the form $g = g_r + dr^2$ (or, more generally a metric on a manifold with center) and as output a metric still of the form $h_r + dr^2$, but which is hyperbolic on a ball centered at the origin.

Hyperbolic forcing is defined as the composition of two other metric deformations: the two variable deformation and warp forcing. We present these first.

In this section $\mathrm{M}^n = (\mathrm{M}^n, g)$ is a Riemannian manifold with center o. As before we identify M with $\mathbf{R}^n$ and $\mathrm{M} - \{o\}$ with $\mathbf{S}^{n-1} \times \mathbf{R}^+$. Therefore on $\mathrm{M} - \{o\} = \mathbf{S}^{n-1} \times \mathbf{R}^+$ we can write $g = g_r + dr^2$. Also B_a and $\bar{\mathrm{B}}_a$ will denote the open and closed balls in $\mathrm{M} = \mathbf{R}^n$ of radius a centered at $o = 0$, respectively.

3.1. *The two variable warping deformation*

Let g'' be a metric on $\mathbf{S}^{n-1}$ and consider the warp product metric $g' = \sinh^2 r g'' + dr^2$ on $\mathbf{S}^{n-1} \times \mathbf{R}^+$. Recall that $\rho : \mathbf{R} \to [0, 1]$ is a fixed smooth function with $\rho(r) = 0$ for $r \leq 0$ and $\rho(r) = 1$ for $r \geq 1$. Given positive numbers a and d define $\rho_{a,d}(r) = \rho(2\frac{r-a}{d})$. Also fix an atlas $\mathcal{A}_{\mathbf{S}^n}$ on $\mathbf{S}^{n-1}$ as before (see Remark 1.5). All norms and boundedness constants will be taken with respect to this atlas. Recall that $\sigma_{\mathbf{S}^{n-1}}$ is the round metric on $\mathbf{S}^{n-1}$. Write

$$g'_r = \big(1 - \rho_{a,d}(r)\big)\sigma_{\mathbf{S}^{n-1}} + \rho_{a,d}(r) g''$$

and define the metric

$$\mathcal{T}_{a,d} g' = \sinh^2 r g'_r + dr^2.$$

We call the correspondence $g' \mapsto \mathcal{T}_{a,d} g'$ the *two variable warping deformation*. By construction we have that $\mathcal{T}_{a,d} g'$ satisfies the following property:

$$\mathcal{T}_{a,d} g' = \begin{cases} \sinh^2 r\sigma_{\mathbf{S}^{n-1}} + dr^2 & \text{on} \quad \bar{\mathrm{B}}_a \\ g' & \text{outside} \quad \mathrm{B}_{a+\frac{d}{2}}. \end{cases}$$

Hence, the two variable warping deformation changes a warp product metric g' inside the ball $\bar{\mathrm{B}}_{a+\frac{d}{2}}$ making it (radially) hyperbolic on the smaller ball $\bar{\mathrm{B}}_a$. The warp product metric g' does not change outside $\mathrm{B}_{a+\frac{d}{2}}$.

Remarks **3.1.1.**

1. Note that if we choose g' to be the warped-by-sinh hyperbolic metric, that is, $g' = \sinh^2 r\sigma_{\mathbf{S}^{n-1}} + dr^2$, then $\mathcal{T}_{a,d} g' = g'$.
2. To be able to define $\mathcal{T}_{a,d} g'$ the metric g' does not need to be a warp product metric everywhere. It only needs to be a warp product metric in the ball $\bar{\mathrm{B}}_{a+\frac{d}{2}}$.
3. Since $\mathcal{T}_{a,d} g' = \sinh^2 r g'_r + dr^2$ the metric $\mathcal{T}_{a,d} g'$ also has $o = 0$ as center.

3.2. *Warp forcing*

Recall that we can write the Riemannian metric g of M on $\mathrm{M} - \{o\} = \mathbf{S}^{n-1} \times \mathbf{R}^+$ as $g = g_r + dr^2$. For a fixed $r_0 > 0$ we can think of the metric g_{r_0} as being obtained from $g = g_r + dr^2$ by "cutting" g along the sphere of radius r_0, so we call g_{r_0} the *spherical cut of g at r_0*. In the same vein, we call the metric

$$\hat{g}_{r_0} = \left(\frac{1}{\sinh^2(r_0)}\right) g_{r_0}$$

the *normalized spherical cut of g at r_0* . Note that in the particular case where $g = g_r + dr^2$ is already a warped-by-sinh metric (that is, $g_r = \sinh^2 r g'$ for some fixed g' independent of r) we have that the spherical cut of $g = \sinh^2 r g' + dt^2$ at r_0 is $\sinh^2(r_0) g'$, and the normalized spherical cut at r_0 is $\hat{g}_{r_0} = g'$.

Fix $r_0 > 0$. We define the warped-by-sinh metric $\bar{g}_{r_0}$ by:

$$\bar{g}_{r_0} = \sinh^2 r \hat{g}_{r_0} + dr^2 = \sinh^2 r \left(\frac{1}{\sinh^2 r_0}\right) g_{r_0} + dr^2.$$

We now force the metric g to be equal to $\bar{g}_{r_0}$ on $\bar{\mathrm{B}}_{r_0}$ and stay equal to g outside $\mathrm{B}_{r_0+\frac{1}{2}}$. For this we define the *warped forced* metric $\mathcal{W}_{r_0} g$ as:

$$\mathcal{W}_{r_0} g = (1 - \rho_{r_0}) \bar{g}_{r_0} + \rho_{r_0} g,$$

where $\rho_{r_0}(t) = \rho(2t - 2r_0)$, and $\rho : \mathbf{R} \to [0, 1]$ is as before (see Section 1). Hence we have

$$\mathcal{W}_{r_0} g = \begin{cases} \bar{g}_{r_0} & \text{on} \quad \bar{\mathrm{B}}_{r_0} \\ g & \text{outside} \quad \mathrm{B}_{r_0+\frac{1}{2}}. \end{cases}$$

We call the process $g \mapsto \mathcal{W} g$ *warp forcing*. Hence warp forcing changes the metric only on $\mathrm{B}_{r_0+\frac{1}{2}}$, making it a warp product metric inside $\bar{\mathrm{B}}_{r_0}$. The metric g does not change outside $\mathrm{B}_{r_0+\frac{1}{2}}$.

Remarks **3.2.1.**

1. Notice that to define $\mathcal{W}_{r_0} g$ we only need g_r to be defined for $r \geq r_0$.
2. Note that if we choose g to be the warped-by-sinh hyperbolic metric, that is, $g = \sinh^2 r \sigma_{\mathbf{S}^{n-1}} + dr^2$, then $\mathcal{W}_{r_0} g = g$.
3. Note that the metric $\mathcal{W}_{r_0} g$ also has $o = 0$ as center.

3.3. *Hyperbolic forcing*

Recall that (M^n, g) has center o, and we are writing $g = g_r + dr^2$. Let $r_0 > d > 0$. We define the metric $\mathcal{H}_{r_0,d} g$ in the following way. First warp-force the metric g, i.e. take $\mathcal{W}_{r_0} g$.

Recall $\mathcal{W}_{r_0}g$ is a warp product metric on B_{r_0} and has o as center (see Remarks 3.2.1(1), (3)). Hence we can use the two variable warping deformation given in Section 3.1 (also see Remark 3.1.1(2)) and define

$$\textbf{(3.3.1)} \qquad \mathcal{H}_{r_0,d}g = \mathcal{T}_{(r_0-d),d}(\mathcal{W}_{r_0}g).$$

The process $g \mapsto \mathcal{H}_{r_0,d}g$ is called *hyperbolic forcing*. Write $h = \mathcal{H}_{r_0,d}g$. Note that h also has the form $h = h_r + dr^2$. In the next results we explicitly describe h_r and give some properties of the metric $h = \mathcal{H}_{r_0,d}g$. These results are proved in [33] (see Propositions 5.1 and 5.2 in [33]).

Remark **3.3.2.** — Let g be a metric on $\mathbf{R}^n$ that can be written in as $g = g_r + dr^2$. To define $\mathcal{H}_{r_0,d}g$ we only need g_r to be defined for $r \geq r_0$ (see Remarks 3.1.1(2), 3.2.1(1)). More generally, if (M, g_0) has center o then we can define $\mathcal{H}_{r_0,d}g$ for any metric on M of the form $g = g_r + dr^2$, where g_r is only defined for $r \geq r_0$. Here we are using g_0 to identify M and $\mathbf{R}^n$. This construction is used in Section 4.

Proposition **3.3.3.** — *Let h_r be as above. Then*

$$h_r = \begin{cases} g_r & r_0 + \frac{1}{2} \leq r \\ (1-\rho_{r_0}(r))\sinh^2 r\hat{g}_{r_0} + \rho_{r_0}(r)g_r & r_0 \leq r \leq r_0 + \frac{1}{2} \\ \sinh^2 r((1-\rho_{(r_0-d),d}(r))\sigma_{\mathbf{S}^{n-1}} + \rho_{(r_0-d),d}(r)\hat{g}_{r_0}) & r_0 - d \leq r \leq r_0 \\ \sinh^2 r\sigma_{\mathbf{S}^{n-1}} & r \leq r_0 - d, \end{cases}$$

where the gluing functions ρ_{r_0} and $\rho_{(r_0-d),d}$ are defined in Sections 3.2 and 3.1, respectively.

Proposition **3.3.4.** — *The metric $h = \mathcal{H}_{r_0,d}g$ has the following properties.*

(i) *The metric h is canonically hyperbolic on $\bar{\mathrm{B}}_{r_0-d}$, i.e. $h = \sinh^2 r\sigma_{\mathbf{S}^n} + dr^2$ on $\bar{\mathrm{B}}_{r_0-d}$.*
(ii) *We have that $g = h$ outside $\mathrm{B}_{r_0+\frac{1}{2}}$.*
(iii) *The metric h coincides with $\mathcal{W}_{r_0}(g_{r_0})$ outside $\mathrm{B}_{r_0-\frac{d}{2}}$.*
(iv) *The metric h coincides with $\mathcal{T}_{(r_0-d),d}\bar{g}_{r_0}$ on $\bar{\mathrm{B}}_{r_0}$.*
(v) *All the g-geodesic rays $r \mapsto ru$, $u \in \mathbf{S}^n$, emanating from the center are geodesic rays of* (M, h). *Hence, the space* (M, h) *has center o. Moreover the function r (distance to the center o) is the same on the spaces* (M, g) *and* (M, h). *In other words, the spaces* (M, g) *and* (M, h) *have the same set of rays.*

Next we discuss the following question:

Is the hyperbolically forced metric $h = \mathcal{H}_{r_0,d}g$ close to hyperbolic, when g is close to hyperbolic?

Notice that from Remarks 3.1.1(1) and 3.2.1(2) it follows that if we choose g to be the warped-by-sinh hyperbolic metric, that is, $g = \sinh^2 t\sigma_{\mathbf{S}^{n-1}} + dt^2$, then $\mathcal{H}_{r_0,d}g = g$.

Therefore one would expect that the answer to the previous question is "yes". So, it is better ask a more quantitative question:

> *To what extend is the hyperbolically forced metric $h = \mathcal{H}_{r_0,d}\, g$ close to hyperbolic, when g is close to hyperbolic?*

The next theorem deals with this question. This theorem is proved in [33] (see Theorem 1.7 in [33]).

Theorem **3.3.5.** — *Let* M^n *have center* o *and metric* $g = g_r + dr^2$*. Assume the normalized spherical cut* $\hat{g}_{r_0}$ *is* c*-bounded. If the metric* g *is radially* ε*-close to hyperbolic outside* $\bar{B}_{r_0-1-\xi}$ *with charts of excess* $\xi > 1$*, then the metric* $\mathcal{H}_{r_0,d}\, g$ *is* (B_{r_0-d}, η)*-close to hyperbolic with charts of excess* $\xi - 1$*, provided*

$$\eta \geq C_1 \left(\frac{1}{d} + e^{-(r_0-d)} \right) + C_2 \varepsilon.$$

Here C_1 *is a constant depending only on* n*,* ξ*,* c*, and* C_2 *depends only on* ξ*.*

Remarks **3.3.6.**

1. An important point here is that by taking r_0 and d large the metric $\mathcal{H}_{r_0,d}\, g$ can be made $2C_2\varepsilon$-close to hyperbolic. How large we have to take d and r_0 depends on c, which is a C^2 bound for $\hat{g}_{r_0}$, the normalized spherical cut of g at r_0 (see Section 3.2).
2. Note that the excess of the charts decreases by 1. This is because of warp forcing.

4. Deformations of families of metrics

In this section we give a one-parameter version of the concepts and results presented in Section 3. Let (M^n, g) be a complete Riemannian manifold with center $o \in M$. As before we identify M with $\mathbf{R}^n$ and $M - \{o\}$ with $\mathbf{S}^{n-1} \times \mathbf{R}^+$. Therefore on $M - \{o\} = \mathbf{S}^{n-1} \times \mathbf{R}^+$ we can write $g = g_r + dr^2$, where r is the distance to o. We will still use the notation B_a and $\bar{B}_a$ for the open and closed balls in $M = \mathbf{R}^n$ of radius a centered at $o = 0$, respectively.

Fix $\xi > 0$, and let $\lambda_0 > 1 + \xi$. We say that the collection $\{g_\lambda\}_{\lambda \geq \lambda_0}$ is a $\odot$*-family of metrics on* M if each g_λ is a metric of the form $g_\lambda = (g_\lambda)_r + dr^2$, where the metrics $(g_\lambda)_r$ are defined (at least) for $r > \lambda - 1 - \xi$.

Remarks **4.1.**

1. Note that we are not demanding the metrics g_λ to be *globally defined*, i.e. that the $(g_\lambda)_r$ are defined for all $r > 0$. The reason is that in the applications (in Section 8) we actually do get an $\odot$-family of metrics that is only *partially defined*, i.e. that $(g_\lambda)_r$

are defined (at least) for all r large. Also we intend to apply hyperbolic forcing $\mathcal{H}_{\lambda,d}$ to each of the g_λ, and for this we only need the $(g_\lambda)_r$ to be defined for $r \geq \lambda$ (see Remark 3.3.2). Moreover, we want a family version of Theorem 3.3.5, this is why we demand a bit more: that the $(g_\lambda)_r$ be defined for $r > \lambda - 1 - \xi$.

2. Since $g_\lambda = (g_\lambda)_r + dr^2$ is defined on the complement of the closed ball $\bar{B}_{\lambda-1-\xi}$ of radius $\lambda - 1 - \xi$, centered at o, g_λ is ray compatible with (M, o), over the complement of $\bar{B}_{\lambda-1-\xi}$, see Remark 1.1. This is why we used the symbol $\odot$, to evoke the idea that all metrics g_λ have, in some sense, a common center and spheres. Note that the property of being an $\odot$-family is actually equivalent to each metric in the collection being ray compatible with (M, o).

We want to give a one-parameter version of Theorem 3.3.5, that is, a version for a $\odot$-family $\{g_\lambda\}$. Since the constant C_1 in Theorem 3.3.5 depends on the bound c there is no uniform C_1 that would work for **every** g_λ. This problem motivates the following definition.

Let $b \in \mathbf{R}$. By cutting each g_λ at $b + \lambda$ we obtain a one-parameter family $\{\widehat{(g_\lambda)}_{\lambda+b}\}_\lambda$ of metrics on the sphere $\mathbf{S}^{n-1}$. Here $\lambda > \max\{\lambda_0, -b\}$, so that the definition makes sense. We say that the $\{g_\lambda\}$ has *cut limit at* b if the family $\{\widehat{(g_\lambda)}_{\lambda+b}\}_\lambda$ C^2 converges. That is, there is a C^2 metric $\hat{g}^b_\infty$ on $\mathbf{S}^{n-1}$ such that

$$\left|\widehat{(g_\lambda)}_{\lambda+b} - \hat{g}^b_\infty\right|_{C^2(\mathbf{S}^{n-1})} \xrightarrow{C^2} 0 \quad \text{as } \lambda \to \infty. \tag{4.2}$$

Remarks **4.3.**

1. Recall that the metric $\widehat{(g_\lambda)}_{\lambda+b}$ is the normalized spherical cut of g_λ at $\lambda + b$. See Section 3.2.
2. The arrow above means convergence in the C^2-norm on the space of C^2 metrics on $\mathbf{S}^{n-1}$. See Remark 1.5.
3. Note that the concept of cut limit at b depends strongly on the indexation of the family.
4. If a family $\{g_\lambda\}$ has cut limits at b, then the family $\{\widehat{(g_\lambda)}_{\lambda+b}\}_\lambda$ is clearly c-bounded, for some c.

Consider the $\odot$-family $\{g_\lambda\}$ and let $d > 0$. Apply hyperbolic forcing to get

$$h_\lambda = \mathcal{H}_{\lambda,d}\, g_\lambda.$$

We say that the family $\{h_\lambda\}$ is the *hyperbolically forced family* corresponding to the $\odot$-family $\{g_\lambda\}$. Note that we can write $h_\lambda = (h_\lambda)_r + dr^2$. Using Proposition 3.3.3 we can explicitly describe $(h_\lambda)_r$:

$$(\mathbf{4.4})\qquad (h_\lambda)_r = \begin{cases} (g_\lambda)_r & \lambda + \frac{1}{2} \le r \\ (1-\rho_\lambda(r))\sinh^2 r\widehat{(g_\lambda)}_\lambda + \rho_\lambda(r)(g_\lambda)_r & \lambda \le r \le \lambda + \frac{1}{2} \\ \sinh^2 r((1-\rho_{(\lambda-d),d}(r))\sigma_{\mathbf{S}^{n-1}} + \rho_{(\lambda-d),d}(r)\widehat{(g_\lambda)}_\lambda) & \lambda - d \le r \le \lambda \\ \sinh^2 r\sigma_{\mathbf{S}^{n-1}} & r \le \lambda - d. \end{cases}$$

The next proposition is a one-parameter version of Proposition 3.3.4. It is proved in [33] (see Proposition 6.3 in [33]).

Proposition **4.5.** — *The metrics h_λ have the following properties.*

(i) *The metrics h_λ are canonically hyperbolic on $\bar{B}_{\lambda-d}$, i.e. equal to $\sinh^2 r\sigma_{\mathbf{S}^{n-1}} + dr^2$ on $\bar{B}_{\lambda-d}$, provided $\lambda > d$.*

(ii) *$g_\lambda = h_\lambda$ outside $B_{\lambda+\frac{1}{2}}$.*

(iii) *The metric h coincides with $\mathcal{W}_\lambda(g_\lambda)$ outside $B_{\lambda-\frac{d}{2}}$.*

(iv) *The metric h coincides with $\mathcal{T}_{(\lambda-d),d}(\overline{(g_\lambda)}_\lambda)$ on $\bar{B}_\lambda$.*

(v) *If the $\odot$-family $\{g_\lambda\}$ has cut limits for $b = 0$ then $\{h_\lambda\}$ has cut limits on $(-\infty, 0]$. In fact we have*

$$\hat{h}^b_\infty = \begin{cases} \hat{g}^0_\infty & b = 0 \\ (1-\rho(2+\frac{2b}{d}))\sigma_{\mathbf{S}^n} + \rho(2+\frac{2b}{d})\hat{g}^0_\infty & -d \le b \le 0 \\ \sigma_{\mathbf{S}^n} & b \le -d, \end{cases}$$

where ρ is as in Section 1.

(vi) *If we additionally assume that $\{g_\lambda\}$ has cut limits on $[0, \frac{1}{2}]$, then $\{h_\lambda\}$ has also cut limits on $[0, \frac{1}{2}]$. In fact, for $b \in [0, \frac{1}{2}]$ we have*

$$\hat{h}^b_\infty = \big(1-\rho(b)\big)\hat{g}^0_\infty + \rho(b)\hat{g}^b_\infty,$$

where ρ is as in Section 3. Of course if $\{g_\lambda\}$ has a cut limit at $b > \frac{1}{2}$ then $\{h_\lambda\}$ has the same cut limit at b (see item (ii)*).*

(vii) *All the rays $r \mapsto ru$, $u \in \mathbf{S}^n$, emanating from the origin are geodesic rays of (M, h_λ). Hence, all spaces (M, h_λ) have center $o \in M$ and have the same geodesic rays emanating from the common center o. Moreover the function r (distance to $o \in M$) is the same on all spaces (M, h_λ). Therefore all spaces (M, h_λ) have the same set of rays as (M, g).*

We now state one of our most important results. It is used in an essential way in smoothing Charney-Davis strict hyperbolizations. It is proved in [33] using Theorem 3.3.5 (see Theorem 1.11 in [33]). Before, we need a definition. We say that an $\odot$-family $\{g_\lambda\}$ is *radially ε-close to hyperbolic, with charts of excess ξ*, if each g_λ is radially ε-close to hyperbolic outside $B_{\lambda-1-\xi}$, with charts of excess ξ.

Theorem **4.6.** — *Let* M *have center* o, $\{g_\lambda\}$ *an* $\odot$*-family on* M, *and* $\varepsilon' > 0$. *Assume that* $\{g_\lambda\}$ *has cut limits at* $b = 0$. *If* $\{g_\lambda\}$ *is radially* ε*-close to hyperbolic, with charts of excess* $\xi > 1$, *then, for every* λ, $\mathcal{H}_{\lambda,d}g_\lambda$ *is* $(\mathrm{B}_{\lambda-d}, \varepsilon' + \mathrm{C}_2\varepsilon)$*-close to hyperbolic, with charts of excess* $\xi - 1$, *provided*

(i) $\lambda - d > \ln(\frac{2\mathrm{C}_1}{\varepsilon'})$.
(ii) $d \geq \frac{2\mathrm{C}_1}{\varepsilon'}$.

Here C_1 *and* C_2 *are as in Theorem* 3.3.5.

Remarks **4.7.**

1. Note that we can take ε' as small as we want, hence we can take $\varepsilon' + \mathrm{C}_2\varepsilon$ as close as $\mathrm{C}_2\varepsilon$ as we desire, provided we choose d and λ sufficiently large.
2. The constant $\mathrm{C}_1 = \mathrm{C}_1(\mathsf{c}, n, \xi)$ in Theorem 4.6 depends on c. Here c is as in Remark 4.3(4), that is, c is such that $\{\widehat{(g_\lambda)}_{\lambda+b}\}_\lambda$ is c-bounded.

4.8. *Cuts limits and hyperbolic extensions*

In (4.2) we gave the definition of a cut limit. More generally, let $\mathrm{I} \subset \mathbf{R}$ be an interval (compact or noncompact). We say the $\odot$-family $\{g_\lambda\}$ has *cut limits on* I if the convergence in (4.2) is uniform with compact supports in the variable $b \in \mathrm{I}$. Explicitly this means: for every $\varepsilon > 0$, and compact $\mathrm{K} \subset \mathrm{I}$ there is λ_* such that $|\widehat{(g_\lambda)}_{\lambda+b'} - \hat{g}_{\infty+b'}|_{\mathrm{C}^2(\mathbf{S}^{n-1})} < \varepsilon$, for $\lambda > \lambda_*$ and $b' \in \mathrm{K}$.

If the $\odot$-family $\{g_\lambda\}$ has cut limits on $\mathbf{R}$ we will just say that $\{g_\lambda\}$ *has cut limits.*

Remark **4.8.1.** — Let $a \in \mathbf{R}$. If $\{g_\lambda\}_\lambda$ has cut limits then so does the reparametrized family $\{g_{\lambda+a}\}_\lambda$.

Here is a natural question:

If $\{h_\lambda\}_\lambda$ *has a cut limits, does* $\{\mathcal{E}_k(h_\lambda)\}_\lambda$ *have cut limits?*

Remark **4.8.2.** — More generally we can ask whether $\{\mathcal{E}_k(h_\lambda)\}_{\lambda'}$ has cut limits, where $\lambda = \lambda(\lambda')$. Of course the answer could depend on the change of variables $\lambda = \lambda(\lambda')$.

The next result gives an affirmative answer to this question provided the family $\{h_\lambda\}$ is, in some sense, nice near the origin. Explicitly, we say that $\{h_\lambda\}$ is *hyperbolic around the origin* if there is a $\mathrm{B} \in \mathbf{R}$ such that

$$\widehat{(h_\lambda)}_{\lambda+b} = \sigma_{\mathbf{S}^{n-1}},$$

for every $b \leq \mathrm{B}$ and every (sufficiently large) λ. Note that this implies that each h_λ is canonically hyperbolic on the ball of radius $\lambda + \mathrm{B}$. Examples of $\odot$-families that are hyperbolic around the origin are families obtained using hyperbolic forcing, as above.

Remark **4.8.3.** — If $\{h_\lambda\}$ is *hyperbolic around the origin* then it is globally defined (see Remark 4.1(1)).

As mentioned before the next result answers affirmatively the question above. Moreover it also says that some reparametrized families $\{\mathcal{E}_k(h_\lambda)\}_{\lambda'}$, for certain change of variables $\lambda = \lambda(\lambda')$, have cut limits as well. Write $\lambda = \lambda(\lambda', \theta) = \sinh^{-1}(\sinh\lambda' \sin\theta)$, for fixed θ. Note that $\lambda = \lambda'$ for $\theta = \pi/2$. We say that $\{\mathcal{E}_k(h_\lambda)\}_{\lambda'}$ is the *θ-reparametrization of* $\{\mathcal{E}_k(h_\lambda)\}_\lambda$. If we consider an hyperbolic right triangle with one angle equal to θ and side (opposite to θ) of length λ, then λ' is the length of the hypotenuse of the triangle. As $\lambda' \to \infty$ all θ-reparametrizations differ by an additive constant, that is, $\lim_{\lambda'\to\infty} \lambda(\lambda') - \lambda' = \ln\sin\theta$, as simple computation shows. The next proposition is proved in [30]; it is the Main Theorem in [30].

Proposition **4.8.4.** — *Let* M *have center* o. *Let* $\{h_\lambda\}_\lambda$ *be* $\odot$*-family of metrics on* M. *If* $\{h_\lambda\}_\lambda$ *is hyperbolic around the origin and has cut limits, then for every* $\theta \in (0, \pi/2]$ *the* θ*-reparametrization* $\{\mathcal{E}_k(h_\lambda)\}_{\lambda'}$ *has cut limits.*

Note that the case $\theta = \pi/2$ gives $\lambda = \lambda'$, answering the question above.

5. Normal neighborhoods on all-right spherical complexes

The goal in this section is to define "natural normal neighborhoods" of simplices in all-right spherical complexes, and give some of its properties.

We use the definition and properties of a spherical complex given in Section 1 of [5]. Recall that a spherical complex is an *all-right spherical complex* if all of its edge lengths are equal to $\pi/2$. We will denote a complex and its realization by the same symbol. In this paper we shall assume that all spherical complexes satisfy the "intersection condition" of simplicial complexes: every two simplices have at most one common face.

Remark **5.0.1.** — Let P be an all-right spherical complex and $\Delta \in$ P. The symbol $\dot\Delta$ denotes the interior of Δ. (If Δ is a point then it is equal to its interior.) In this paper we will use the three definitions of link $\mathsf{Link}(\Delta, \mathrm{P})$ of Δ in P. The *geometric link* $\mathsf{Link}(\Delta, \mathrm{P})$ is the union of the end points of geodesic segments of small length $\beta > 0$ emanating perpendicularly (to Δ) from some point $x \in \dot\Delta$. If we want to specify β and x we say that $\mathsf{Link}_\beta(\Delta, \mathrm{P})$ is the β-link *based at* x. The *geometric star* $\mathsf{Star}(\sigma, \mathrm{K})$ is the union of the corresponding segments. The *simplicial link* is the subcomplex of P formed by all simplices Δ' such that (1) Δ' is disjoint from Δ, (2) Δ' and Δ span a simplex (this simplex is the join $\Delta * \Delta' \in$ P, and Δ' is the *opposite face* of Δ in $\Delta * \Delta'$). Note that if we continue a geodesic $[x, u]$, with u in the geometric β-link at x, we will hit a unique point in Δ'. This radial geodesic projection gives a relationship between geometric links and simplicial links. The *simplicial star* is the subcomplex of P formed by all simplices Δ' that contain Δ, together with its faces. For $x \in \dot{\Delta}^k$ the *direction link of* Δ *in* P *at* x is the set of all vectors at x perpendicular to Δ^k. Using geodesics emanating from x perpendicularly to Δ we also get a relationship between geometric links and the direction links. These different definitions of link all come with natural all-right spherical metrics: the simplicial link with the induced metric, the direction link with the angle metric, and the geometric link $\mathsf{Link}_\beta(\Delta, \mathrm{P})$ with the induced metric from P rescaled by $1/\beta^2$. The relationships between the different definitions of link mentioned above all respect the metrics.

Remark **5.0.2.** — Let P be an all-right spherical complex and $\Delta \in \mathrm{P}$. The simplices of $\mathsf{Link}(\Delta, \mathrm{P})$ are of the form $\Delta' \cap \mathsf{Link}(\Delta, \mathrm{P})$ where $\Delta' \in \mathrm{P}$ and $\Delta \subset \Delta'$. Here by $\mathsf{Link}(\Delta, \mathrm{P})$ we mean either the geometric or the simplicial link. Alternatively, the simplices of $\mathsf{Link}(\Delta, \mathrm{P})$ are $\mathsf{Link}(\Delta, \Delta')$, $\Delta' \in \mathrm{P}$, $\Delta \subsetneq \Delta'$. Again, here $\mathsf{Link}(\Delta, \mathrm{P})$ is either the geometric or the simplicial link. Note that if we write $\Delta' = \Delta * \Delta''$, where Δ'' is opposite to Δ in Δ', then $\Delta'' = \mathsf{Link}(\Delta, \Delta')$. In this last equality Link is the simplicial link.

5.1. *Sequences of widths of normal neighborhoods on the sphere* $\mathbf{S}^m$

The 2^{m+1} quadrants of $\mathbf{R}^{m+1}$ intersect the unit sphere $\mathbf{S}^m$ centered at the origin in the *canonical m-simplices*. We consider the m-sphere $\mathbf{S}^m$ with its *canonical all-right spherical structure* formed by the canonical m-simplices together its faces. For $\Delta \in \mathbf{S}^m$ we will denote by $\dot{\Delta}$ its interior.

Remark **5.1.1.** — Let $\Delta^k \in \mathbf{S}^m$ be an all-right k-simplex in $\mathbf{S}^m$, and $p \in \Delta^k$. The *perpendicular sphere* $\mathrm{S}_{\Delta^k,p}$ *to* Δ^k *at* p is the union of (images of) geodesics in $\mathbf{S}^m$ emanating from p and perpendicular to Δ^k. Note that this makes sense even if p is in the boundary of Δ because the tangent space to Δ at p is well defined. We can identify this sphere with $\mathbf{S}^{m-k}$ in such a way that the set $\{\Delta \cap \mathrm{S}_{\Delta^k,p}\}_{\Delta\in\mathbf{S}^m}$ corresponds to the canonical all-right spherical structure of $\mathbf{S}^{m-k}$. The proof of this fact is straightforward.

Let $\beta \in (0, \pi/2]$ and $\Delta^k \in \mathbf{S}^m$, $0 \le k < m$. The *closed normal neighborhood of* Δ^k *in* $\mathbf{S}^m$ *of width* β is the union of (images of) geodesics of length β emanating perpendicularly from Δ^k. It will be denoted by $\mathsf{N}_\beta(\Delta^k, \mathbf{S}^m)$. For the special case $\dim \Delta = m$ by definition we take $\mathsf{N}_\beta(\Delta^m, \mathbf{S}^m) = \Delta^m$, for any β.

Let $\mathsf{B} = \{\beta_k\}_{k=0,1,2...}$ be a (finite or infinite) sequence of real numbers with $\beta_k \in (0, \pi/2)$ and $\beta_{k+1} < \beta_k$. We write $\mathsf{B}(m) = \{\beta_0, \dots, \beta_{m-1}, \}$. The set B determines *the set of spherical* B*-neighborhoods* $\mathsf{N}_\mathsf{B}(\mathbf{S}^m) = \mathsf{N}_{\mathsf{B}(m)}(\mathbf{S}^m) = \{\mathsf{N}_{\beta_k}(\Delta^k, \mathbf{S}^m)\}_{\Delta^k\in\mathbf{S}^m, k<m}$, for any sphere $\mathbf{S}^m$ (of any dimension). Note that the normal neighborhoods of all k-simplices Δ^k have the same width β_k. The set B is called a *sequence of widths of spherical normal neighborhoods* or simply *a sequence of widths*. The sequence $\mathsf{B}(m)$ is a *finite sequence of widths of length m*. The definitions above still make sense if we replace $\mathbf{S}^m$ by $\mathbf{S}^m_\mu$, the m-sphere of radius μ (for small β_k's).

We are interested in pairs of sequences of widths (B, A), $\mathsf{B} = \{\beta_k\}$ and $\mathsf{A} = \{\alpha_j\}$, having the following **D***isjoint* **N***eighborhood* **P***roperty*:

$$(\mathbf{5.1.2}) \qquad \mathbf{DNP}: \quad \text{For every } k < m \text{ any two sets in the following collection are disjoint}$$

$$\Big\{ \mathsf{N}_{\beta_k}\big(\Delta^k, \mathbf{S}^m\big) - \bigcup_{j<k} \mathsf{N}_{\alpha_j}\big(\Delta^j, \mathbf{S}^m\big) \Big\}_{\Delta^k \in \mathbf{S}^m}.$$

The disjoint neighborhood property obtained by fixing k and m above will be denoted by $\mathbf{DNP}(k, m)$. In this case we allow the sequences of widths to be finite, and of length at least $k+1$. It is straightforward to verify that $\mathbf{DNP}(k, m)$ is equivalent to the following property. For fixed k and m we have: for different k-simplices Δ^k_1 and Δ^k_2 we have

$$(\mathbf{5.1.2'}) \qquad \mathsf{N}_{\beta_k}\big(\Delta^k_1, \mathbf{S}^m\big) \bigcap \mathsf{N}_{\beta_k}\big(\Delta^k_2, \mathbf{S}^m\big) \subset \bigcup_{j<k} \mathsf{N}_{\alpha_j}\big(\Delta^j, \mathbf{S}^m\big).$$

The same is true for **DNP**. We define the A-*neighborhood of the* $(k-1)$-*skeleton* as $\bigcup_{j<k} \mathsf{N}_{\alpha_j}(\Delta^j, \mathbf{S}^m)$. Then (5.1.2') says that the B-neighborhoods of different k-simplices intersect only inside the A-neighborhood of the $(k-1)$-skeleton.

Proposition **5.1.3.** — *The pair of (finite or infinite) sequences of widths* (B, A) *satisfy* $\mathbf{DNP}(k, m)$ *if and only if* $\frac{\sin \beta_k}{\sin \alpha_{k-1}} \leq \frac{\sqrt{2}}{2}$.

Note that the inequality condition is independent of m. The proposition follows directly from Lemmas 5.1.4 (taking $k = l$ and $\beta = \gamma$) and 5.1.5 given below, and the fact that $\{\alpha_k\}$ is decreasing.

Lemma **5.1.4.** — *Let* Δ^k, $\Delta^l \in \mathbf{S}^m$ *and* $\Delta^j = \Delta^k \cap \Delta^l$. *Let* $\alpha, \beta, \gamma \in (0, \pi/2)$ *such that* $\frac{\sin\beta}{\sin\alpha}, \frac{\sin\gamma}{\sin\alpha} \leq \frac{\sqrt{2}}{2}$. *Then*

$$\mathsf{N}_\beta\big(\Delta^k, \mathbf{S}^m\big) \cap \mathsf{N}_\gamma\big(\Delta^l, \mathbf{S}^m\big) \subset \mathsf{N}_\alpha\big(\Delta^j, \mathbf{S}^m\big).$$

Proof. — In this proof $\mathsf{Link}(\Delta, \mathbf{S}^m)$ shall denote the simplicial link and $\mathsf{Star}(\Delta, \mathbf{S}^m)$ the simplicial star (see Remark 5.0.1). Note that $\mathsf{N}_\beta(\Delta, \mathbf{S}^m) \subset \mathsf{Star}(\Delta, \mathbf{S}^m)$, for every $\Delta \in \mathbf{S}^m$. Write $\mathsf{S} = \mathsf{Link}(\Delta^j, \mathbf{S}^m)$, $\Delta'_1 = \mathsf{S} \cap \Delta^k$ and $\Delta'_2 = \mathsf{S} \cap \Delta^l$. Then Δ'_i is a simplex in the all-right triangulation of S. Also Δ'_1 and Δ'_2 are disjoint. Hence their distance in S is at least $\frac{\pi}{2}$.

Take $q \in \mathsf{N}_\beta(\Delta^k, \mathbf{S}^m) \cap \mathsf{N}_\gamma(\Delta^l, \mathbf{S}^m)$. Since both of these neighborhoods lie in $\mathsf{Star}(\Delta^j, \mathbf{S}^m)$ there is a geodesic segment $[p, q]$ in $\mathsf{Star}(\Delta^j, \mathbf{S}^m)$ with $p \in \Delta^j$ and $[p, q]$ perpendicular to Δ^j. Since $q \in \mathsf{N}_\beta(\Delta^k, \mathbf{S}^m) \subset \mathsf{Star}(\Delta^k, \mathbf{S}^m)$ there is $\Delta_1 \in \mathbf{S}^m$ with $q \in \Delta_1 \supset \Delta^k$. Note that $p \in \Delta^j \subset \Delta^k \subset \Delta_1$, hence $[p, q] \subset \Delta_1$. Because $\Delta^j \subset \Delta_1$ we can write $\Delta_1 = \Delta^j * \Delta_1''$, where Δ_1'' is opposite to Δ^j in Δ_1; notice that $\Delta_1'' \in \mathsf{S}$. Analogously, replacing Δ^k by Δ^l above we have that there are Δ_2 and Δ_2'' with $\Delta_2 = \Delta^j * \Delta_2''$, $\Delta_2'' \in \mathsf{S}$, and $[p, q] \subset \Delta_2$.

Write $\alpha' = \text{length}([p, q])$. We have to prove $\alpha' \le \alpha$. We assume $\alpha' > \alpha$ and get a contradiction. Let q_1 be the closest point in Δ^k to q, and q_2 be the closest point in Δ^l to q. We have $a_1 = \text{length}([q_1, q]) \le \beta$ and $a_2 = \text{length}([q_2, q]) \le \gamma$. Note that $[q_i, q] \subset \Delta_i$. It is straightforward to show that $[q_i, p]$ is perpendicular to Δ^j at p (the simplex $\{p\} * \Delta_1''$ is convex in Δ_1 and perpendicular to Δ^k; similarly for Δ^l). We get a right (at q_i) spherical triangle $\Delta(q, q_i, p)$ with one side equal to a_i and hypotenuse equal to α'. Let θ_i be the angle at p, that is, the angle opposite to the side of length a_i. Then by the spherical law of sines we get

$$\sin\theta_1 = \frac{\sin a_1}{\sin\alpha'} < \frac{\sin\beta}{\sin\alpha} \le \frac{\sqrt{2}}{2}.$$

Consequently $\theta_1 < \frac{\pi}{4}$. Similarly we get $\theta_2 < \frac{\pi}{4}$. Let z_i be the intersection of S with the ray at p with direction q_i. Analogously let q' be the intersection of S with the ray at p with direction q. Note that $z_i \in \Delta_i'$ and $q' \in \Delta_1'' \cap \Delta_2''$. Since $\Delta_i' \subset \Delta_i''$ we get segments $[q', z_i] \subset \Delta_i''$. Because the angle at p of the triangle $\Delta(q, q_i, p) \subset \Delta_i$ is θ_i, we get $\text{length}([q', z_i]) = \theta_i$. Therefore

$$\frac{\pi}{2} \le d_{\mathsf{S}}\big(\Delta_1', \Delta_2'\big) \le d_{\mathsf{S}}(z_1, z_2) \le d_{\mathsf{S}}\big(z_1, q'\big) + d_{\mathsf{S}}\big(q', z_2\big) \le \theta_1 + \theta_2.$$

Hence $\frac{\pi}{2} \le \theta_1 + \theta_2 < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$ which is a contradiction. □

Lemma **5.1.5.** — *Let Δ_1^k, $\Delta_2^k \in \mathbf{S}^m$ be two different k-simplices, and $\Delta^{k-1} = \Delta_1^k \cap \Delta_2^k$. Moreover the Δ_i^k are k-faces of a $k+1$-simplex. Let $\alpha, \beta \in (0, \pi/2)$. Suppose that $\mathsf{N}_\beta(\Delta_1^k, \mathbf{S}^m) \cap \mathsf{N}_\beta(\Delta_2^k, \mathbf{S}^m) \subset \mathsf{N}_\alpha(\Delta^{k-1}, \mathbf{S}^m)$. Then $\frac{\sin\beta}{\sin\alpha} \le \frac{\sqrt{2}}{2}$.*

Proof. — The lemma is certainly true for $\mathbf{S}^1$. Using the spherical law of sines it is straightforward to verify the lemma for $\mathbf{S}^2$. The case $\mathbf{S}^m$, $m > 2$, can be reduced to the case $m = 2$ in the following way. First note that Δ_1^k and Δ_2^k span a simplex Δ^{k+1} which is contained in a canonical $(k+1)$-sphere $\mathbf{S}^{k+1} \subset \mathbf{S}^m$. Now the case $m > 2$ can be reduced to the case $m = 2$ using the orthogonal sphere $\mathrm{S} = \mathrm{S}_{\Delta^{k-1}, p}$ in $\mathbf{S}^{k+1}$ (see Remark 5.1.1), where p is the barycenter of Δ^{k-1}. □

The next result says that **DNP** implies a seemingly stronger version of itself (see (5.1.2')).

Lemma **5.1.6.** — *Suppose the pair of sequences of widths* (B, A) *satisfies* **DNP**. *Let $\Delta^j = \Delta^k \cap \Delta^l$, $j < k, l$. Then*

$$\mathsf{N}_{\beta_k}\big(\Delta^k, \mathbf{S}^m\big) \bigcap \mathsf{N}_{\beta_l}\big(\Delta^l, \mathbf{S}^m\big) \subset \bigcup_{i \le j} \mathsf{N}_{\alpha_i}\big(\Delta^i, \mathbf{S}^m\big).$$

Remark **5.1.7.** — Note that the condition $\Delta^j = \Delta^k \cap \Delta^l$, $j < k, l$, is equivalent to $\Delta^k \not\subset \Delta^l$ and $\Delta^l \not\subset \Delta^k$, where the empty set is considered a simplex of dimension -1.

Proof of Lemma 5.1.6. — From Proposition 5.1.3 we have $\frac{\sin\beta_l}{\sin\alpha_j}, \frac{\sin\beta_k}{\sin\alpha_j} < \frac{\sqrt{2}}{2}$. The lemma now follows from Lemma 5.1.4. □

5.2. *Natural neighborhoods on the sphere* $\mathbf{S}^m$

Let $\Delta = \Delta^k \in \mathbf{S}^m$. In this section the β-geometric link at the barycenter of Δ (see Remark 5.0.1) will be called *the link sphere of* Δ *of radius* β, and will be denoted by $\mathsf{S}_\Delta = \mathsf{S}_\Delta^\beta$. Rescaling gives an identification between S_{Δ^k} and $\mathbf{S}^{m-k-1}$, thus we will consider S_Δ as an all-right spherical complex (alternatively we can consider S_{Δ^k} with the angle metric). The simplices of S_Δ are $\mathsf{S}_\Delta \cap \Delta'$, $\Delta' \supset \Delta$.

Let $\Delta = \Delta^k$, where $\Delta^k \subset \Delta^j \in \mathbf{S}^m$, and $\gamma \in (0, \pi/2)$. It is straightforward to verify that there is $\beta' \in (0, \pi/2]$ such that

$$\mathsf{S}_\Delta^\beta \cap \mathsf{N}_\gamma\big(\Delta^j, \mathbf{S}^m\big) = \mathsf{N}_{\beta'}\big(\mathsf{S}_\Delta^\beta \cap \Delta^j, \mathsf{S}_\Delta^\beta\big),$$

where the last term is the β'-normal neighborhood of the simplex $\mathsf{S}_\Delta^\beta \cap \Delta^j$ in S_Δ^β. Recall that we are identifying S_Δ with $\mathbf{S}^{m-k-1}$, or, using the angle metric. Therefore the equality above says that the set on the left of the equality is equal, after rescaling, to the right side of the equality. The next lemma gives a relationship between β, β' and γ. Note that when $\gamma \geq \beta$ then $\beta' = \pi/2$.

Lemma **5.2.1.** — *Let* β, β' *and* γ *be as above, with* $\gamma < \beta$. *Then* $\sin\beta' = \frac{\sin\gamma}{\sin\beta}$.

Proof. — Let $p \in \mathsf{S}_\Delta^\beta \cap \mathsf{N}_\gamma(\Delta^j, \mathbf{S}^m)$, where $\Delta = \Delta^k \subset \Delta^j$. Then there is a $q \in \Delta^j$ such that $d = d_{\mathbf{S}^m}(p, q) = d_{\mathbf{S}^m}(p, \Delta^j) \leq \gamma$. We are interested in the case when d is maximum, so we assume $d = \gamma$. Let o be the barycenter of Δ. Since $d_{\mathbf{S}^m}(o, p) = \beta$ we get a right (at q) spherical triangle with one side equal to γ and hypotenuse equal to β. The angle opposite to the side of length γ is β'. Then, by the spherical law of sines we get $\frac{\sin\beta}{1} = \frac{\sin\gamma}{\sin\beta'}$. □

Let $\mathsf{B} = \{\beta_k\}$ be a sequence of widths. Let $\Delta = \Delta^k \in \mathbf{S}^m$ and $\mathsf{S}_\Delta = \mathsf{S}_\Delta^{\beta_k}$ be the link sphere of Δ of radius β_k. By intersecting S_Δ with each element of the set $\mathsf{N}_\mathsf{B}(\mathbf{S}^m)$ we get the set $\mathsf{N}(\mathsf{S}_\Delta, \mathsf{B}) = \{\mathsf{S}_\Delta \cap \mathsf{N}_{\beta_j}(\Delta^j, \mathbf{S}^m)\}_{\Delta^j \in \mathbf{S}^m}$. It follows from Lemma 5.2.1 that for simplices Δ^j, with $\Delta \subsetneq \Delta^j$, there are decreasing $\beta'_{j-k-1} > 0$ such that

$$\mathsf{S}_\Delta \cap \mathsf{N}_{\beta_j}\big(\Delta^j, \mathbf{S}^m\big) = \mathsf{N}_{\beta'_{j-k-1}}\big(\mathsf{S}_\Delta \cap \Delta^j, \mathsf{S}_\Delta\big).$$

Since the β_i's are decreasing, we have $\beta'_l < \pi/2$. Hence we can write $\mathsf{N}(\mathsf{S}_\Delta, \mathsf{B}) = \mathsf{N}_{\mathsf{B}'(m-k-1)}(\mathsf{S}_\Delta)$ where $\mathsf{B}'(m-k-1) = \{\beta'_0, \ldots, \beta'_{m-k-2}\}$ and we also say that $\mathsf{N}_{\mathsf{B}'(m-k-1)}(\mathsf{S}_\Delta)$ is the set of $\mathsf{B}'(m-k-1)$-neighborhoods of S_Δ. Note that $\mathsf{B}'(m-k-1)$ depends only on

B and the dimension k of Δ^k. The next corollary, which is immediate from Lemma 5.2.1, gives this relation explicitly.

Corollary **5.2.2.** — *For* $l = 0, \ldots, m-k-2$ *we have* $\sin(\beta'_l) = \frac{\sin(\beta_{k+l+1})}{\sin(\beta_k)}$.

Let $\mathsf{B} = \{\beta_i\}_{i=0,1,\ldots}$ be a sequence of widths. We say that B is a *natural set of neighborhood widths for all spheres* if $\mathsf{B}(m-k-1) = \mathsf{B}'(m-k-1)$ for all m and k with $m > k$.

Corollary **5.2.3.** — *The sequence of widths* $\mathsf{B} = \{\beta_i\}$ *is natural if and only if* $\sin(\beta_i) = \sin^{i+1}(\beta_0)$ *and* $\beta_0 < \pi/4$.

Proof. — It follows from Corollary 5.2.2 with $l = 0$ that $\sin(\beta_{k+1}) = \sin(\beta_k)\sin(\beta_0)$. □

Given $\varsigma \in (0,1)$ we define $\mathsf{B}(\varsigma) = \{\beta_i\}$ by $\beta_i = \sin^{-1}(\varsigma^{i+1})$. Hence the corollary says that B is natural if and only if $\mathsf{B} = \mathsf{B}(\varsigma)$, for some $\varsigma \in (0,1)$. In fact, in this case we have $\varsigma = \sin(\beta_0)$.

Let $\varsigma \in (0,1)$ and $c > 1$. We denote by $\mathsf{B}(\varsigma; c) = \{\gamma_i\}$ the set defined by $\gamma_i = \sin^{-1}(c\varsigma^{i+1})$. Note that $\mathsf{B}(\varsigma; c)$ is a sequence of widths provided $c\varsigma < 1$. Proposition 5.1.3 implies the next corollary.

Corollary **5.2.4.** — *The pair of sequence of widths* $(\mathsf{B}(\varsigma; c), \mathsf{B}(\varsigma; c'))$ *satisfy* **DNP** *provided* $\frac{c}{c'}\varsigma < \frac{\sqrt{2}}{2}$.

If $c = c' = 1$, then a natural sequence of widths satisfies **DNP** with $\mathsf{A} = \mathsf{B} = \mathsf{B}(\varsigma)$.

5.3. *Neighborhoods in piecewise spherical complexes*

This subsection is essentially a version of Section 5.1 in which we replace $\mathbf{S}^m$ by an arbitrary all-right spherical complex. Let P be an all-right spherical complex and $\Delta^j \in$ P. As before $\dot{\Delta}$ is interior of Δ. We can write $\mathsf{Link}(\Delta^j, \mathrm{P}) = \bigcup_{\Delta^j \subset \Delta^k \in \mathrm{P}} \mathsf{Link}(\Delta^j, \Delta^k)$ as sets and complexes (see Remark 5.0.2). The set $\{\Delta^k\}_{\Delta^j \subsetneq \Delta^k \in \mathrm{P}}$ is in one-to-one correspondence with the set of spherical simplices of $\mathsf{Link}(\Delta^j, \mathrm{P})$, that is Δ^k corresponds to $\mathsf{Link}(\Delta^j, \Delta^k)$, which is an all-right spherical simplex of dimension $k-j-1$ in $\mathsf{Link}(\Delta^j, \mathrm{P})$. The all-right spherical metric on $\mathsf{Link}(\Delta^j, \mathrm{P})$ will be denoted by $\sigma_{\mathsf{Link}(\Delta^j,\mathrm{P})}$.

Remark **5.3.1.** — The above paragraph is valid regardless of the type of link, see Remarks 5.0.1, 5.0.2, and all definitions of $\mathsf{Link}(\Delta, \mathrm{P})$ are equivalent as metric complexes because P has the "intersection condition". Any of these will lead to the corresponding definition of $\mathsf{Link}(\Delta, \mathrm{P})$, but they are all equivalent as metric complexes (note that we are assuming P has the "intersection condition"). We will use any of the definitions depending on the situation.

Lemma **5.3.2.** — *Let* $\Delta^j \subsetneq \Delta^k \in \mathrm{P}$. *Then*

$$\mathsf{Link}\big(\mathsf{Link}\big(\Delta^j, \Delta^k\big), \mathsf{Link}\big(\Delta^j, \mathrm{P}\big)\big) = \mathsf{Link}\big(\Delta^k, \mathrm{P}\big).$$

Remark **5.3.3.** — The equation in Lemma 5.3.2 is an equality of all-right spherical metric complexes. If we use the simplicial definition of link it is an equality of sets. In this case the lemma takes the form $\mathsf{Link}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P})) = \mathsf{Link}(\Delta^k, \mathrm{P})$, where $\Delta^l = \mathsf{Link}(\Delta^j, \Delta^k)$ is opposite to Δ^j in Δ^k (see Remark 5.0.2).

Proof. — Let $\Delta^j \subsetneq \Delta^k$. Let Δ^l be the opposite face of Δ^j in Δ^k. We use the simplicial definition of link, so we have to prove: $\mathsf{Link}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P})) = \mathsf{Link}(\Delta^k, \mathrm{P})$ (see Remark 5.3.3). We have that $\Delta^i \in \mathsf{Link}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P}))$ if and only the following two statements hold: (i) $\Delta^i \cap \Delta^l = \emptyset$, (ii) $\Delta^i \cup \Delta^l$ is contained in a simplex in $\mathsf{Link}(\Delta^j, \mathrm{P})$. But statements (i) and (ii) hold if and only if the following four statements hold: (recall $\Delta^j \cap \Delta^l = \emptyset$) (1) $\Delta^i \cap \Delta^j = \emptyset$, (2) $\Delta^i \cup \Delta^j$ is contained in a simplex, (3) $\Delta^i \cap \Delta^l = \emptyset$, (4) $\Delta^i \cup \Delta^l \cup \Delta^j$ is contained in a simplex. On the other hand $\Delta^i \in \mathsf{Link}(\Delta^k, \mathrm{P})$ if and only if the following two statements are true: (a) $\Delta^i \cap \Delta^k = \emptyset$, (b) $\Delta^i \cup \Delta^k$ is contained in a simplex. Since Δ^l is opposite to Δ^j in Δ^k we have that statements (a) and (b) are true if and only if statements (1) to (4) are true. □

Let $\Delta^j \subset \Delta^k$. Define the *closed normal neighborhood of* Δ^j *in* Δ^k *of width* β as $\mathsf{N}_\beta(\Delta^j, \Delta^k) = \mathsf{N}_\beta(\Delta^j, \mathbf{S}^m) \cap \Delta^k$. Note that this subset of Δ^k does not depend on the particular isometric embedding $\Delta^k \hookrightarrow \mathbf{S}^m$. If Δ^j is a simplex in the all-right spherical complex P, we define the *closed normal neighborhood of* Δ^j *in* P *of width* β as

$$\mathsf{N}_\beta\big(\Delta^j, \mathrm{P}\big) = \bigcup_{\Delta^j \subset \Delta^k \in \mathrm{P}} \mathsf{N}_\beta\big(\Delta^j, \Delta^k\big).$$

Hence $\mathsf{N}_\beta(\Delta^j, \mathrm{P})$ is the union of (images of) geodesics of length β emanating perpendicularly from Δ^j.

Let $\mathsf{B} = \{\beta_k\}$ be a sequence of widths. Then, for any all-right spherical complex P the set B induces the set of neighborhoods $\mathsf{N}_\mathsf{B}(\mathrm{P}) = \{\mathsf{N}_{\beta_k}(\Delta^k, \mathrm{P})\}_{\Delta^k \in \mathrm{P}}$. The next lemma is the spherical complex version of Lemma 5.1.4.

Corollary **5.3.4.** — *Let* $\Delta^k, \Delta^l \in \mathrm{P}$ *and* $\Delta^j = \Delta^k \cap \Delta^l$. *Let* $\alpha, \beta, \gamma \in (0, \pi/2)$ *such that* $\frac{\sin\beta}{\sin\alpha}, \frac{\sin\gamma}{\sin\alpha} \leq \frac{\sqrt{2}}{2}$. *Then* $\mathsf{N}_\beta(\Delta^k, \mathrm{P}) \cap \mathsf{N}_\gamma(\Delta^l, \mathrm{P}) \subset \mathsf{N}_\alpha(\Delta^j, \mathrm{P})$.

Proof. — The proof is the same as the proof of Lemma 5.1.4, just replace $\mathbf{S}^m$ by P. Recall that we are assuming P to have the intersection condition. □

As in Section 5.1, the next two results follow directly from Corollary 5.3.4. The first is a version of **DNP** (see (5.1.2)) for P, obtained by replacing $\mathbf{S}^m$ by P.

Corollary **5.3.5.** — *Let the pair of sequences of widths* (B, A) *satisfy* **DNP**. *Then for any all-right spherical complex* P *and* k *the following sets are disjoint*

$$\left\{ \mathsf{N}_{\beta_k}(\Delta^k, \mathrm{P}) - \bigcup_{j<k} \mathsf{N}_{\beta_j}(\Delta^j, \mathrm{P}) \right\}_{\Delta^k \in \mathrm{P}}.$$

The next is a version of Lemma 5.1.6 for general P.

Corollary **5.3.6.** — *Let the pair of sequences of widths* (B, A) *satisfy* **DNP**. *Then*

$$\mathsf{N}_{\beta_k}(\Delta^k, \mathrm{P}) \bigcap \mathsf{N}_{\beta_l}(\Delta^l, \mathrm{P}) \subset \bigcup_{i \le j} \mathsf{N}_{\alpha_i}(\Delta^i, \mathrm{P}),$$

for any all-right spherical complex P *and* $\Delta^j = \Delta^k \cap \Delta^l$, $j < k, l$, *simplices in* P.

6. Normal neighborhoods on hyperbolic cones

In this section we give some properties of neighborhoods of faces in hyperbolic cones, and define some special type of neighborhoods. Hyperbolic cones are cones over all-right spherical complexes; they admit a canonical piecewise hyperbolic metric. To define the special type of neighborhoods on hyperbolic cones we will use the objects and results of Section 5.

6.1. *Neighborhoods in piecewise hyperbolic cones*

We write $\mathbf{R}^{k+1}_+ = (0, \infty)^{k+1}$, $\bar{\mathbf{R}}^{k+1}_+ = [0, \infty)^{k+1}$ and $\bar{\mathbf{H}}^{k+1}_+ = \mathbf{B}^{k+1}_{\mathbf{H}} \cap \bar{\mathbf{R}}^{k+1}_+$, where $\mathbf{B}^{k+1}_{\mathbf{H}}$ is the disc model of $\mathbf{H}^{k+1}$. The canonical all-right spherical k-simplex is $\Delta_{\mathbf{S}^k} = \mathbf{S}^k \cap \bar{\mathbf{R}}^{k+1}_+$. We denote the origin of $\mathbf{H}^{k+1}$ by $o = o_{\mathbf{H}^{k+1}}$. We can identify $\bar{\mathbf{H}}^{k+1}_+ - \{o\}$ with $\Delta_{\mathbf{S}^k} \times \mathbf{R}^+$ with the metric $\sinh^2 s \sigma_{\mathbf{S}^k} + ds^2$, where s is the distance to the "vertex" o. We say that $\bar{\mathbf{H}}^{k+1}_+$ is the *infinite hyperbolic cone of* $\Delta_{\mathbf{S}^k}$ and write $\mathrm{C}\Delta_{\mathbf{S}^k} = \bar{\mathbf{H}}^{k+1}_+$.

Let P be an all-right spherical complex. The piecewise spherical path metric on P will be denoted by σ_{P}. Recall that P is constructed by gluing the all-right spherical simplices $\Delta \in \mathrm{P}$ via isometries, where each $\Delta = \Delta^k$ is a copy of $\Delta_{\mathbf{S}^k}$, for k depending on Δ. The *infinite piecewise hyperbolic cone of* P is the space CP obtained by gluing the hyperbolic cones $\mathrm{C}\Delta$, $\Delta \in \mathrm{P}$ using the same rules used for obtaining P. The gluings of the $\mathrm{C}\Delta$ use the identity on $[0, \infty)$. Note that all vertex points of the $\mathrm{C}\Delta$ get glued to a unique *vertex* $o = o_{\mathrm{CP}}$. The cones $\mathrm{C}\Delta$, $\Delta \in \mathrm{P}$, are the *cone simplices of* CP and the *faces* of the cone simplex $\mathrm{C}\Delta$ are of the form $\mathrm{C}\Delta'$ where $\Delta' \subset \Delta$. The set of all cone simplices will also be denoted by CP. The complex CP (i.e. CP together with its cone faces) is an *all-right hyperbolic cone complex.*

The piecewise hyperbolic metric on CP shall be denoted by σ_{CP} and its corresponding geodesic metric by d_{CP}. Note that CP is smooth and hyperbolic outside the cone of the codimension 2 skeleton of P. All (constant speed) rays emanating from o are length minimizing geodesics defined on $[0,\infty)$. Then we can identify $\mathrm{CP}-\{o\}$ with $\mathrm{P}\times\mathbf{R}^+$ with warp product metric $\sinh^2 s\sigma_{\mathrm{P}}+ds^2$, where r is the distance to the vertex o. Even though σ_{CP} is not (generally) smooth, the set of speed one geodesic rays emanating from the vertex o_{CP} gives a well defined set of rays as in Section 1.

Remark **6.1.1.** — It also makes sense to consider the concept of a partially defined metric f on $\mathrm{U}\subset\mathrm{CP}$ being ray equivalent with (CP,o), see Remark 1.1. Moreover, the remarks in 1.1 are still valid in this context.

For $s\geq 0$ we denote the *open ball of radius* s *of* CP *centered at* o by $\mathbf{B}_s(\mathrm{CP})$. Note that this ball is the "finite open cone" $\mathrm{P}\times(0,s)\cup\{o\}$, where we are using the identification above. The *closed ball* will be denoted by $\bar{\mathbf{B}}_s(\mathrm{CP})$ and the *sphere of radius* s, $s>0$, will be denoted by $\mathbf{S}_s(\mathrm{CP})$, which we shall sometimes identify with $\mathrm{P}\times\{s\}$ or simply with P.

Let $\Delta\in\mathrm{P}$. In this section $\mathsf{Star}(\Delta,\mathrm{P})$ and $\mathsf{Link}(\Delta,\mathrm{P})$ will denote the simplicial star and link of Δ in P, respectively. Since $\mathsf{Star}(\Delta,\mathrm{P})$ is an all-right spherical complex then $\mathrm{C}(\mathsf{Star}(\Delta,\mathrm{P}))$ is a well defined all-right hyperbolic cone complex, which we could interpret as the simplicial star of $\mathrm{C}\Delta$ in CP. To save parentheses we will write $\mathrm{C}\mathsf{Star}(\Delta,\mathrm{P})$ instead of $\mathrm{C}(\mathsf{Star}(\Delta,\mathrm{P}))$ and $\mathrm{C}\mathsf{Link}(\Delta,\mathrm{P})$ instead of $\mathrm{C}(\mathsf{Link}(\Delta,\mathrm{P}))$. Note that $\mathrm{C}\mathsf{Link}(\Delta,\mathrm{P}),\ \mathrm{C}\mathsf{Star}(\Delta,\mathrm{P})\subset\mathrm{CP}$.

Items 6.1.2, 6.1.3, and 6.1.4 below, will be very useful in what follows of Section 6 and in Section 8. We will use the notation $\mathcal{E}_{\mathrm{A}}(\mathrm{S})$ given at the beginning of Section 2.

6.1.2. Let $\Delta^j\subset\Delta^k$ and let Δ^l be the opposite face of Δ^j in Δ^k. Thus $\Delta^k=\Delta^j*\Delta^l$. We have $l=k-j-1$. As we mentioned at the beginning of Section 6.1 we can write $\mathrm{C}\Delta^j=\bar{\mathbf{H}}^{j+1}_+\subset\mathbf{H}^{j+1}$, $\mathrm{C}\Delta^l=\bar{\mathbf{H}}^{l+1}_+\subset\mathbf{H}^{l+1}$. Also, $\mathbf{H}^{k+1}=\mathbf{H}^{j+1}\times\mathbf{H}^{l+1}$ with warp product metric $\cosh^2 r\sigma_{\mathbf{H}^{j+1}}+\sigma_{\mathbf{H}^{l+1}}$, where r is the distance in $\mathbf{H}^{l+1}$ to a fixed center o. Equivalently $\mathbf{H}^{k+1}=\mathcal{E}_{j+1}\mathbf{H}^{l+1}$. Therefore we can write

$$\mathrm{C}\Delta^k=\mathrm{C}\Delta^j\times\mathrm{C}\Delta^l\subset\mathbf{H}^{j+1}\times\mathbf{H}^{l+1}=\mathcal{E}_{j+1}\mathbf{H}^{l+1},$$

with warp product metric $\cosh^2 r\sigma_{\mathbf{H}^{j+1}}+\sigma_{\mathbf{H}^{l+1}}$, where r is the distance in $\mathbf{H}^{l+1}$ to o. Thus we can write $\mathrm{C}\Delta^k=\mathcal{E}_{\mathrm{C}\Delta^j}(\mathrm{C}\Delta^l)$. (Here we using the definition of partial hyperbolic extension $\mathcal{E}_{\mathrm{A}}(\mathrm{S})$ given at the beginning of Section 2.) Note that the order of the decomposition here is important (see Proposition 2.5, Remark 2.6(1)). The identification above can be done explicitly in the following way. Let $p\in\mathrm{C}\Delta^k\subset\mathbf{H}^{k+1}=\mathcal{E}_{\mathbf{H}^{j+1}}(\mathbf{H}^{l+1})$. We use the functions (or coordinates) given in Section 2: s, r, t, y, v, x, u, w. Then $p=sx\in\mathrm{C}\Delta^k$, $(s,x)\in\mathbf{R}^+\times\Delta^k$, corresponds to $(y,v)=(tw,ru)\in\mathrm{C}\Delta^j\times\mathrm{C}\Delta^l$, $(t,w)\in\mathbf{R}^+\times\Delta^j$, $(r,u)\in\mathbf{R}^+\times\Delta^l$. Note that $x=[w,u](\beta)$, where β is as in Section 2, i.e. it is the angle between w and x, and $[w,u]$ is the spherical segment in $\Delta^k=\Delta^j*\Delta^l$ from $w\in\Delta^j$ to $u\in\Delta^l$.

6.1.3. Fix $\Delta = \Delta^j \in \mathrm{P}$. Then the cone simplices in $\mathsf{CStar}(\Delta, \mathrm{P})$ are $\mathrm{C}\Delta^k$, where $\Delta^k \supset \Delta^j$ (thus $\Delta^k = \Delta^j * \Delta^l$, where Δ^l is opposite to Δ^j in Δ^k). We can now apply the identification in 6.1.2 to each cone $\mathrm{C}\Delta^k$ where $\Delta^k \supset \Delta^j$. Gluing all these identifications we obtain the following important identification:

$$\mathsf{CStar}(\Delta, \mathrm{P}) = \mathrm{C}\Delta \times \mathsf{CLink}(\Delta, \mathrm{P}),$$

where we consider the term on the right $\mathrm{C}\Delta \times \mathsf{CLink}(\Delta, \mathrm{P})$ with the metric $\cosh^2 r\sigma_{\mathbf{H}^{j+1}} + \sigma_{\mathsf{CLink}(\Delta,\mathrm{P})}$, and r is the distance in $\mathsf{CLink}(\Delta, \mathrm{P})$ to the vertex $o \in \mathsf{CLink}(\Delta, \mathrm{P})$. This identification will be used many times. Note that the vertex of $\mathsf{CStar}(\Delta, \mathrm{P})$ is identified with (o', o''), where o', o'' are the vertices of $\mathrm{C}\Delta$ and $\mathsf{CLink}(\Delta, \mathrm{P})$, respectively. The identification here is an identification of all-right hyperbolic cone complexes. Explicitly using the coordinates s, r, t, y, v, x, u, w given in Section 2 we see that an element $p = sx \in \mathrm{C}\Delta^k \in \mathsf{CStar}(\Delta, \mathrm{P})$, where $\Delta^k = \Delta^j * \Delta^l$, can be written as $(tw, ru) \in \mathrm{C}\Delta^j \times \mathrm{C}\Delta^l \subset \mathrm{C}\Delta^j \times \mathsf{CLink}(\Delta, \mathrm{P})$, using that Δ^l is a simplex in $\mathsf{Link}(\Delta, \mathrm{P})$. Since we can write $x = [w, u](\beta)$, where β is the angle between w and x, the identification is given by $s[w, u](\beta) = (tw, ru)$.

6.1.4. As mentioned above, even though $\sigma_{\mathsf{CLink}(\Delta,\mathrm{P})}$ is not in general smooth it has a well defined set of rays, where we are taking $o_{\mathrm{CP}} = o_{\mathsf{CLink}(\Delta,\mathrm{P})}$ as the *center* of $\mathsf{CLink}(\Delta, \mathrm{P})$. Hence it makes sense to consider, as in Section 2, the *hyperbolic extension* $\mathcal{E}_j(\mathsf{CLink}(\Delta, \mathrm{P})) = \mathrm{C}\Delta \times \mathsf{CLink}(\Delta, \mathrm{P})$ with the metric $\cosh^2 r\sigma_{\mathbf{H}^{j+1}} + \sigma_{\mathsf{CLink}(\Delta,\mathrm{P})}$. Therefore, using 6.1.3, we can write

$$\mathsf{CStar}(\Delta, \mathrm{P}) = \mathcal{E}_{\mathrm{C}\Delta}\big(\mathsf{CLink}(\Delta, \mathrm{P})\big) \subset \mathcal{E}_j\big(\mathsf{CLink}(\Delta, \mathrm{P})\big),$$

where we consider $\mathsf{CStar}(\Delta, \mathrm{P}) \subset \mathrm{CP}$ with the metric σ_{CP} and $\mathsf{CLink}(\Delta, \mathrm{P})$ with the metric $\sigma_{\mathsf{CLink}(\Delta,\mathrm{P})}$.

6.1.5. Note that the cone (or center) point $o = o_{\mathrm{CP}}$ of CP belongs to $\mathsf{CLink}(\Delta, \mathrm{P})$; furthermore, every geodesic $s \mapsto sx$ in CP (with the metric σ_{CP}) emanating from o, with $x \in \mathsf{Star}(\Delta, \mathrm{P})$ is contained in $\mathsf{CStar}(\Delta, \mathrm{P})$. Therefore, it follows from 6.1.4 that these geodesics coincide with the geodesics in $\mathcal{E}_{\mathrm{C}\Delta}(\mathsf{CLink}(\Delta, \mathrm{P}))$, emanating from $o \in \mathsf{CStar}(\Delta, \mathrm{P}) \subset \mathrm{CP}$. In other words, the metric on $\mathsf{CStar}(\Delta, \mathrm{P})$ is ray compatible with both, (CP, o) and $(\mathcal{E}_{\mathrm{C}\Delta}(\mathsf{CLink}(\Delta, \mathrm{P})), o)$, see Remarks 6.1.1 and 1.1.

Remark. — The set $\mathsf{CStar}(\Delta, \mathrm{P})$ is not open in CP, nor in $\mathcal{E}_{\mathrm{C}\Delta}(\mathsf{CLink}(\Delta, \mathrm{P}))$. The second condition given in (1) of Remark 1.1 in our contex here means that the geodesics are perpendicular to the spheres on each cone $\mathrm{C}\Delta$, for $\Delta \in \mathsf{Star}(\Delta, \mathrm{P})$.

The next lemma tells us how the identification in 6.1.3 behaves when we pass to subsimplices.

Lemma **6.1.6.** — *Let* $\Delta^j \subset \Delta^k \in \mathrm{P}$. *Then*

$$\mathsf{CStar}(\Delta^k, \mathrm{P}) = \mathrm{C}\Delta^k \times \mathsf{CLink}(\Delta^k, \mathrm{P}) = \mathrm{C}\Delta^j \times \mathrm{C}\Delta^l \times \mathsf{CLink}(\Delta^k, \mathrm{P})$$
$$= \mathrm{C}\Delta^j \times \mathsf{CStar}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P})) \subset \mathrm{C}\Delta^j \times \mathsf{CLink}(\Delta^j, \mathrm{P}),$$

where $\Delta^l = \mathsf{Link}(\Delta^j, \Delta^k)$ *is the opposite face of* Δ^j *in* $\Delta^k = \Delta^j * \Delta^l$.

Proof. — The first equality is given in 6.1.3 above. The last inclusion follows from the inclusion $\mathsf{Star}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P})) \subset \mathsf{Link}(\Delta^j, \mathrm{P})$. The two middle equalities in the statement of the lemma are equalities of hyperbolic cone complexes. We have

$$\mathrm{C}\Delta^k \times \mathsf{CLink}(\Delta^k, \mathrm{P}) = (\mathrm{C}\Delta^j \times \mathrm{C}\Delta^l) \times \mathsf{CLink}(\Delta^k, \mathrm{P})$$
$$= \mathrm{C}\Delta^j \times (\mathrm{C}\Delta^l \times \mathsf{CLink}(\Delta^k, \mathrm{P}))$$
$$= \mathrm{C}\Delta^j \times (\mathrm{C}\Delta^l \times \mathsf{CLink}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P})))$$
$$= \mathrm{C}\Delta^j \times \mathsf{CStar}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P}))$$

where the first equality follows from 6.1.2, and the third one from Lemma 5.3.2, and the fact that $\Delta^l = \mathsf{Link}(\Delta^j, \Delta^k)$. Finally the fourth equality follows from 6.1.3. □

Here is a metric version of Lemma 6.1.6. Let Δ^j, Δ^k, and Δ^l be as in Lemma 6.1.6. Fix a homeomorphism $h : \mathbf{S}^{m-k-1} \to \mathsf{Link}(\Delta^k, \mathrm{P})$ and consider its cone $\mathrm{C}h : \mathbf{R}^{m-k} \to \mathsf{CLink}(\Delta^k, \mathrm{P})$. Let f' be a metric on $\mathbf{R}^{m-k}$ of the form $f' = f'_r + dr^2$. Thus f' and $\sigma_{\mathbf{R}^{m-k}}$ have the same set of rays. The metric $f = (\mathrm{C}h)_* f'$ is a metric on $\mathsf{CLink}(\Delta^k, \mathrm{P})$ in the smooth structure induced by $\mathrm{C}h$, and it has the same set of rays as $\sigma_{\mathsf{CLink}(\Delta^k, \mathrm{P})}$. We can consider the (restriction of the) metric $\mathcal{E}_k(f)$ defined on $\mathcal{E}_k(\mathsf{CLink}(\Delta^k, \mathrm{P}))$ to $\mathcal{E}_{\mathrm{C}\Delta^k}(\mathsf{CLink}(\Delta^k, \mathrm{P})) = \mathrm{C}\Delta^k \times \mathsf{CLink}(\Delta^k, \mathrm{P})$. And, since we have $\mathsf{Link}(\Delta^k, \mathrm{P}) = \mathsf{Link}(\Delta^l, \mathsf{CLink}(\Delta^j, \mathrm{P}))$ (see Lemma 5.3.2) the metric f is also a metric on $\mathsf{CLink}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P}))$, and we can consider the metric $\mathcal{E}_j(\mathcal{E}_l(f))$ on $\mathcal{E}_{\mathrm{C}\Delta^j}(\mathcal{E}_{\mathrm{C}\Delta^l}(\mathsf{CLink}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P})))) = \mathrm{C}\Delta^j \times \mathrm{C}\Delta^l \times \mathsf{CLink}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P}))$.

Corollary **6.1.7.** — *Using the identification in Lemma* 6.1.6 *we get* $\mathcal{E}_k(f) = \mathcal{E}_j(\mathcal{E}_l(f))$.

Proof. — The proof follows from Proposition 2.5 and the proof of Lemma 6.1.6. □

Taking $f = \sigma_{\mathsf{CLink}(\Delta^k, \mathrm{P})}$ gives the following corollary which follows from 6.1.4 and Corollary 6.1.7.

Corollary **6.1.8.** — *Let* Δ^k, Δ^j, Δ^l *as in Lemma* 6.1.6. *Then*

$$\mathsf{CStar}(\Delta^k, \mathrm{P}) = \mathcal{E}_{\mathrm{C}\Delta^k}(\mathsf{CLink}(\Delta^k, \mathrm{P})) = \mathcal{E}_{\mathrm{C}\Delta^j}(\mathcal{E}_{\mathrm{C}\Delta^l}(\mathsf{CLink}(\Delta^k, \mathrm{P})))$$
$$= \mathcal{E}_{\mathrm{C}\Delta^j}(\mathcal{E}_{\mathrm{C}\Delta^l}(\mathsf{CLink}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P})))),$$

where $\mathsf{CStar}(\Delta, \mathrm{P}) \subset \mathrm{CP}$ *is considered with the metric* σ_{CP}*,* $\mathrm{C}(\mathsf{Link}(\Delta, \mathrm{P}))$ *with the metric* $\sigma_{\mathsf{CLink}(\Delta,\mathrm{P})}$*, and* $\mathsf{CLink}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P}))$ *with the metric* $\sigma_{\mathsf{CLink}(\Delta^l,\mathsf{Link}(\Delta^j,\mathrm{P}))}$.

For a cone simplex $\mathrm{C}\Delta \in \mathrm{CP}$, we define its *closed normal neighborhood of width s* by

$$\textbf{(6.1.9)} \qquad \mathsf{N}_s(\mathrm{C}\Delta, \mathrm{CP}) = \mathrm{C}\Delta \times \bar{\mathbf{B}}_s\big(\mathsf{CLink}(\Delta, \mathrm{P})\big) \subset \mathsf{CStar}(\Delta, \mathrm{P}),$$

where we are using the identification given in 6.1.3. Hence $\mathsf{N}_s(\mathrm{C}\Delta, \mathrm{CP})$ is the union of (the images of) all geodesics of length s emanating perpendicularly from $\mathrm{C}\Delta$. The *open normal neighborhood of width s* will be denoted by $\overset{\circ}{\mathsf{N}}_s(\mathrm{C}\Delta, \mathrm{CP})$. For $\mathrm{A} \subset \Delta$ we will write $\mathsf{N}_s(\mathrm{CA}, \mathrm{CP}) = \mathrm{CA} \times \bar{\mathbf{B}}_s(\mathsf{CLink}(\Delta, \mathrm{P})) \subset \mathsf{N}_s(\Delta, \mathrm{CP})$.

The next two results will be needed in Section 6.2.

Lemma **6.1.10.** — *Let* $\Delta^j \subset \Delta^k \in \mathrm{P}$*. Then*

$$\mathsf{N}_s\big(\mathrm{C}\Delta^k, \mathrm{CP}\big) = \mathrm{C}\Delta^j \times \mathsf{N}_s\big(\mathrm{C}\Delta^l, \mathsf{CLink}\big(\Delta^j, \mathrm{P}\big)\big),$$

where $\Delta^l = \mathsf{Link}(\Delta^j, \Delta^k)$*. A similar statement holds if we replace* N *by* $\overset{\circ}{\mathsf{N}}$.

Remark **6.1.11.** — Note that $\mathsf{N}_s(\mathrm{C}\Delta^k, \mathrm{CP})$ is a subset of $\mathsf{CStar}(\Delta^k, \mathrm{P})$. The right-hand side is a subset of $\mathrm{C}\Delta^j \times \mathsf{CLink}(\Delta^j, \mathrm{P})$. By Lemma 6.1.6 we can write $\mathsf{CStar}(\Delta^k, \mathrm{P}) \subset \mathrm{C}\Delta^j \times \mathsf{CLink}(\Delta^j, \mathrm{P})$. Lemma 6.1.10 says that under this inclusion $\mathsf{N}_s(\mathrm{C}\Delta^k, \mathrm{CP})$ corresponds to $\mathrm{C}\Delta^j \times \mathsf{N}_s(\mathrm{C}\Delta^l, \mathsf{CLink}(\Delta^j, \mathrm{P}))$.

Proof. — We have

$$\begin{aligned}\mathsf{N}_s\big(\mathrm{C}\Delta^k, \mathrm{CP}\big) &= \mathrm{C}\Delta^j \times \big(\mathrm{C}\Delta^l \times \bar{\mathbf{B}}_s\big(\mathsf{CLink}\big(\Delta^k, \mathrm{P}\big)\big)\big)\\ &= \mathrm{C}\Delta^j \times \mathsf{N}_s\big(\mathrm{C}\Delta^l, \mathsf{CLink}\big(\Delta^j, \mathrm{P}\big)\big),\end{aligned}$$

where the first equality follows from (6.1.9) and 6.1.2 and the last from Lemma 5.3.2 and (6.1.9). □

Lemma **6.1.12.** — *Let* $s > 0$*,* $\beta \in (0, \pi/2)$ *and* $\Delta \in \mathrm{P}$*. Then*

$$\mathsf{N}_{s_\beta}(\mathrm{C}\Delta, \mathrm{CP}) \cap \mathbf{S}_s(\mathrm{CP}) = \mathsf{N}_\beta(\Delta, \mathrm{P}) \times \{s\} \subset \mathrm{P} \times \{s\} = \mathbf{S}_s(\mathrm{CP})$$

where $s_\beta = \sinh^{-1}(\sinh s \sin\beta)$.

Proof. — Recall that in the identification $\mathsf{CStar}(\Delta, \mathrm{P}) = \mathrm{C}\Delta \times \mathsf{CLink}(\Delta, \mathrm{P})$ given in 6.1.3 the vertex o of $\mathsf{CStar}(\Delta, \mathrm{P})$ is identified with (o', o''), where o', o'' are the vertices of $\mathrm{C}\Delta$ and $\mathsf{CLink}(\Delta, \mathrm{P})$, respectively.

Let $p \in \mathsf{N}_t(\Delta, \mathrm{P}) \cap \mathbf{S}_s(\mathrm{CP})$, for some t, $0 < t < s$. Then $d_{\mathrm{CP}}(o, p) = s$. From (6.1.9) we have that $\mathsf{N}_t(\Delta, \mathrm{P}) \subset \mathsf{CStar}(\Delta, \mathrm{P})$. Hence there is $\Delta^k \in \mathrm{P}$, where $\Delta \subset \Delta^k$, such that $p \in C\Delta^k$. Write $\Delta^k = \Delta * \Delta^l$, thus $C\Delta^k = C\Delta \times C\Delta^l \subset C\Delta \times \mathsf{CLink}(\Delta, P)$ (see 6.1.2 and 6.1.3). Therefore we can write $p = (x, y) \in C\Delta \times C\Delta^l$. Since $p \in \mathsf{N}_t(\Delta, P) = C\Delta \times \bar{\mathbf{B}}_t(\mathsf{CLink}(\Delta, P))$ (see (6.1.9)), we have that $y \in C\Delta^l \cap \bar{\mathbf{B}}_t(\mathsf{CLink}(\Delta, P)) = \bar{\mathbf{B}}_t(C\Delta^l)$. Consider the geodesic segments $A = [o, p]$, $B = [(x, o'), p]$ and $C = [o, (x, o')]$. These three geodesic segments lie in $C\Delta^k$. Since $C\Delta^k = \mathcal{E}_{C\Delta^j}(C\Delta^l)$ the slices $\{x\} \times C\Delta^l \subset \{x\} \times \mathsf{CLink}(\Delta, P)$ are totally geodesic in $C\Delta^k$ (see Section 2). Therefore B lies in $\{x\} \times C\Delta^l$. Also, since $C\Delta$ is totally geodesic in $C\Delta^k$, we have that C lies in $C\Delta \times \{o'\}$. The length of A is s, and the length b of B is $\leq t$. Therefore we get a hyperbolic geodesic triangle $\triangle$ with sides s, b, $c = \text{length}\, C$, whose angle at (x, o') is $\pi/2$ (because $C\Delta \times \{o'\}$ and $\{x\} \times C\Delta^l$). Let β' be the angle at o. By the hyperbolic law of sines applied to the right triangle $\triangle$ we have $b = \sinh^{-1}(\sinh s \sin \beta')$. We have shown that $p \in \mathsf{N}_t(\Delta, P) \cap \mathbf{S}_s(CP)$ if and only if $b = b(p) \leq t$. On the other hand $p \in \mathsf{N}_\beta(\Delta, P) \times \{s\}$ if and only if $\beta' = \beta'(p) \leq \beta$. These last two equivalences, together with the identity $b = \sinh^{-1}(\sinh s \sin \beta')$, prove the lemma. □

6.2. *Construction of the fundamental neighborhoods in hyperbolic cones*

In this section we construct the fundamental sets $\mathcal{Y}$ and $\mathcal{X}$ on the cone of a given all-right spherical complex P. These sets are meticulously constructed objects that depend on a number of pre-fixed variables; they are key objects which will be used in Section 8 to smooth the metric σ_{CP} on CP. In Section 8 the idea is to define metrics on each of the $\mathcal{X}$ and $\mathcal{Y}$, and then glue all these metrics using the properties given in this section, specifically Propositions 6.2.1, 6.2.3, and 6.2.5.

Let $\xi > 0$, $\varsigma \in (0, 1)$ and $c > 1$ with $c\varsigma < e^{-6-2\xi}$. Let $\mathsf{B} = \mathsf{B}(\varsigma; c) = \{\beta_i\}$ and $\mathsf{A} = \mathsf{B}(\varsigma) = \{\alpha_i\}$ be sequence of widths as in Section 5.2. We have $\sin \beta_i = c\varsigma^{i+1}$, $\sin \alpha_i = \varsigma^{i+1}$. Since $e^{-6-2\xi} < \frac{\sqrt{2}}{2}$, the condition $c\varsigma < e^{-6-2\xi}$ together with Corollary 5.2.4 imply that (B, A) and (B, B) satisfy condition **DNP** in Section 5.1.

Given a number $r > 0$ and an integer $k \geq 0$ we define $r_k = r_k(r) = \sinh^{-1}(\frac{\sinh r}{\sin \alpha_k})$. By convention we also set $r_{-1} = r$. Alternatively, we could restrict to widths $\{\alpha_k\}$ with $\alpha_{-1} = \pi/2$. Let k and m be integers with $m \geq 2$ and $0 \leq k \leq m-2$. Define $s_{m,k} = \sinh^{-1}(\frac{\sinh r \sin \beta_k}{\sin \alpha_{m-2}}) = \sinh^{-1}(\sinh r_{m-2} \sin \beta_k)$. We write $r_{m,k} = r_{m-k-3}$. Note that $r_{m,k} < s_{m,k}$.

Let $P = P^m$ be a finite all-right spherical complex of dimension m, with $m \leq \xi$, and let $r > 6 + 2\xi$. For every $\Delta^k \in P$, $0 \leq k \leq m-2$, define the following subsets of CP:

$$\mathcal{Y}\big(P, \Delta^k, r, \xi, (c, \varsigma)\big) = \mathring{\mathsf{N}}_{s_{m,k}}\,(C\Delta^k, CP) - \Big(\bigcup_{j<k} \mathsf{N}_{r_{m,j}}(C\Delta^j, CP)\Big) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(CP)$$

$$\mathcal{Y}\big(P, r, \xi, (c, \varsigma)\big) = CP - \Big(\bigcup_{j<m-1} \mathsf{N}_{r_{m,j}}(C\Delta^j, CP)\Big) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(CP).$$

Since ξ, c and ς will remain constant, in the rest of this section we will write $\mathcal{Y}(\mathrm{P}, \Delta^k, r)$ and $\mathcal{Y}(\mathrm{P}, r)$ instead of $\mathcal{Y}(\mathrm{P}, \Delta^k, r, \xi, (c, \varsigma))$ and $\mathcal{Y}(\mathrm{P}, r, \xi, (c, \varsigma))$, respectively. Recall that $\dot{\Delta}$ is the interior of $\Delta \in \mathrm{P}$.

Proposition **6.2.1.** — *For* $r > 6 + 2\xi$ *and* $0 \leq k \leq m - 2$ *the following properties hold*

(i) $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \subset \overset{\circ}{\mathsf{N}}_{s_{m,k}} (\mathrm{C}\dot{\Delta}^k, \mathrm{CP}) \subset \mathrm{int}\, \mathrm{C}\mathsf{Star}(\Delta^k, \mathrm{P})$.
(ii) $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \cap \mathsf{N}_{r_{m,j}}(\mathrm{C}\Delta^j, \mathrm{CP}) = \emptyset$ *for* $j < k$.
(iii) $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \cap \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP}) = \emptyset$.
(iv) $\mathrm{CP} - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP}) = \mathcal{Y}(\mathrm{P}, r) \cup \bigcup_{\Delta^k \in \mathrm{P}, k \leq m-2} \mathcal{Y}(\mathrm{P}, \Delta^k, r)$.
(v) $\Delta^j \cap \Delta^k = \emptyset$ *implies*

$$\left[\mathsf{N}_{s_{m,j}}\left(\mathrm{C}\Delta^j, \mathrm{CP}\right) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})\right] \cap \left[\mathsf{N}_{s_{m,k}}\left(\mathrm{C}\Delta^k, \mathrm{CP}\right) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})\right] = \emptyset.$$

(vi) $\Delta^j \cap \Delta^k = \emptyset$ *implies* $\mathcal{Y}(\mathrm{P}, \Delta^j, r) \cap \mathcal{Y}(\mathrm{P}, \Delta^k, r) = \emptyset$.
(vii) $\Delta^j = \Delta^k \cap \Delta^l$, *with* $j < k, l$, *implies*

$$\left[\mathsf{N}_{s_{m,j}}\left(\mathrm{C}\Delta^j, \mathrm{CP}\right) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})\right] \cap \left[\mathsf{N}_{s_{m,k}}\left(\mathrm{C}\Delta^k, \mathrm{CP}\right) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})\right] \subset \bigcup_{i \leq j} \mathsf{N}_{r_{m,i}}\left(\mathrm{C}\Delta^i, \mathrm{CP}\right).$$

(viii) $\Delta^k = \Delta^i \cap \Delta^j$, *with* $k < i, j$, *implies* $\mathcal{Y}(\mathrm{P}, \Delta^i, r) \cap \mathcal{Y}(\mathrm{P}, \Delta^j, r) = \emptyset$.
(ix) $\mathcal{Y}(\mathrm{P}, r) \cap \mathsf{N}_{r_{m,j}}(\mathrm{C}\Delta^j, \mathrm{CP}) = \emptyset$, *for* $j < m - 1$.

Proof. — The statements (ii), (iii), and (ix) follow from the definition of $\mathcal{Y}$. We prove (i). The second inclusion holds because $\overset{\circ}{\mathsf{N}}_{s_{m,k}} (\mathrm{C}\dot{\Delta}^k, \mathrm{CP})$ is open. We prove the first inclusion. By definition we have $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \subset \overset{\circ}{\mathsf{N}}_{s_{m,k}} (\mathrm{C}\Delta^k, \mathrm{CP})$. If a point $p \in \overset{\circ}{\mathsf{N}}_{s_{m,k}} (\mathrm{C}\Delta^k, \mathrm{CP}) - \overset{\circ}{\mathsf{N}}_{s_{m,k}} (\mathrm{C}\dot{\Delta}^k, \mathrm{CP})$ then its distance to $\mathrm{C}\partial\Delta^k$ is $< s_{m,k}$. Hence $p \in \overset{\circ}{\mathsf{N}}_{s_{m,k}} (\mathrm{C}\Delta^j, \mathrm{CP})$ for some $\Delta^j \subset \partial\Delta^k$; thus $j < k$. But it can be checked that $r_{m,j} > s_{m,k}$, $j < k$ (this follows from $c\varsigma < e^{-6-2\xi} < 1$). Therefore $p \in \overset{\circ}{\mathsf{N}}_{r_{m,j}} (\mathrm{C}\Delta^j, \mathrm{CP})$, which implies $p \notin \mathcal{Y}(\mathrm{P}, \Delta^k, r)$. This proves (i). Next we prove (iv). Using $r_{m,j} < s_{m,j}$ and the definition of $\mathcal{Y}(\mathrm{P}, r)$ we have

$$\mathrm{CP} - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP}) \subset \mathcal{Y}(\mathrm{P}, r) \cup \bigcup_{j \leq m-2} \mathsf{N}_{r_{m,j}}\left(\mathrm{C}\Delta^j, \mathrm{CP}\right) \subset \mathcal{Y}(\mathrm{P}, r) \cup \bigcup_{j \leq m-2} \mathsf{N}_{s_{m,j}}\left(\mathrm{C}\Delta^j, \mathrm{CP}\right).$$

This together with (iii) imply that we can prove (iv) by showing, by induction on k, that $\mathrm{U} = \bigcup_{l \leq m-2} \mathcal{Y}(\mathrm{P}, \Delta^l, r)$ contains $\overset{\circ}{\mathsf{N}}_{s_{m,k}} (\mathrm{C}\Delta^k, \mathrm{CP}) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})$ for every k-simplex of P, $k \le m-2$. For $k=0$ this statement holds because $\mathcal{Y}(\Delta^0, \mathrm{P}) = \mathring{\mathsf{N}}_{s_{m,0}}(\mathrm{C}\Delta^0, \mathrm{CP}) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})$. Assume U contains every $\mathring{\mathsf{N}}_{s_{m,j}}(\mathrm{C}\Delta^j, \mathrm{CP}) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})$, for all $j<k$. By the definition of $\mathcal{Y}(\Delta^k, \mathrm{P})$ we have that $\mathring{\mathsf{N}}_{s_{m,k}}(\mathrm{C}\Delta^k, \mathrm{CP}) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})$ is contained in

$$\Big[\mathcal{Y}\big(\Delta^k, \mathrm{P}\big) \cup \bigcup_{j<k} \mathsf{N}_{r_{m,j}}\big(\mathrm{C}\Delta^j, \mathrm{CP}\big)\Big] - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP}).$$

This together with the fact that $s_{m,k} > r_{m,k}$ and the inductive hypothesis imply that $\mathring{\mathsf{N}}_{s_{m,k}}(\mathrm{C}\Delta^k, \mathrm{CP}) - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP}) \subset \mathrm{U}$. This proves (iv). To prove the other statements we need a lemma.

Lemma **6.2.2.** — *For $t \ge r_{m-2} - (4+2\xi)$ and $r > 6+2\xi$ the following hold (see Lemma 6.1.12)*

$$\mathsf{N}_{r_{m,k}}(\mathrm{C}\Delta, \mathrm{CP}) \cap \mathbf{S}_t(\mathrm{CP}) = \mathsf{N}_{\theta_{m,k}(t)}(\Delta, \mathrm{P}) \times \{t\}$$
$$\mathsf{N}_{s_{m,k}}(\mathrm{C}\Delta, \mathrm{CP}) \cap \mathbf{S}_t(\mathrm{CP}) = \mathsf{N}_{\phi_{m,k}(t)}(\Delta, \mathrm{P}) \times \{t\},$$

where $\theta_{m,k}(t)$ and $\phi_{m,k}(t)$ are defined by the equations $\sin(\theta_{m,k}(t)) = c'' \sin\alpha_k$, $\sin(\phi_{m,k}(t)) = c'' \sin\beta_k$, with $c'' = \frac{\sinh r_{m-2}}{\sinh t} < 2e^{4+2\xi}$. Moreover $\theta_{m,k}(t)$ and $\phi_{m,k}(t)$ are well defined and less that $\pi/4$.

Proof. — Lemma 6.1.12 says that the first equation above holds if the variables $r_{m,k}$, t and $\theta_{m,k}(t)$ satisfy certain relationship. Similarly for the second equation with the variables $s_{m,k}$, t and $\phi_{m,k}(t)$. These relationships are the first equalities on the left below:

$$\sin\big(\theta_{m,k}(t)\big) = \frac{\sinh r_{m,k}}{\sinh t} = \frac{\sinh r_{m-2}}{\sinh t}\,\frac{\sinh r_{m,k}}{\sinh r_{m-2}} = c'' \sin\alpha_k$$

and

$$\sin\big(\phi_{m,k}(t)\big) = \frac{\sinh s_{m,k}}{\sinh t} = \frac{\sinh r_{m-2}}{\sinh t}\,\frac{\sinh s_{m,k}}{\sinh r_{m-2}} = c'' \sin\beta_k,$$

where we are using the definitions of $r_{m,k}$ and $s_{m,k}$ in the second equalities, and the definitions of α_k and β_k in the third equalities. Since $\xi > 0$, a simple calculation shows that $c'' < 2e^{4+2\xi}$, provided $t \ge r_{m-2} - (4+2\xi)$, $r > 6+2\xi$ (thus $r_{m-2} > 6+2\xi$). Hence the definitions of α_k and β_k and the condition $c\varsigma < e^{-6-2\xi}$ given at the beginning of this section imply $c'' \sin\alpha_k = c''\varsigma^{k+1} < \frac{\sqrt{2}}{2}$ and $c''\sin\beta_k = c'' c\varsigma^{k+1} < \frac{\sqrt{2}}{2}$. □

We now finish the proof of Proposition 6.2.1. Statement (v) follows from Lemma 6.2.2 and the fact that β-neighborhoods, $\beta < \pi/4$, of disjoint simplices in an all-right spherical complex are disjoint. Note that to apply Lemma 6.2.2 we need the condition $t \geq r_{m-2} - (4+2\xi)$; this is why we have the terms $\bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})$ in (v). Statement (vi) follows from (v) and the definition of the sets $\mathcal{Y}$. Next we prove (vii). Note that $c'' = c''(t)$ (r and m are fixed). Using items (i), (ii), and Lemma 6.2.2 and Corollary 5.3.6 it is enough to prove that, for fixed t, the pair of sequences of widths $(\{\phi_{m,k}(t)\}, \{\theta_{m,k}(t)\})$ satisfies **DNP**. But from the definitions we have $\{\phi_{m,k}(t)\} = \mathsf{B}(\varsigma, cc'')$ and $\{\theta_{m,k}(t)\} = \mathsf{B}(\varsigma, c'')$. Therefore Corollary 5.2.4 and the condition $c\varsigma < e^{-6-2\xi}$ imply $(\{\phi_{m,k}(t)\}, \{\theta_{m,k}(t)\})$ satisfies **DNP**. This proves (vii). Item (viii) follows from (vii) and the definition of the sets $\mathcal{Y}$. □

Define the sets

$$\mathcal{X}\big(\mathrm{P}^m, \Delta^k, r\big) = \mathcal{Y}\big(\mathrm{P}^m, \Delta^k, r\big) - \bar{\mathbf{B}}_{r_{m-2}}\big(\mathrm{CP}^m\big),$$
$$\mathcal{X}\big(\mathrm{P}^m, r\big) = \mathcal{Y}\big(\mathrm{P}^m, r\big) - \bar{\mathbf{B}}_{r_{m-2}}\big(\mathrm{CP}^m\big).$$

Alternatively, we can define $\mathcal{X}(\mathrm{P}^m, \Delta^k, r)$ by the same formula that defines $\mathcal{Y}(\mathrm{P}^m, \Delta^k, r)$ with just one change: in the last term replace the radius $r_{m-2} - (4+2\xi)$ by r_{m-2}, and similarly for $\mathcal{X}(\mathrm{P}^m, r)$.

Proposition **6.2.3.** — *For $\Delta^j \subset \Delta^k \in \mathrm{P}$ the following holds*

$$\mathcal{Y}\big(\mathrm{P}, \Delta^k, r\big) \subset \mathrm{C}\Delta^j \times \mathcal{X}\big(\mathsf{Link}\big(\Delta^j, \mathrm{P}\big), \Delta^l, r\big),$$

where $\Delta^l = \Delta^k \cap \mathsf{Link}(\Delta^j, \mathrm{P})$ is opposite to Δ^j in Δ^k.

Remark **6.2.4.** — The left term in the proposition is a subset of $\mathsf{CStar}(\Delta^k, \mathrm{P})$, thus also a subset of $\mathsf{CStar}(\Delta^j, \mathrm{P})$. The right term is a subset $\mathrm{C}\Delta^j \times \mathsf{CLink}(\Delta^j, \mathrm{P})$ and, by 6.1.3, we can write $\mathrm{C}\Delta^j \times \mathsf{CLink}(\Delta^j, \mathrm{P}) = \mathsf{CStar}(\Delta^j, \mathrm{P})$. Proposition 6.2.3 says that $\mathcal{Y}(\mathrm{P}, \Delta^k, r)$ is a subset of $\mathrm{C}\Delta^j \times \mathcal{X}(\mathsf{Link}(\Delta^j, \mathrm{P}), \Delta^l, r)$ under this identification.

Proof. — By the (alternative) definition of $\mathcal{X}$, it is enough to prove the following three statements

(1) $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \subset \mathrm{C}\Delta^j \times \overset{\circ}{\mathsf{N}}_{s_{m-j-1,l}} (\mathrm{C}\Delta^l, \mathsf{CLink}(\Delta^j, \mathrm{P}))$.
(2) For $\Delta^i \in \mathsf{Link}(\Delta^j, \mathrm{P})$, $i < l = k - j - 1$, we have

$$\mathcal{Y}\big(\mathrm{P}, \Delta^k, r\big) \cap \big[\mathrm{C}\Delta^j \times \mathsf{N}_{r_{m-j-1,i}}\big(\mathrm{C}\Delta^i, \mathsf{CLink}\big(\Delta^j, \mathrm{P}\big)\big)\big] = \emptyset.$$

(3) $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \cap [\mathrm{C}\Delta^j \times \bar{\mathbf{B}}_{r_{m-j-3}}(\mathsf{CLink}(\Delta^j, \mathrm{P}))] = \emptyset$.

Statement (1) follows from (i) of Proposition 6.2.1, Lemma 6.1.10 and the equalities $s_{m,k} = s_{m-j-1,k-j-1}$ and $l = k - j - 1$. Statement (2) follows from (ii) of Proposition 6.2.1, Lemma 6.1.10 and the statements $r_{m,i+j+1} = r_{m-j-1,i}$, $i + j + 1 < k$. For (3) note that (6.1.9) and the definition of $r_{m,j}$ imply

$$\mathrm{C}\Delta^j \times \bar{\mathbf{B}}_{r_{m-j-3}}\big(\mathsf{CLink}\big(\Delta^j, \mathrm{P}\big)\big) = \mathsf{N}_{r_{m,j}}\big(\mathrm{C}\Delta^j, \mathrm{CP}\big).$$

This together with (ii) of Proposition 6.2.1 imply (3). □

Proposition **6.2.5.** — *For* $\Delta^k \in \mathrm{P}$, $k \leq m-2$, *we have*

$$\mathcal{Y}(\mathrm{P}, r) \cap \mathcal{Y}\big(\mathrm{P}, \Delta^k, r\big) \subset \mathrm{C}\Delta^k \times \mathcal{X}\big(\mathsf{Link}\big(\Delta^k, \mathrm{P}\big), r\big).$$

Proof. — Using the definition of $\mathcal{X}(\mathsf{Link}(\Delta^k, \mathrm{P}), r)$, it is enough to prove the following three statements

(1) $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \subset \mathrm{C}\Delta^k \times \mathsf{CLink}(\Delta^k, \mathrm{P})$.
(2) For $\Delta^j \in \mathrm{P}$, $\Delta^k \subset \Delta^j$, $l \leq m-k-3$, and Δ^l opposite to Δ^k in Δ^j, we have

$$\mathcal{Y}(\mathrm{P}, r) \cap \big[\mathrm{C}\Delta^k \times \mathsf{N}_{r_{m-k-1,l}}\big(\mathrm{C}\Delta^l, \mathsf{CLink}\big(\Delta^k, \mathrm{P}\big)\big)\big] = \emptyset.$$

(3) $\mathcal{Y}(\mathrm{P}, r) \cap [\mathrm{C}\Delta^k \times \bar{\mathbf{B}}_{r_{m-k-3}}(\mathsf{CLink}(\Delta^k, \mathrm{P}))] = \emptyset$.

Statement (1) follows from (i) of Proposition 6.2.1, and 6.1.3. Statement (2) follows from Proposition 6.2.1(ix), Lemma 6.1.10, the identities $r_{m-k-1,j-k-1} = r_{m,j}$, $k+l+1 = j$, the fact that $l \leq m-k-3$ if and only if $j \leq m-2$, and the definition of $\mathcal{Y}(\mathrm{P}, r)$. Finally (3) follows from (6.1.9), Proposition 6.2.1(ix), the definition of $r_{m,k}$ and the definition of $\mathcal{Y}(\mathrm{P}, r)$. □

We will need one more property of the sets $\mathcal{Y}(\Delta, \mathrm{P}) \subset \mathsf{Star}(\mathsf{CLink}(\Delta, \mathrm{P})) \subset \mathrm{CP}$.

6.3. *Radial stability of the sets* $\mathcal{Y}(\mathrm{P}, \Delta^k, r)$

In Section 8 we will need a certain stability property for the sets $\mathcal{Y}$. We use the objects and notation in Section 6.2. Recall that $\mathsf{Star}(\Delta, \mathrm{P})$ is the simplicial star of Δ in P, and that an element in CP can be written as sx, $s \in [0, \infty)$, $x \in \mathrm{P}$. Let $\theta \in (0, \pi/2)$, and write $a(s) = a_\theta(s) = \sinh^{-1}(\sinh s \sin\theta)$.

Lemma **6.3.1.** — *Let* $b \in \mathbf{R}$, $\Delta^k \in \mathrm{P}$, *and* $x \in \mathsf{Star}(\Delta^k, \mathrm{P})$. *Then for every* $s > 0$

$$(s+b)x \in \mathsf{N}_{a(s)}\big(\mathrm{C}\Delta^k, \mathrm{CP}\big)0 \quad \textit{if and only if} \quad \frac{\sinh(s+b)}{\sinh s}\sin\gamma \leq \sin\theta,$$

where $\gamma = \gamma(x) = d_{\mathrm{P}}(x, \Delta^k)$.

Proof. — Note that γ is the angle opposite to the cathetus of length $d(s) = d_{\mathrm{CP}}((s+b)x, \mathrm{C}\Delta^k)$ of the right hyperbolic triangle with hypotenuse $(s+b)$. We want $d(s) \leq a(s)$; equivalently $\sinh d(s) \leq \sinh a(s)$. By the hyperbolic law of sines $\sinh d(s) = \sin\gamma \sinh(s+b)$, hence $\sinh d(s) \leq \sinh a(s)$ is equivalent to $\sinh(s+b)\sin\gamma \leq \sinh s \sin\theta$. □

Note that the lemma also holds if we replace N by $\mathring{\mathsf{N}}$ and $\leq$ by $<$. Write $\mathrm{R}(s) = \mathrm{R}_{x,b}(s) = (s+b)x$.

Lemma **6.3.2.** — *Let* Δ^k, P *and* x *as in Lemma* 6.3.1. *There are three mutually exclusive cases:*

C1: $e^b \sin\gamma < \sin\theta$*, which implies that* $\mathrm{R}(s) \in \mathring{\mathsf{N}}_{a(s)}(\mathrm{C}\Delta, \mathrm{CP})$*, for all* $s \geq s_0$*, for some* s_0.
C2: $e^b \sin\gamma > \sin\theta$*, which implies that* $\mathrm{R}(s) \notin \mathsf{N}_{a(s)}(\mathrm{C}\Delta, \mathrm{CP})$*, for all* $s \geq s_0$*, for some* s_0.
C3: $e^b \sin\gamma = \sin\theta$.

Proof. — The lemma follows from Lemma 6.3.1 and the equation $\lim_{s\to\infty} \frac{\sinh(s+b)}{\sinh s} = e^b$. ☐

From the definition of r_k given at the beginning of Section 6.2 we have $r_{m-2} = r_{m-2}(r) = \sinh^{-1}(\frac{\sinh r}{\sin\alpha_{m-2}})$, hence we can write $r = r(r_{m-2}) = \sinh^{-1}(\sinh r_{m-2} \sin\alpha_{m-2})$. Therefore we can write $r_{m,k} = r_{m,k}(r)$ and $s_{m,k} = s_{m,k}(r)$ in terms of the new variable r_{m-2}, and a calculation shows that $r_{m,k} = a_{\alpha_k}(r_{m-2})$ and $s_{m,k} = a_{\beta_k}(r_{m-2})$. We will use these identities in the proof of the next result.

Proposition **6.3.3.** — *Given* $b \in \mathbf{R}$ *and* $x \in \mathrm{P}$ *there is* $r' \in \mathbf{R}$ *such that at least one of the following conditions holds.*

(1) *There is* Δ^k, $k \leq m-2$*, such that* $\mathrm{R}_{x,b}(r_{m-2}) \in \mathcal{Y}(\mathrm{P}, \Delta^k, r(r_{m-2}))$*, for all* $r_{m-2} > r'$*, for some* r'.
(2) $\mathrm{R}_{x,b}(r_{m-2}) \in \mathcal{Y}(\mathrm{P}, r(r_{m-2}))$*, for all* $r_{m-2} > r'$*, for some* r'.

Moreover, these two conditions are stable in the following sense. If x' *and* b' *are sufficiently close to* x *and* b*, respectively, and* $\mathrm{R}_{x,b}$ *satisfies* (i) *then* $\mathrm{R}_{x',b'}$ *also satisfies* (i) *(with the same* r'*). Similarly for condition* (ii).

Proof. — By induction. Suppose C1 of Lemma 6.3.2 holds for $\mathrm{R} = \mathrm{R}_{x,b}$ with $\theta = \alpha_0$, for some Δ^0. Then, since $\mathsf{N}_{a_{\alpha_0}(r_{m-2})}(\Delta^0, \mathrm{P}) = \mathsf{N}_{r_{m,0}(r_{m-2})}(\Delta^0, \mathrm{P}) \subset \mathcal{Y}(\mathrm{P}, \Delta^0, r)$ we see that R satisfies (1) for $\mathcal{Y}(\mathrm{P}, \Delta^0, r)$ and we are done. Suppose C3 holds with $\theta = \alpha_0$, for some Δ^0. Then $x \in \mathsf{Star}(\Delta^0, \mathrm{P})$ and $e^b \sin\gamma = \sin\alpha_0$, where $\gamma = \gamma(x)$. Since $\alpha_k < \beta_k$, we have $e^b \sin\gamma < \sin\beta_0$, hence by Lemma 6.3.2 (with $\theta = \beta_0$) we have that $\mathrm{R}(r_{m-2}) \in \mathring{\mathsf{N}}_{s_{m,0}(r_{m-2})}(\Delta^0, \mathrm{P})$, for large r_{m-2}, and follows that R satisfies (1) for $\mathcal{Y}(\mathrm{P}, \Delta^0, r)$ and we are done. Now, if neither C1 nor C2 hold for all Δ^0 then we have that C2 happens for all Δ^0, with $\theta = \alpha_0$ (and some s_0 independent of Δ^0, which is possible because P is finite). As before we have three possibilities. First C1 holds for $\mathrm{R} = \mathrm{R}_{x,b}$ with $\theta = \alpha_1$, for some Δ^1. This, together with the assumption that C2 holds for all Δ^0 (with $\theta = \alpha_0$), and the definition of $\mathcal{Y}(\mathrm{P}, \Delta^1, r)$ imply that R satisfies (1) for $\mathcal{Y}(\mathrm{P}, \Delta^1, r)$ and we are done. Suppose C3 holds for R and Δ^1 (with $\theta = \alpha_1$), for some Δ^1. Using the same argument as in the Δ^0 case (when we assumed C3 some Δ^0) we get that R satisfies (1) for $\mathcal{Y}(\mathrm{P}, \Delta^1, r)$ and we are done. The third case is that C2 happens for R and all Δ^1. Proceeding in this way we obtain that either R satisfies (1), for some Δ^k, $k \leq m-2$ or C2 holds for R and all Δ^k, $k \leq m-2$ (with $\theta = \alpha_k$). Hence (2) holds for R. Moreover it does so stably. ☐

7. Smooth structures on cube and all-right spherical complexes

For the basic definitions and results about cube complexes see for instance [4]. Given a (cube or all-right spherical) complex K we use the same notation K for the complex itself (the collection of all closed cubes or simplices) and its realization (the union of all cubes or simplices). For $\sigma \in K$ we denote its interior by $\dot{\sigma}$.

Let M^n be a smooth manifold of dimension n. A *smooth cubulation* of M is a pair (K, f), where K is a cube complex and $f : K \to M$ is a non-degenerate PD homeomorphism [24], that is, for all $\sigma \in K$ we have $f|_\sigma$ is a smooth embedding. Sometimes we will write K instead of (K, f). The smooth manifold M together with a smooth cubulation is a *smooth cube manifold* or a *smooth cube complex*. A *smooth all-right-spherical triangulation* and a *smooth all-right-spherical manifold* (or complex) is defined analogously.

In this section $\mathsf{Link}(\sigma^j, K)$ means the *geometric link* of an open j-cube or j-all-right simplex σ^j, defined as the union of the end points of straight (geodesic) segments of small length $\varepsilon > 0$ emanating perpendicularly (to $\dot{\sigma}^j$) from some point $x \in \dot{\sigma}^j$. The star $\mathsf{Star}(\sigma, K)$ is the union of such segments. We can identify the star with the cone of the link $\mathsf{CLink}(\sigma, K)$ (or ε-cone) defined as $\mathsf{CLink}(\sigma, K) = \mathsf{Link}(\sigma, K) \times [0, \varepsilon)/\mathsf{Link}(\sigma, K) \times \{0\}$. Thus a point x in $\mathsf{CLink}(\sigma, K)$, different from the cone point $o = o_{\mathsf{CLink}(\sigma,K)}$, can be written as $x = tu$, $t \in (0, \varepsilon)$, $u \in \mathsf{Link}(\sigma, K)$. For $s > 0$ we get the *cone homothety* $x \mapsto sx = (st)u$ (partially defined if $s > 1$). If we want to make explicit the dependence of the link or the cone on ε we shall write $\mathsf{Link}_\varepsilon(\sigma, K)$ or $C_\varepsilon\mathsf{Link}(\sigma, K)$ respectively.

Remark **7.0.1.** — As usual we shall identify the normal ε-neighborhood of $\dot{\sigma}$ in K with $C_\varepsilon\mathsf{Link}(\sigma, K) \times \dot{\sigma}$ which we may denote $\mathsf{CLink}(\sigma, K) \times \dot{\sigma}$.

In what follows we assume that $f : K \to M$ is a smooth cubulation (or an all-right spherical triangulation) of the smooth manifold M. Since the PL structure on M induced by f equals the PL structure induced be the given smooth structure on M (see Theorem 10.5 in [24]) we have that the link $\mathsf{Link}(\sigma^i, K)$ is PL homeomorphic to $\mathbf{S}^{n-i-1}$. A *link smoothing for* $\dot{\sigma}^i$ (or σ^i) is a homeomorphism $h_{\sigma^i} : \mathbf{S}^{n-i-1} \to \mathsf{Link}(\sigma^i, K)$. The *cone* of h_{σ^i} is the map $Ch_{\sigma^i} : \mathbf{D}^{n-i} \longrightarrow \mathsf{CLink}(\sigma^i, K)$ given by $tx = [x, t] \mapsto th_{q^i}(x) = [h_{q^i}(x), t]$, where we are canonically identifying the ε-cone of $\mathbf{S}^{n-i-1}$ with the disc $\mathbf{D}^{n-i}$. We remark that we are not assuming h_{σ^i} to be smooth. A link smoothing h_{σ^i} induces the following smoothing of the normal neighborhood of $\dot{\sigma}^i$:

$$h^\bullet_{\sigma^i} = f \circ (Ch_{\sigma^i} \times 1_{\dot{\sigma}^i}) : \mathbf{D}^{n-i} \times \dot{\sigma}^i \longrightarrow M.$$

The pair $(h^\bullet_{\sigma^i}, \mathbf{D}^{n-i} \times \dot{\sigma}^i)$, or simply $h^\bullet_{\sigma^i}$, is a *normal chart* on M. Note that the collection $\mathcal{A} = \{(h^\bullet_{\sigma^i}, \mathbf{D}^{n-i} \times \dot{\sigma}^i)\}_{\sigma^i \in K}$ is a topological atlas for M. Sometimes will just write $\mathcal{A} = \{h^\bullet_{\sigma^i}\}_{\sigma^i \in K}$. The topological atlas $\mathcal{A}$ is called a *normal atlas*. It depends uniquely on the complex K, the map f and the collection of link smoothings $\{h_\sigma\}_{\sigma \in K}$. To express the dependence of the atlas on the set of links smoothings we shall write $\mathcal{A} = \mathcal{A}(\{h_\sigma\}_{\sigma \in K})$ (this is different from $\mathcal{A} = \{h^{\bullet}_{\sigma^i}\}_{\sigma^i \in \mathrm{K}}$, as written above). The most important feature about these normal atlases is that they preserve the radial and sphere (link) structure given by K.

Note that not every collection of link smoothings induce a smooth atlas. But when the induced atlas is smooth we call $\mathcal{A}$ a *normal smooth atlas on* M *with respect to* K and the corresponding smooth structure $\mathcal{S}'$ a *normal smooth structure on* M *with respect to* K. In this case we say that the set of link smoothings $\{h_\sigma\}_{\sigma\in\mathrm{K}}$ is *smooth*. The following theorem is proved in [28]; it is the Main Theorem in [28].

Theorem **7.1.** — *Let* M *be a smooth cube or all-right spherical manifold, with smooth structure* $\mathcal{S}$. *Then* M *admits a normal smooth structure* $\mathcal{S}'$ *diffeomorphic to* $\mathcal{S}$.

Hence if M^n is a smooth manifold with smooth structure $\mathcal{S}$ and K is a smooth cubulation (or all-right spherical triangulation) of M, then there are link smoothings h_σ, for all $\sigma \in \mathrm{K}$, such that the atlas $\mathcal{A} = \mathcal{A}(\{h_\sigma\}_{\sigma\in\mathrm{K}})$ is smooth or equivalently, $\{h_\sigma\}_{\sigma\in\mathrm{K}}$ is smooth. Moreover the normal smooth structure $\mathcal{S}'$, induced by $\mathcal{A}$, is diffeomorphic to $\mathcal{S}$.

7.2. *Induced link smoothings*

Let K be a cubical or all-right spherical complex. Then the links of $\sigma \in \mathrm{K}$ are all-right-spherical complexes. We explain here how to obtain from a given collection of link smoothings for K (and its corresponding normal atlas and structure) a collection of links smoothings for a link in K (and its corresponding normal atlas and structure).

The all-right-spherical structure on $\mathsf{Link}(\sigma,\mathrm{K})$ induced by K has all-right-spherical simplices $\{\tau \cap \mathsf{Link}(\sigma,\mathrm{K}), \tau \in \mathrm{K}\}$. Note that $\tau \cap \mathsf{Link}(\sigma,\mathrm{K})$ is non-empty only when $\sigma \subsetneq \tau$, hence we can write

$$\mathsf{Link}(\sigma,\mathrm{K}) = \big\{\tau \cap \mathsf{Link}(\sigma,\mathrm{K}), \sigma \subsetneq \tau \in \mathrm{K}\big\}.$$

Since $\tau \cap \mathsf{Link}(\sigma,\mathrm{K})$ is a simplex in the all-right spherical complex $\mathsf{Link}(\sigma,\mathrm{K})$ we can consider its link $\mathsf{Link}(\tau \cap \mathsf{Link}(\sigma,\mathrm{K}), \mathsf{Link}(\sigma,\mathrm{K}))$. By definition we have:

$$\mathsf{Link}\big(\tau \cap \mathsf{Link}(\sigma,\mathrm{K}), \mathsf{Link}(\sigma,\mathrm{K})\big) = \mathsf{Link}(\tau,\mathrm{K}) \tag{7.2.1}$$

provided we choose the radii and bases of the links properly. In the formula above radii and bases are not specified but the radii are certainly not equal. The simple relationship between these radii is given by Equation (1) in the proof of Lemma 1.2 [28] (or the corresponding one in the spherical case; see Remark 1 after the proof of Lemma 1.3 [28]). By (7.2.1) we can say that the set of link smoothings $\{h_\sigma\}_{\sigma\in\mathrm{K}}$ for K induces, by restriction, a set of link smoothings for $\mathsf{Link}(\sigma,\mathrm{K})$, $\sigma \in \mathrm{K}$. That is, we set $h_{\tau\cap\mathsf{Link}(\sigma,\mathrm{K})} = h_\tau$, $\sigma \subsetneq \tau \in \mathrm{K}$. The next result is proved in [31] (see Corollary 1.3.5 in [31]).

Proposition **7.2.2.** — *Let* $\{h_\sigma\}_{\sigma\in\mathrm{K}}$ *be a set of link smoothings on* K, *and let* $\sigma^k \in \mathrm{K}$. *Assume* $\{h_\sigma\}_{\sigma\in\mathrm{K}}$ *is smooth; that is, the atlas* $\mathcal{A} = \mathcal{A}(\{h_\sigma\}_{\sigma\in\mathrm{K}})$ *is smooth. Let* $\mathcal{S}'$ *be the normal smooth structure on* K *induced by* $\mathcal{A}$. *Then:*

(1) *The set of link smoothings* $\{h_{\sigma^i\cap \mathsf{Link}(\sigma,\mathrm{K})}\}_{\sigma^k\subsetneq\sigma^i}$ *for the links of* $\mathsf{Link}(\sigma^k,\mathrm{K})$ *is smooth; that is, the atlas* $\mathcal{A}_{\sigma^k}=\mathcal{A}_{\mathsf{Link}(\sigma^k,\mathrm{K})}=\{h^\bullet_{\sigma^i\cap\mathsf{Link}(\sigma^k,\mathrm{K})}\}_{\sigma^k\subsetneq\sigma^i}$ *is a smooth normal atlas on* $\mathsf{Link}(\sigma^k,\mathrm{K})$.

(2) *The link smoothing*

$$h_{\sigma^k}:\mathbf{S}^{n-k-1}\to\big(\mathsf{Link}\big(\sigma^k,\mathrm{K}\big),\mathcal{S}_{\sigma^k}\big)$$

is a diffeomorphism. Here $\mathcal{S}_{\sigma^k}$ *is the smooth structure induced by the atlas* $\mathcal{A}_{\sigma^k}$.

(3) *The link* $\mathsf{Link}(\sigma^k,\mathrm{K})$ *is a smooth submanifold of* $(\mathrm{K},\mathcal{S}')$. *Moreover*

$$\mathcal{S}'\big|_{\mathsf{Link}(\sigma^k,\mathrm{K})}=\mathcal{S}_{\sigma^k}$$

where $\mathcal{S}'|_{\mathsf{Link}(\sigma^k,\mathrm{K})}$ *denotes the restriction of* $\mathcal{S}'$ *to* $\mathsf{Link}(\sigma^k,\mathrm{K})$.

7.3. *The case of manifolds with codimension zero singularities*

Here we treat the case of manifolds with a one point singularity. The case of manifolds with many (isolated) point singularities is similar.

Let Q be a smooth manifold with a one point singularity q, that is $\mathrm{Q}-\{q\}$ is a smooth manifold and there is a topological embedding $\mathrm{CN}\to\mathrm{Q}$, with $o_{\mathrm{CN}}\mapsto q$, that is a smooth embedding outside the vertex o_{CN}. Here $\mathrm{N}=(\mathrm{N},\mathcal{S}_{\mathrm{N}})$ is a closed smooth manifold (with smooth structure $\mathcal{S}_{\mathrm{N}}$). Also CN is the (closed) cone of N and we identify $\mathrm{CN}-\{o_{\mathrm{CN}}\}$ with $\mathrm{N}\times(0,1]$. We write $\mathrm{CN}\subset\mathrm{Q}$. We say that the *singularity* q *of* Q *is modeled on* CN.

Assume (K,f) is a smooth cubulation of Q, that is

(i) K is a cubical complex.
(ii) $f:\mathrm{K}\to\mathrm{Q}$ is a homeomorphism. Write $f(p)=q$ and $\mathrm{L}=\mathsf{Link}(p,\mathrm{K})$.
(iii) $f|_\sigma$ is a smooth embedding for every cube σ not containing p.
(iv) $f|_{\sigma-\{p\}}$ is a smooth embedding for every cube σ containing p.
(v) L is PL homeomorphic to $(\mathrm{N},\mathcal{S}_{\mathrm{N}})$.

Many of the definitions and results given before for smooth cube manifolds still hold (with minor changes) in the case of manifolds with a one point singularity:

(1) A *link smoothing* for $\mathrm{L}=\mathsf{Link}(p,\mathrm{K})$ (or p) is a homeomorphism $h_p:\mathrm{N}\to\mathrm{L}$.

(2) Given a set of link smoothings for K we get a set of normal charts as before. For the vertex p we have the cone map $h^\bullet_p=f\circ \mathrm{C}h_p:\mathrm{CN}\to\mathrm{Q}$. We will also denote the restriction of $h^\bullet_p$ to $\mathrm{CN}-\{o_{\mathrm{CN}}\}$ by $h^\bullet_p$. As before $\{h^\bullet_\sigma\}_{\sigma\in\mathrm{K}}$ is a *(topological) normal atlas on* Q *with respect to* K. The atlas on Q is *smooth* if all transition functions are smooth, where for the case $h^\bullet_p:\mathrm{CN}-\{o_{\mathrm{CN}}\}\to\mathrm{Q}-\{q\}$ we are identifying $\mathrm{CN}-\{o_{\mathrm{CN}}\}$ with $\mathrm{N}\times(0,1]$ with the product smooth structure obtained from some smooth structure $\tilde{\mathcal{S}}_{\mathrm{N}}$ on N. A smooth normal atlas on Q with respect to K induces, by restriction, a smooth normal structure on $\mathrm{Q}-\{q\}$ with respect to $\mathrm{K}-\{p\}$ (this makes sense even though $\mathrm{K}-\{p\}$ is not, strictly speaking, a cube complex).

(3) We say that the set $\{h_\sigma\}$ is *smooth* if the atlas $\mathcal{A} = \{h^\bullet_\sigma\}_{\sigma\in\mathrm{K}}$ is smooth. If $\{h_\sigma\}$ is smooth and the associated smooth structure $\tilde{\mathcal{S}}_{\mathrm{N}}$ is diffeomorphic to $\mathcal{S}_{\mathrm{N}}$, then we say that the smooth atlas $\mathcal{A}$ (or the induced smooth structure, or the set $\{h_\sigma\}$) is *correct with respect to* N.

(4) Also it is straightforward to verify that Proposition 7.2.2 holds in our present case.

(5) In [28] the following version of Theorem 7.1 is proved (see Theorem 2.1 in [28]):

Theorem **7.3.1.** — *Let* Q *be a smooth manifold with one point singularity* q *modeled on* CN, *where* N *is a closed smooth manifold. Let* (K, f) *be a smooth cubulation of* Q*. Then* Q *admits a normal smooth structure with respect to* K*, whose restriction to* $\mathrm{Q}-\{q\}$ *is diffeomorphic to* $\mathrm{Q}-\{q\}$*. Moreover this normal smooth structure is correct with respect to* N *if*

(a) $\dim \mathrm{N} \leq 4$.

(b) $\dim \mathrm{N} \geq 5$ *and the Whitehead group* $\mathrm{W}h(\mathrm{N})$ *of* N *vanishes.*

8. Smoothing hyperbolic cones

Given an all-right spherical complex P^m of dimension m and a compatible smooth structure $\mathcal{S}_{\mathrm{P}}$ on P, by Theorem 7.1 we can assume that $\mathcal{S}_{\mathrm{P}}$ is a normal smooth structure, and $\mathcal{S}_{\mathrm{P}}$ has a normal atlas $\mathcal{A}_{\mathrm{P}}$. The atlas $\mathcal{A}_{\mathrm{P}}$ and its induced differentiable structure $\mathcal{S}_{\mathrm{P}}$ are constructed (canonicaly) from a set of link smoothings $\mathcal{L}_{\mathrm{P}} = \{h_\Delta\}_{\Delta\in\mathrm{P}}$. To express this dependence we will sometimes write $\mathcal{A}_{\mathrm{P}} = \mathcal{A}_{\mathrm{P}}(\mathcal{L}_{\mathrm{P}})$ and $\mathcal{S}_{\mathrm{P}} = \mathcal{S}_{\mathrm{P}}(\mathcal{L}_{\mathrm{P}})$.

Recall that the cone CP has a piecewise hyperbolic metric induced by the piecewise spherical metric on P. We denote these metrics by σ_{CP} and σ_{P} respectively. As mentioned in Remark 6.1.1 the piecewise hyperbolic metric σ_{CP} has a well defined set of rays.

(**8.0.1**) Consider the following data.

1. A positive number ξ.
2. A sequence $\mathsf{d} = \{d_2, d_3, \ldots.\}$ of real numbers, with $d_i > 6 + 2\xi$. We write $\mathsf{d}(k) = \{d_2, d_3, \ldots, d_k\}$.
3. A positive number r, with $r > 2d_i$, $i = 2, \ldots, m+1$, and m as in item 5.
4. Real numbers $\varsigma \in (0, 1)$, $c > 1$, with $c\varsigma < e^{-6-2\xi}$. This defines the sequences of widths (see Sections 5.2 and 5.3) $\mathsf{A} = \mathsf{B}(\varsigma) = \{\alpha_i\}$ and $\mathsf{B} = \mathsf{B}(\varsigma; c) = \{\beta_i\}$, where $\sin\alpha_i = \varsigma^{i+1}$, $\sin\beta_i = c\varsigma^{i+1}$. Recall that (B, A) satisfies **DNP** (see Section 6.2 or Corollary 5.2.4).
5. An all-right spherical complex P^m, $\dim \mathrm{P} = m$, with smooth normal atlas $\mathcal{A}_{\mathrm{P}}(\mathcal{L}_{\mathrm{P}})$, where $\mathcal{L}_{\mathrm{P}}$ is a smooth set of link smoothings on P.

6. A diffeomorphism $\phi_{\mathrm{P}} = \phi_{\mathrm{P},\mathcal{L}_{\mathrm{P}}} : (\mathrm{P}, \mathcal{S}_{\mathrm{P}}(\mathcal{L}_{\mathrm{P}})) \to \mathbf{S}^m$ to the standard m-sphere. The map ϕ_{P} is called a *global smoothing for* P, *with respect to* $\mathcal{S}_{\mathrm{P}}$ *(or* $\mathcal{A}_p$*, or* $\mathcal{L}_{\mathrm{P}}$*)*. For $m = 1$ the diffeomorphism ϕ_{P} will be defined canonically (that is, depending only on P) in Section 8.1.

The smooth atlas $\mathcal{A}_{\mathrm{P}}(\mathcal{L}_{\mathrm{P}})$ on P induces, by coning, a smooth atlas on $\mathrm{CP} - \{o_{\mathrm{CP}}\}$, and, by item 6, this atlas together with the coning $\mathrm{C}\phi_{\mathrm{P}} : \mathrm{CP} \to \mathbf{R}^{m+1}$ of the map ϕ_{P} induce a smooth atlas $\mathcal{A}_{\mathrm{CP}} = \mathcal{A}_{\mathrm{CP}}(\mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}})$ on CP. We denote the corresponding smooth structure by $\mathcal{S}_{\mathrm{CP}} = \mathcal{S}_{\mathrm{CP}}(\mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}})$. Note that we get a diffeomorphism $\mathrm{C}\phi_{\mathrm{P}} : (\mathrm{CP}, \mathcal{S}_{\mathrm{CP}}) \to \mathbf{R}^{m+1}$.

With the data given in items 1–6 in (8.0.1) above we will construct the *smoothed Riemannian metric* $\mathcal{G}(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}}, r, \xi, \mathsf{d}, (c, \varsigma))$ on the cone CP of P, where we consider CP with smooth structure $\mathcal{S}_{\mathrm{CP}}$. This construction will be done by induction on m.

In Sections 8.1 and 8.2 we will assume ξ, d, c, ς fixed. In particular we shall assume A, B fixed. So, to simplify our notation, we shall denote the smoothed metric by $\mathcal{G}(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}}, r)$ or just $\mathcal{G}(\mathrm{P}, r)$ or $\mathcal{G}(\mathrm{P})$. In Sections 8.3 and 8.4 we need to make explicit the dependence of the smoothed metric on the other variables, and we will show that, given $\varepsilon > 0$, we can choose r and d_i, $i = 2, \ldots, m$, large so that $\mathcal{G}(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}}, r, \xi, \mathsf{d}, (c, \varsigma))$ has curvatures near -1, provided the variables satisfy certain conditions. Before we begin with dimension 1 we need to discuss induced structures.

Let $\Delta = \Delta^k \in \mathrm{P}$. The *restriction of* $\mathcal{L}_{\mathrm{P}}$ *to* $\mathsf{Link}(\Delta, \mathrm{P})$ is the set $\mathcal{L}_{\mathrm{P}}|_{\mathsf{Link}(\Delta,\mathrm{P})} = \{h_{\Delta'}\}_{\Delta \subsetneq \Delta'}$, see Section 7.2. Sometimes we will just write $\mathcal{L}_{\mathsf{Link}(\Delta,\mathrm{P})}$ or, more specifically, $\mathcal{L}_{\mathsf{Link}(\Delta,\mathrm{P})}(\mathcal{L}_{\mathrm{P}})$. The corresponding induced atlas on $\mathsf{Link}(\Delta, \mathrm{P})$ is $\mathcal{A}_{\mathsf{Link}(\Delta,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}) = \{h^{\bullet}_{\Delta'}\}_{\Delta \subsetneq \Delta'}$, and sometimes we will simply write $\mathcal{A}_{\mathsf{Link}(\Delta,\mathrm{P})}$. The smooth structure on $\mathsf{Link}(\Delta, \mathrm{P})$ induced by $\mathcal{A}_{\mathsf{Link}(\Delta,\mathrm{P})}$ will be denoted by $\mathcal{S}_{\mathsf{Link}(\Delta,\mathrm{P})}(\mathcal{L}_{\mathrm{P}})$, or simply by $\mathcal{S}_{\mathsf{Link}(\Delta,\mathrm{P})}$. By Proposition 7.2.2 we have that, for $\Delta \in \mathrm{P}$, the link smoothing h_Δ is a global smoothing for $\mathsf{Link}(\Delta, \mathrm{P})$ with respect to $\mathcal{S}_{\mathsf{Link}(\Delta,\mathrm{P})}$. Write $\phi_{\mathsf{Link}(\Delta,\mathrm{P})} = \phi_{\mathsf{Link}(\Delta,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}) = h_\Delta$. Therefore we obtain the following *restriction rule*:

$$\textbf{(8.0.2)} \qquad \mathcal{L}_{\mathrm{P}} \longrightarrow \big(\mathcal{L}_{\mathsf{Link}(\Delta,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}), \phi_{\mathsf{Link}(\Delta,\mathrm{P})}(\mathcal{L}_{\mathrm{P}})\big),$$

where $\mathcal{L}_{\mathrm{P}}$ satisfies 5 in (8.0.1) for P, and the objects $\mathcal{L}_{\mathsf{Link}(\Delta,\mathrm{P})}$, $\phi_{\mathsf{Link}(\Delta,\mathrm{P})}$ satisfy 5, 6 of (8.0.1) for $\mathsf{Link}(\Delta, \mathrm{P})$. The smooth structure on $\mathsf{CLink}(\Delta, \mathrm{P})$ constructed from the data $(\mathcal{L}_{\mathsf{Link}(\Delta,\mathrm{P})}, \phi_{\mathsf{Link}(\Delta,\mathrm{P})})$ will be denoted by $\mathcal{S}_{\mathsf{CLink}(\Delta,\mathrm{P})}(\mathcal{L}_{\mathrm{P}})$, or $\mathcal{S}_{\mathsf{CLink}(\Delta,\mathrm{P})}(\mathcal{L}_{\mathsf{Link}(\Delta,\mathrm{P})}, \phi_{\mathsf{Link}(\Delta,\mathrm{P})})$, or simply by $\mathcal{S}_{\mathsf{CLink}(\Delta,\mathrm{P})}$. The next lemma says that the restriction rule (8.0.2) is transitive, that is, it respects the identity $\mathsf{Link}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P})) = \mathsf{Link}(\Delta^k, \mathrm{P})$, where $\Delta^l = \mathsf{Link}(\Delta^j, \Delta^k)$ (see Lemma 5.3.2).

Lemma **8.0.3.** — *Let* $\Delta^j \subset \Delta^k \in \mathrm{P}$ *and let* $\Delta^l = \mathsf{Link}(\Delta^j, \Delta^k)$*. Then*

$$\mathcal{L}_{\mathsf{Link}(\Delta^l,\mathsf{Link}(\Delta^j,\mathrm{P}))}\big(\mathcal{L}_{\mathsf{Link}(\Delta^j,\mathrm{P})}(\mathcal{L}_{\mathrm{P}})\big) = \mathcal{L}_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}),$$

$$\phi_{\mathsf{Link}(\Delta^l,\mathsf{Link}(\Delta^j,\mathrm{P}))}\big(\mathcal{L}_{\mathsf{Link}(\Delta^j,\mathrm{P})}(\mathcal{L}_{\mathrm{P}})\big) = \phi_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}).$$

Proof. — If we use the simplicial definition of link the identity $\mathsf{Link}(\Delta^l, \mathsf{Link}(\Delta^j, \mathrm{P})) = \mathsf{Link}(\Delta^k, \mathrm{P})$ is an equality of sets; hence the lemma follows from the definition of $\mathcal{L}$ and ϕ. □

Recall that we have an identification $\mathsf{CStar}(\Delta, \mathrm{P}) = \mathrm{C}\Delta \times \mathsf{CLink}(\Delta, \mathrm{P})$ (see 6.1.3). The "open" version of this identification is $\mathrm{C}(\overset{\circ}{\mathsf{Star}}(\dot{\Delta}, \mathrm{P})) = \mathrm{C}\dot{\Delta} \times \mathsf{CLink}(\Delta, \mathrm{P})$. Here $\overset{\circ}{\mathsf{Star}}(\dot{\Delta}, \mathrm{P}) = \overset{\circ}{\mathsf{N}}_{\pi/2}(\dot{\Delta}, \mathrm{P})$. Define the open set $\mathsf{C}\overset{\circ}{\mathsf{Star}}(\Delta, \mathrm{P})$ to be the set $\mathrm{C}(\overset{\circ}{\mathsf{Star}}(\dot{\Delta}, \mathrm{P}))$ with the cone point deleted. Note that $\mathsf{C}\overset{\circ}{\mathsf{Star}}(\Delta, \mathrm{P})$ as an open subset of CP has the induced smooth structure $\mathcal{S}_{\mathrm{CP}}|_{\mathsf{C}\overset{\circ}{\mathsf{Star}}(\Delta,\mathrm{P})}$, and, for simplicity, we will just write $\mathcal{S}_{\mathrm{CP}}$. Define the set $\mathrm{C}_0(\dot{\Delta})$ to be $\mathrm{C}(\dot{\Delta})$ with the cone point deleted. Then $\mathrm{C}_0\dot{\Delta} = \mathrm{C}_0\dot{\Delta}^k$ as an open set of $\mathbf{H}^{k+1}$ has the natural smooth structure $\mathcal{S}_{\mathbf{H}^{k+1}}$, and $\mathsf{CLink}(\dot{\Delta}, \mathrm{P})$ has the smooth structure $\mathcal{S}_{\mathsf{CLink}(\Delta,\mathrm{P})}$. Therefore we can give $\mathrm{C}_0\dot{\Delta} \times \mathsf{CLink}(\Delta, \mathrm{P})$ the "product" smooth structure $\mathcal{S}_{\times} = \mathcal{S}_{\mathrm{C}_0\dot{\Delta}\times\mathsf{CLink}(\Delta,\mathrm{P})}$.

Lemma **8.0.4.** — *The following identification is a diffeomorphism*

$$\big(\mathsf{C}\overset{\circ}{\mathsf{Star}}(\Delta, \mathrm{P}), \mathcal{S}_{\mathrm{CP}}\big) = \big(\mathrm{C}_0\dot{\Delta} \times \mathsf{CLink}(\Delta, \mathrm{P}), \mathcal{S}_{\times}\big).$$

Proof. — We use the variables s, t, r, y, v, x, w, u, β defined in Section 2. We also use the notation from 6.1.2, 6.1.3 and Remark 7.0.1. Using rescaling we can assume that the image of the chart $h^{\bullet}_{\Delta}$ is $\overset{\circ}{\mathsf{N}}_{\pi/2}(\Delta, \mathrm{P})$. Again by rescaling, and using the notation in 6.1.2 and 6.1.3 we can write

$$\begin{array}{rccc} h^{\bullet}_{\Delta}: & \mathbf{D}^{m-k}(\pi/2) \times \dot{\Delta} & \longrightarrow & \mathrm{P} \\ & (\beta u', w) & \mapsto & \big[w, h_{\Delta}(u')\big](\beta), \end{array} \tag{1}$$

where $\mathbf{D}^{m-k}(\pi/2)$ is the disc of radius $\pi/2$, and we are expressing and element $\mathbf{D}^{m-k}(\pi/2)$ as $\beta u'$, with $\beta \in [0, \pi/2)$, $u' \in \mathbf{S}^{m-k-1}$. A chart for $(\mathsf{C}\overset{\circ}{\mathsf{Star}}(\Delta, \mathrm{P}), \mathcal{S}_{\mathrm{CP}})$ is the cone of $h^{\bullet}_{\Delta}$, which we shall denote by h^{*}_{Δ}. Explicitly, from (1) we have (see Remark 7.0.1)

$$\begin{array}{rccc} h^{*}_{\Delta}: & \mathbf{R}_{+} \times \mathbf{D}^{m-k}(\pi/2) \times \dot{\Delta} & \longrightarrow & \mathrm{CP} \\ & (s, \beta u', w) & \mapsto & s\big[w, h_{\Delta}(u')\big](\beta). \end{array} \tag{2}$$

And for $(\mathrm{C}\dot{\Delta} \times \mathsf{CLink}(\Delta, \mathrm{P}), \mathcal{S}_{\times})$ we can take the following chart

$$\begin{array}{rccc} h^{\dagger}_{\Delta}: & \mathbf{R}_{+} \times \mathbf{R}^{m-k} \times \dot{\Delta} & \longrightarrow & \mathrm{C}\dot{\Delta} \times \mathsf{CLink}(\Delta, \mathrm{P}) \\ & (t, ru', w) & \mapsto & \big(tw, rh_{\Delta}(u')\big), \end{array} \tag{3}$$

where we write an element in $\mathbf{R}^{m-k}$ as ru', $r \in [0, \infty)$, $u' \in \mathbf{S}^{m-k-1}$. From (2) and (3) and 6.1.3 we get

$$\big(h^{\dagger}_{\Delta}\big)^{-1} \circ h^{*}_{\Delta}\big(s, \beta u', w\big) = \big(t, ru', w\big), \tag{4}$$

where the relationship between the variables s, β, t, r is the following (see Section 2). There is a right hyperbolic triangle with catheti of length t, r, hypotenuse of length s and angle β opposite to the cathetus of length r. Using hyperbolic trigonometry we can find an invertible transformation $(s,\beta) \to (t,r)$. In particular $r = \sinh^{-1}(\sin\beta \sinh s)$. The variables s and t are never zero, but β and r could vanish. Note that $\beta = 0$ if and only if $r = 0$. To get differentiability at $\beta = 0$ note that the map $(s, \beta u') \to ru'$ can be rewritten as $(s,z) \to (\frac{r(s,\beta)}{\beta} z)$, $\beta = |z|$, which is smooth because $\frac{r(s,\beta)}{\beta}$ is a smooth even function on β. Similarly, the smoothness of the inverse of the map in (4) follows from the fact that the map $(t,r) \to \frac{\beta(r,t)}{r}$ is a smooth even function on r. □

8.1. *Dimension one*

An all-right spherical complex P^1 of dimension one satisfying item 6 of (8.0.1) is formed by a finite number k' of segments of length $\pi/2$ glued successively forming a circle. Hence P is isometric to $\mathbf{S}^1$ with metric $k^2\sigma_{\mathbf{S}^1}$, $k = k'/4$ (i.e. a circle of length $2\pi k$). Let $\phi = \phi_{\mathrm{P}} : \mathrm{P} \to (\mathbf{S}^1, k^2\sigma_{\mathbf{S}^1})$ be an isometry. Consequently we can identify CP with $\mathbf{R}^2$, and $\mathrm{CP} - \{o_{\mathrm{CP}}\}$ to $\mathbf{R}^2 - \{0\}$ with hyperbolic metric $\sigma_{\mathrm{CP}} = \sinh^2 s k^2 \sigma_{\mathbf{S}^1} + ds^2$. Notice that this metric is smooth on $\mathbf{R}^2$ away from the cone point $o_{\mathrm{CP}} = 0 \in \mathbf{R}^2$, and it does have a singularity at 0 unless $k = 1$.

As promised after (8.0.1) we now construct the metric $\mathcal{G}(\mathrm{P})$ when P is one-dimensional.

Let ρ be as in Section 1. Define

$$\mu(s) = \mu_{d_2,r,k}(s) = k^2 \rho\left(\frac{s}{d_2} - \frac{r-d_2}{d_2}\right) + \left(1 - \rho\left(\frac{s}{d_2} - \frac{r-d_2}{d_2}\right)\right).$$

Hence $\mu(s) = 1$, for $s \le r - d_2$ and $\mu(s) = k^2$ for $s \ge r$. Define

$$\mathcal{G}(\mathrm{P}, r) = \sinh^2 s \mu(s) \sigma_{\mathbf{S}^1} + ds^2.$$

Since the metric $\mathcal{G}(\mathrm{P}, r)$ is equal to the canonical hyperbolic warp product metric $\sinh^2 s \sigma_{\mathbf{S}^1} + ds^2$ on the ball of radius $r - d_2$, we can extend $\mathcal{G}(\mathrm{P}, r)$ to the cone point $o_{\mathrm{CP}} = 0 \in \mathbf{R}^2$. It is straightforward to verify that $\mathcal{G}(\mathrm{P}, r)$ satisfies the following three properties:

- P'1. The metrics $\mathcal{G}(\mathrm{P}, r)$ and σ_{CP} have the same set of rays.
- P'2. The metric $\mathcal{G}(\mathrm{P}, r)$ coincides with σ_{CP} outside the ball of radius r.
- P'3. The metric $\mathcal{G}(\mathrm{P}, r)$ coincides with $\sinh^2 s \sigma_{\mathbf{S}^1} + ds^2$ on the ball of radius $r - d_2$.
- P'4. The family of metrics $\{\mathcal{G}(\mathrm{P}, r)\}_{r > d_2}$ has cut limits (see Section 4). Here we think of d_2 as fixed while r is the index of the family.

The cut limit of $\mathcal{G}(p, r)$ at b is

$$(\mathbf{8.1.2}) \qquad \left(\lim_{r\to\infty} \mu_{d_2,r,k}(r+b)\right)\sigma_{\mathbf{S}^1} = \left(1 + (k^2 - 1)\rho\left(1 + \frac{b}{d_2}\right)\right)\sigma_{\mathbf{S}^1}.$$

8.2. *The inductive step*

In this section we follow notations of (8.0.1). We fix d, ξ, c, ς and hence A, B. With the data ξ, A, B, $r > 0$ and an all-right spherical complex P we defined in Section 6.2 the numbers $r_k = r_k(r)$ and for every Δ^k constructed the sets $\mathcal{Y}(\mathrm{P}, \Delta^k, r)$, $\mathcal{Y}(\mathrm{P}, r)$, $\mathcal{X}(\mathrm{P}, \Delta^k, r)$, $\mathcal{X}(\mathrm{P}, r)$, where $\Delta^k \in \mathrm{P}$. The inverse of the function $r_k = r_k(r)$ shall be denoted by $r = r(r_k)$. Recall also that in 6.1.4 we identified $\mathsf{CStar}(\Delta^k, \mathrm{P})$, with the metric $\sigma_{\mathrm{CP}}|_{\mathsf{CStar}(\Delta^k,\mathrm{P})}$, with $\mathcal{E}_{\mathrm{C}\Delta^k}(\mathsf{CLink}(\Delta^k))$, with the metric $\mathcal{E}_k(\sigma_{\mathsf{CLink}(\Delta^k,\mathrm{P})})$. We will use these objects in this section.

Inductive hypothesis

Let $m \geq 2$ and suppose that for every triple $(\mathrm{P}, \mathcal{L}_\mathrm{P}, \phi_\mathrm{P})$, $j = \dim \mathrm{P} \leq m - 1$, as in items 5 and 6 of (8.0.1), and $r > d_i$, $i = 2, \ldots, m+1$ there are two Riemannian metrics: the *smoothed metric* $\mathcal{G}(\mathrm{P}, \mathcal{L}_\mathrm{P}, \phi_\mathrm{P}, r, \xi, \mathsf{d}, (c, \varsigma))$, and the *patched metric* $\wp(\mathrm{P}, \mathcal{L}_\mathrm{P}, r)$. Sometimes we will use the notation $\mathcal{G}(\mathrm{P}, \mathcal{L}_\mathrm{P}, \phi_\mathrm{P}, r)$, or even $\mathcal{G}(\mathrm{P}, r)$, for the smoothed metric, and $\wp(\mathrm{P}, r)$ for the patched metric. We demand these metrics satisfy the following properties

P1. The smoothed metric $\mathcal{G}(\mathrm{P}, r)$ is a Riemannian metric defined on the whole of $(\mathrm{CP}, \mathcal{S}_{\mathrm{CP}})$, and it has the same set of rays as σ_{CP}.

P2. The patch metric $\wp(\mathrm{P}, r)$ is a Riemannian metric defined outside the ball in CP of radius $r_{j-2} - (4 + 2\xi)$ (with smooth structure $\mathcal{S}_{\mathrm{CP}}$), and it is ray compatible with (CP, o).

P3. On $\mathcal{Y}(\mathrm{P}, \Delta^k, r)$, $k \leq j - 2 = \dim \mathrm{P} - 2$, the patched metric $\wp(\mathrm{P}, r)$ coincides with the metric

$$\mathcal{E}_{\mathrm{C}\Delta^k}\big(\mathcal{G}\big(\mathsf{Link}\big(\Delta^k, \mathrm{P}\big), r\big)\big),$$

where $\mathcal{G}(\mathsf{Link}(\Delta^k, \mathrm{P}), r) = \mathcal{G}(\mathsf{Link}(\Delta^k, \mathrm{P}), \mathcal{L}_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_\mathrm{P}), \phi_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_\mathrm{P}), r)$ is defined on $(\mathsf{CLink}(\Delta, \mathrm{P}), \mathcal{S}_{\mathrm{CP}})$. (Recall $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \subset \mathsf{CStar}(\Delta^k, \mathrm{P}) = \mathrm{C}\Delta^k \times \mathsf{CLink}(\Delta^k, \mathrm{P})$, see 6.1.3, Propositions 6.2.1, and Lemma 8.0.4.)

P4. On $\mathcal{Y}(\mathrm{P}, r)$ the patched metric $\wp(\mathrm{P}, r)$ coincides with σ_{CP} (which is hyperbolic on $\mathcal{Y}(\mathrm{P}, r)$).

P5. The metrics $\mathcal{G}(\mathrm{P}, r)$ and $\wp(\mathrm{P}, r)$ coincide outside the ball in CP of radius r_{j-2}.

Note that the patched metric $\wp(\mathrm{P}, \mathcal{L}_\mathrm{P}, r)$ does not depend on ϕ_P.

Remark **8.2.1.** — Here is a subtle point. In the Inductive Hypothesis we are assuming the existence of the metrics $\mathcal{G}(\mathrm{P}, \mathcal{L}_\mathrm{P}, \phi_\mathrm{P}, r)$, $\wp(\mathrm{P}, \mathcal{L}_\mathrm{P}, r)$ for every abstract all-right spherical complex P of dimension $\leq m - 1$. On the other hand in P3 we are considering $\mathsf{Link}(\Delta, \mathrm{P})$ as a subcomplex of P. We will identify the abstract complex $\mathsf{Link}(\Delta, \mathrm{P})^{abstract}$ with the subcomplex $\mathsf{Link}(\Delta, \mathrm{P})$ of P using the other data given in (8.0.1):

$$\mathsf{Link}(\Delta, \mathrm{P})^{abstract} \xrightarrow{\phi_{\mathsf{Link}(\Delta,\mathrm{P})^{abstract}}} \mathbf{S}^i \xrightarrow{h_\Delta} \mathsf{Link}(\Delta, \mathrm{P}) \subset \mathrm{P},$$

where $i = \dim \mathrm{P} - \dim \Delta - 1$, and $h_\Delta \in \mathcal{L}_\mathrm{P}$ is the given link smoothing of $\mathsf{Link}(\Delta, \mathrm{P})$ in P. Lemma 8.0.3 implies that these identifications are transitive, that is, they preserve the identification given in Lemma 5.3.2.

Properties P3, P4, P5 and the definition of the sets $\mathcal{X}(\mathrm{P}, \Delta^k, r)$, $\mathcal{X}(\mathrm{P}, r)$ imply

P6. On $\mathcal{X}(\mathrm{P}, \Delta^k, r)$, $k \le j - 2 = \dim \mathrm{P} - 2$, the smoothed metric $\mathcal{G}(\mathrm{P}, r)$ coincides with the metric

$$\mathcal{E}_{\mathrm{C}\Delta^k}\big(\mathcal{G}\big(\mathsf{Link}\big(\Delta^k, \mathrm{P}\big), r\big)\big),$$

where $\mathcal{G}(\mathsf{Link}(\Delta^k, \mathrm{P}), r) = \mathcal{G}(\mathsf{Link}(\Delta^k, \mathrm{P}), \mathcal{L}_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_\mathrm{P}), \phi_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_\mathrm{P}), r)$ is defined on $(\mathsf{CStar}(\Delta, \mathrm{P}), \mathcal{S}_{\mathrm{CP}})$.

P7. On $\mathcal{X}(\mathrm{P}, r)$ the smoothed metric $\mathcal{G}(\mathrm{P}, r)$ coincides with the metric σ_{CP}.

Note that the metrics $\mathcal{G}(\mathrm{P}^1, r)$ constructed for spherical all-right 1-complexes in Section 8.1, together with the choice $\wp(\mathrm{P}^1, r) = \sigma_{\mathrm{CP}^1}$ satisfy properties P1–P5. Indeed P1' implies P1, P2' implies P5 (recall $r_{-1} = r$, see Section 6.2) and P2, P3, P4 are trivially satisfied.

Inductive step

Now, assume we are given the data: P, $\dim \mathrm{P} = m$, $\mathcal{L}_\mathrm{P}$, ϕ_P, r as items 5 and 6 in (8.0.1). We define the patched metric $\wp(\mathrm{P}, r) = \wp(\mathrm{P}, \mathcal{L}_\mathrm{P}, r)$ as in P3 and P4 above. That is, we define $\wp(\mathrm{P}, r)$ by demanding that:

P"3. On $\mathcal{Y}(\mathrm{P}, \Delta^k, r)$, $k \le \dim \mathrm{P} - 2$, $\wp(\mathrm{P}, r)$ coincides with the metric $\mathcal{E}_{\mathrm{C}\Delta^k}(\mathcal{G}(\mathsf{Link}(\Delta^k, \mathrm{P}), r))$.

P"4. On $\mathcal{Y}(\mathrm{P}, r)$, the patched metric $\wp(\mathrm{P}, r)$ coincides with the metric σ_{CP}.

Lemma **8.2.2.** — *The patched metric* $\wp(\mathrm{P}, r)$ *defined by properties* P"3 *and* P"4 *is well defined.*

Proof. — The metric $\wp(\mathrm{P}, r)$ is defined on the "patches" $\mathcal{Y}(\mathrm{P}, \Delta, r)$, $\Delta \in \mathrm{P}$, and $\mathcal{Y}(\mathrm{P}, r)$. We have to prove that these definitions coincide on the intersections $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \cap \mathcal{Y}(\mathrm{P}, \Delta^j, r)$, $\mathcal{Y}(\mathrm{P}, r) \cap \mathcal{Y}(\mathrm{P}, \Delta^j, r)$. If $\Delta^j \cap \Delta^k = \emptyset$ then (vi) of Proposition 6.2.1 implies $\mathcal{Y}(\mathrm{P}, \Delta^j, r) \cap \mathcal{Y}(\mathrm{P}, \Delta^k, r) = \emptyset$. Also if $\Delta^j \not\subset \Delta^k$ and $\Delta^k \not\subset \Delta^j$ by (viii) of Proposition 6.2.1, we also get $\mathcal{Y}(\mathrm{P}, \Delta^j, r) \cap \mathcal{Y}(\mathrm{P}, \Delta^k, r) = \emptyset$. Therefore we assume $\Delta^j \subset \Delta^k, j < k$.

Recall that $\mathcal{Y}(\mathrm{P}, \Delta^j, r) \subset \mathsf{CStar}(\Delta^j, r)$ and $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \subset \mathsf{CStar}(\Delta^k, r)$ (see Proposition 6.2.1(i)). The metrics

$$h = \mathcal{E}_{\mathrm{C}\Delta^j}\big(\mathcal{G}\big(\mathsf{Link}\big(\Delta^j, \mathrm{P}\big), \mathcal{L}_{\mathsf{Link}(\Delta^j,\mathrm{P})}(\mathcal{L}_\mathrm{P}), \phi_{\mathsf{Link}(\Delta^j,\mathrm{P})}(\mathcal{L}_\mathrm{P}), r\big)\big), \qquad (\mathbf{1})$$

$$g = \mathcal{E}_{\mathrm{C}\Delta^k}\big(\mathcal{G}\big(\mathsf{Link}\big(\Delta^k, \mathrm{P}\big), \mathcal{L}_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_\mathrm{P}), \phi_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_\mathrm{P}), r\big)\big), \qquad (\mathbf{2})$$

are defined on the whole of $\mathsf{CStar}(\Delta^j, \mathrm{P})$ and $\mathsf{CStar}(\Delta^k, \mathrm{P})$, respectively. From 6.1.3 we have that $\mathsf{CStar}(\Delta^j, \mathrm{P}) = \mathrm{C}\Delta^j \times \mathsf{CLink}(\Delta^j, \mathrm{P})$. And from Proposition 6.2.3 we have that $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \subset \mathrm{C}\Delta^j \times \mathcal{X}(\mathsf{Link}(\Delta^j, \mathrm{P}), \Delta^l, r)$, where $\Delta^l = \Delta^k \cap \mathsf{Link}(\Delta^j, \mathrm{P})$ (alternatively Δ^l is opposite to Δ^j in Δ^k, or $\Delta^l = \mathsf{Link}(\Delta^j, \Delta^k)$). Hence it is enough to prove that the metrics h and g coincide on $\mathrm{C}\Delta^j \times \mathcal{X}(\mathsf{Link}(\Delta^j, \mathrm{P}), \Delta^l, r)$. But (2) and (the second equality in) Corollary 6.1.8 imply

$$g = \mathcal{E}_{\mathrm{C}\Delta^j}\big[\mathcal{E}_{\mathrm{C}\Delta^l}\big(\mathcal{G}\big(\mathsf{Link}\big(\Delta^k, \mathrm{P}\big), \mathcal{L}_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}), \phi_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}), r\big)\big)\big]. \tag{3}$$

Note that the inductive hypothesis (specifically property P6, which is implied by P3, P5) applied to the data $\mathsf{Link}(\Delta^j, \mathrm{P})$ and Δ^l gives us that on the set $\mathcal{X}(\mathsf{Link}(\Delta^j, \mathrm{P}), \Delta^l, r)$ we have

$$\mathcal{G}\big(\mathsf{Link}\big(\Delta^j, \mathrm{P}\big), \mathcal{L}_{\mathsf{Link}(\Delta^j,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}), \phi_{\mathsf{Link}(\Delta^j,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}), r\big) = \mathcal{E}_{\mathrm{C}\Delta^l}(f), \tag{4}$$

where

$$f = \mathcal{G}\big(\mathsf{Link}\big(\Delta^l, \mathsf{Link}\big(\Delta^j, \mathrm{P}\big)\big), \mathcal{L}_{\mathsf{Link}(\Delta^l,\mathsf{Link}(\Delta^j,\mathrm{P}))}(\mathcal{L}_{\mathsf{Link}(\Delta^j,\mathrm{P})}), \phi_{\mathsf{Link}(\Delta^l,\mathsf{Link}(\Delta^j,\mathrm{P}))}(\mathcal{L}_{\mathsf{Link}(\Delta^j,\mathrm{P})}). \tag{5}$$

Using Lemma 5.3.2 (and Remark 5.3.3) together with the transitivity of the restriction rule (Lemma 8.0.4) in (5) we get

$$f = \mathcal{G}\big(\mathsf{Link}\big(\Delta^k, \mathrm{P}\big), \mathcal{L}_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}), \phi_{\mathsf{Link}(\Delta^k,\mathrm{P})}(\mathcal{L}_{\mathrm{P}}), r\big). \tag{6}$$

Putting together (1), (4) and (6) we obtain an equation with the same right-hand side as in (3) but with h instead of g on the left-hand side. This proves that $g = h$ on $\mathcal{Y}(\mathrm{P}, \Delta^j, r) \cap \mathcal{Y}(\mathrm{P}, \Delta^k, r)$.

The proof that the patched metric is well defined on $\mathcal{Y}(\mathrm{P}, \Delta^k, r) \cap \mathcal{Y}(\mathrm{P}, r)$ uses a similar argument and it follows from Proposition 6.2.5, the inductive hypothesis applied to $\mathsf{Link}(\Delta^k, \mathrm{P})$ (that is, properties P4, P5 which imply P7) and Corollary 6.1.8. □

By construction, the patch metric $\wp(\mathrm{P}, r)$ we just constructed satisfies P3 and P4. We next prove it also satisfies P2.

Lemma **8.2.3.** — *The patch metric* $\wp(\mathrm{P}, r)$ *satisfies* P2.

Proof. — Follows from (iv) of Proposition 6.2.1 that the patch metric $\wp(\mathrm{P}^j, r)$ is defined outside the closed ball in CP of radius $r_{j-2} - (4 + 2\xi)$. The ray compatibility property (see Remark 1.1) is proved by induction on the dimension of the complex P. It is clearly true for $\dim \mathrm{P} = 1$. Assume is true for complexes of dimension $< j$, and take P with $\dim \mathrm{P} = j$. We have to show that $\wp(\mathrm{P}, r)$ is ray compatible with (CP, o) over the complement of the closed ball in CP of radius $r_{j-2} - (4 + 2\xi)$. By P"4 (that is, by construction) this is true over $\mathcal{Y}(\mathrm{P}, r)$. And 6.1.5 together with P3, P2 (for complexes of dimension $< j$) imply that this is also true over $\mathcal{Y}(\mathrm{P}, \Delta^i, r)$, $i < j-1$. By Remark 1.1(5) the patch metric is ray compatible with (CP, o) over the union of all the $\mathcal{Y}$ sets, which is, by Proposition 6.2.1(iv), the complement of the closed ball in CP of radius $r_{j-2} - (4+2\xi)$. □

We now define the smoothed metric $\mathcal{G}(\mathrm{P}, r)$. Recall that $r_{m-2} = r_{m-2}(r)$. Let $r = r(r_{m-2})$ be the inverse, where we consider r_{m-2} as a large real variable. For $\mathrm{P} = \mathrm{P}^m$ using $\mathrm{C}\phi_{\mathrm{P}}$ we get an identification between CP and $\mathbf{R}^{m+1}$. Therefore we can consider the family of metrics $\{\wp(\mathrm{P}, r(r_{m-2}))\}_{r_{m-2}-\frac{1}{2}}$ as a family of metrics on $\mathbf{R}^{m+1}$. Lemma 8.2.3 (see also Remark 4.1(2)) implies that this family is an $\odot$-family of metrics. We define

$$\textbf{(8.2.4)} \qquad \mathcal{G}(\mathrm{P}, r) = \mathcal{H}_{r_{m-2}-\frac{1}{2}, d_{m+1}-\frac{1}{2}} \wp\big(\mathrm{P}, r(r_{m-2})\big).$$

Property P5 for $\mathcal{G}(\mathrm{P}, r)$ holds by construction and by (ii) of Proposition 4.5. Property P1 follows from P2, P5 and (i) of Proposition 4.5.

Remarks **8.2.5.**

1. The terms $\frac{1}{2}$ above are introduced to "correct" the $\frac{1}{2}$ term that appears in hyperbolic forcing (see Section 3.3 and Proposition 4.5). Without the term $\frac{1}{2}$ property P5 would appear with radius $r_{j-2} + \frac{1}{2}$ instead of just r_{j-2}, so that P7 would not be true, and the last part of the proof of Lemma 8.2.2 would fail.
2. We want to apply Proposition 4.5 to the family $\{\wp(\mathrm{P}, r(r_{m-2}))\}$; this is why we are considering this family indexed by $r_{m-2} - \frac{1}{2}$ instead of r_{m-2}.
3. Note that because of the way we constructed the patch metric $\wp(\mathrm{P}, r)$, it does not depend on the map ϕ_{P}; but the smoothed metric $\mathcal{G}(\mathrm{P}, r)$ does depend on ϕ_{P}.

By construction and Proposition 4.5(i) we have the following property.

P8. The smoothed metric $\mathcal{G}(\mathrm{P}^m, r)$ is hyperbolic on $\mathbf{B}_{r_{m-2}-d_{m+1}}(\mathrm{CP})$.

Note that the patched metric $\wp(\mathrm{P}^m, r)$ does not depend on d_i, $i > m$. Also the smoothed metric $\mathcal{G}(\mathrm{P}^m, r)$ does not depend on d_i, $i > m+1$.

This concludes the construction of the smoothed metric $\mathcal{G}(\mathrm{P}, r) = \mathcal{G}(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}}, r, \xi, \mathsf{d}, (c, \varsigma))$, and the patch metric $\wp(\mathrm{P}, r) = \mathcal{G}(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, r, \xi, \mathsf{d}, (c, \varsigma))$.

8.3. *On the dependence of* $\mathcal{G}(\mathrm{P}, r)$ *on the variable* c

In this section we show that the smoothed metric $\mathcal{G}(\mathrm{P}, r) = \mathcal{G}(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}}, \xi, r, (c, \varsigma))$ does not depend on the variable c, provided $c\varsigma$ is small enough. In the next section we will show that, assuming d and r large, the metric $\mathcal{G}(\mathrm{P}, r)$ is ε-close to hyperbolic. However the excess of the ε-close to hyperbolic charts does depend on c. In the next result assume ς, ξ and d fixed. We shall write $\mathcal{G}(\mathrm{P}, r, c) = \mathcal{G}(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}}, r, \xi, (c, \varsigma))$ and similarly for the patch metric.

Proposition **8.3.1.** — *Let $c' > c > 1$ be such that $c'\varsigma < e^{-6-2\xi}$. Then $\wp(\mathrm{P}, r, c') = \wp(\mathrm{P}, r, c)$ on $\mathrm{CP} - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})$. Also $\mathcal{G}(\mathrm{P}, r, c') = \mathcal{G}(\mathrm{P}, r, c)$ on* CP.

Proof. — Write $\mathsf{A}' = \mathsf{B}(c', \varsigma)$. Denote by $\mathcal{Y}'(\mathrm{P}, \Delta, r) = \mathcal{Y}(\mathrm{P}, \Delta, r, \xi, (c', \varsigma))$ the sets obtained by replacing c in the definition of $\mathcal{Y}(\mathrm{P}, \Delta, r) = \mathcal{Y}(\mathrm{P}, r, \xi, (c, \varsigma))$ (see Section 6.2) by c'. Similarly we obtain $\mathcal{Y}'(\mathrm{P}, r)$. Also, let $s'_{m,k}$ be obtained from $s_{m,k}$ by replacing c by c' (see Section 6.2). Then $s'_{m,k} > s_{m,k}$. Since $c' > c$ we have

$$\mathcal{Y}(\mathrm{P}, \Delta, r) \subset \mathcal{Y}'(\mathrm{P}, \Delta, r) \quad \text{and} \quad \mathcal{Y}(\mathrm{P}, r) \subset \mathcal{Y}'(\mathrm{P}, r). \tag{1}$$

We will prove the proposition by induction on the dimension m of P^m. It can be checked from Section 8.1 that when $m = 1$ the metrics are independent of the variable c. Assume $\mathcal{G}(\mathrm{P}^k, r, c') = \mathcal{G}(\mathrm{P}^k, r, c)$, for every P^k, $k < m$. Consider P^m. First we prove that the corresponding patched metrics $\wp(\mathrm{P}, r, c')$ and $\wp(\mathrm{P}, r, c)$ coincide. But it follows from properties P3 and P4 applied to both metrics, the inductive hypothesis and (1) that $\wp(\mathrm{P}^m, r, c') = \wp(\mathrm{P}^m, r, c)$ on $\mathcal{Y}(\mathrm{P}, \Delta^k, r)$, for all $\Delta^k \in \mathrm{P}$, $k \le m-2$, and on $\mathcal{Y}(\mathrm{P}, r)$. Therefore, by Proposition 6.2.1(iv), the metrics $\wp(\mathrm{P}^k, r, c')$, $\wp(\mathrm{P}^k, r, c)$ coincide on $\mathrm{CP} - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})$. Finally note that the smoothed metrics $\mathcal{G}(\mathrm{P}, r, c)$, $\mathcal{G}(\mathrm{P}, r, c')$ are obtained from the corresponding patched metrics by using the hyperbolic forcing process of Section 4. But this process depends only on d and $r_{m-2} = \sinh^{-1}(\frac{\sinh r}{\sin \alpha_{m-2}})$. The former is fixed and the latter, since $\sin \alpha_{m-2} = \varsigma^{m-1}$ (see Section 6.2), is independent of c and c'. □

In the next section we will need the following result. We use the notation in the proof of the previous proposition. Recall $s'_{m,k}$ is obtained from $s_{m,k}$ by replacing c by c'.

Lemma **8.3.2.** — $(s'_{m,k} - s_{m,k}) > \ln(\frac{c'}{c}) - 1$, *provided $r > 1$ and $c' > c$.*

Proof. — A simple calculation shows that the function $t \mapsto \sinh^{-1}(c't) - \sinh^{-1}(ct)$ is increasing. And another calculation shows that the value of this function at $t = 1$ has value at least $\ln c' - \ln c - 1$. Hence

$$\sinh^{-1}\big(c't\big) - \sinh^{-1}(ct) \ge \ln c' - \ln c - 1, \quad \text{for } t \ge 1 \tag{1}$$

From the definition at the beginning of Section 6.2 we have $s_{m,k} = \sinh^{-1}(c\frac{\sinh r}{\varsigma^{m-k-2}})$ and $s'_{m,k} = \sinh^{-1}(c'\frac{\sinh r}{\varsigma^{m-k-2}})$. Take $t = \frac{\sinh r}{\varsigma^{m-k-2}}$ in inequality (1). □

8.4. *On the ε-close to hyperbolicity of $\mathcal{G}(\mathrm{P}, r)$*

In this section we prove that the smoothed metrics on CP^m are ε-close to hyperbolic, provided $d_2, \ldots, d_{m+1}$ and r are large enough. Recall that an element of CP can be written as sx, $s \ge 0$, $x \in \mathrm{P}$.

Lemma **8.4.2.** — *The family of metrics* $\{\wp(\mathrm{P}^m, r(r_{m-2}))\}_{r_{m-2}}$ *has cut limits on* $[-1,\infty)$. *Also, the family of metrics* $\{\mathcal{G}(\mathrm{P}^m, r(r_{m-2}))\}_{r_{m-2}}$ *has cut limits on* $\mathbf{R}$.

Proof. — First note Propositions 4.5(v), 4.5(vi) and (8.2.4) imply that the first statement in the lemma implies the second. We prove the first statement by induction on the dimension m of P^m. For $m=1$ the lemma follows P4' and (8.1.2) in Section 8.1.

Claim. — *Suppose the* $\odot$*-family of metrics* $\{\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P}^m), r(r_{m-k-3}))\}_{r_{m-k-3}}$ *has cut limits on* $\mathbf{R}$. *Then the* $\odot$*-family of metrics* $\{\mathcal{E}_{\mathrm{C}\Delta^k}(\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P})), r(r_{m-2}))\}_{r_{m-2}}$ *also has cut limits on* $\mathbf{R}$.

Proof of Claim. — By construction (see Proposition 4.5(i) or P8), the family $\{\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P}), r(r_{m-k-3}))\}_{r_{m-k-3}}$ satisfies the hypothesis of Proposition 4.8.4: the family is hyperbolic around the origin. Since $r_{m-k-3}=\sinh^{-1}(\sinh r_{m-2}\sin\alpha_k)$ the claim follows from Proposition 4.8.4. □

We continue with the proof of Lemma 8.4.2. Assume the lemma holds for P^k, $k<m$. Let $\mathrm{P}=\mathrm{P}^m$ Suppose that the lemma does not hold for the family $\mathcal{F}=\{\wp(\mathrm{P}^m, r(r_{m-2}))\}_{r_{m-2}}$. We will show a contradiction. To simplify our notation write $s=r_{m-2}$ and $g_s=\mathcal{G}(\mathrm{P}^m, r(s))$. We have $g_s=\sinh^2 r(g_s)_t+dt^2$, where t is the distance to o_{CP^m}. Since $\mathcal{F}$ does not have cut limits on $[-1,\infty)$ there is a bounded closed interval $\mathrm{I}\subset[-1,\infty)$ such that $\mathcal{F}$ does not have cut limits on I. For $(x,b)\in\mathrm{P}^m\times\mathrm{I}$ write $f(s,x,b)=\widehat{(g_s)}_{s+b}(x)$. Note that $\sinh^2(s+b)f(s,x,b)+dt^2=g_s((s+b)x)$. Since we are assuming that $\mathcal{F}$ does not have cut limits on I we have that the family $\{f(s,x,b)\}_s$ defined for $(x,b)\in\mathrm{P}\times\mathrm{I}$ does not converge in the C^2 topology as $s\to\infty$. Hence there is a derivative ∂^{J}, for some multi-index of order ≤ 2, and sequences $s_n\to\infty$, $x_n\to x$, $b_n\to b$ such that $|\partial^{\mathrm{J}}f(s_n,x_n,b_n)-\partial^{\mathrm{J}}f(s_{n+1},x_n,b_n)|\geq a$ for some fixed $a>0$, and n even. By Proposition 6.3.3 we have that $\mathrm{R}_{x,b}(s)=(s+b)x\in\mathcal{Y}(\mathrm{P},\Delta^k,r(s))$, for some Δ^k, $k\leq m-2$, and $s>s'$, for some s'; or $\mathrm{R}_{x,b}(s)=(s+b)x\in\mathcal{Y}(\mathrm{P},r(s))$, $s>s'$, for some s'. Consider the first case: $\mathrm{R}_{x,b}(s)=(s+b)x\in\mathcal{Y}(\mathrm{P},\Delta^k,r(s))$, for some Δ^k, $k\leq m-2$. Moreover, also by Proposition 6.3.3, we can assume $\mathrm{R}_{s_n,b_n}(s)=(s+b_n)x_n\in\mathcal{Y}(\mathrm{P},\Delta^k,r(s))$, for $s>s'$. But by property P3, on $\mathcal{Y}(\mathrm{P},\Delta^k,r(s))$ the metric g_s is equal to $\mathcal{E}_{\mathrm{C}\Delta^k}(\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P}),r(s)))$. Consequently the family of metrics $\{\mathcal{E}_{\mathrm{C}\Delta^k}(\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P}),r(s))\}_s$ does not have cut limits on I either. But the claim, together with the inductive hypothesis, imply that this family does have cut limits, which leads to a contradiction.

Now consider the second case in Proposition 6.3.3, that is, $\mathrm{R}_{x,b}(s)=(s+b)x\in\mathcal{Y}(\mathrm{P},r(s))$, $s>s'$, for some s'. But by P4 the metric $\wp(\mathrm{P}^m, r(r_{m-2}))$ coincides with σ_{CP}, hence f is constant on s (s large) near (x,b). This is a contradicts the assumption $|\partial^{\mathrm{J}}f(s_n,x_n,b_n)-\partial^{\mathrm{J}}f(s_{n+1},x_n,b_n)|\geq a$. □

For a positive real number ξ and a positive integer write $\xi_k=\xi-k+\frac{1}{k}$. Note that $\xi_1=\xi$.

Proposition **8.4.3.** — *Let* $\varsigma \in (0,1)$, $\xi > 0$, $c > 1$, *and consider* $(\mathrm{P}^m, \mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}})$. *Assume*

(i) $c\varsigma < e^{-10-3\xi}$
(ii) $c \geq e^{4+\xi}$
(iii) $m+1 \leq \xi$.

Let $\varepsilon > 0$. *Then we have that* $\mathcal{G}(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, \phi_{\mathrm{P}}, \xi, r, \mathsf{d}, (c, \varsigma))$ *is* $(\mathrm{B}_a, \varepsilon)$*-close to hyperbolic* $(a = r_{m-2} - d_{m+1})$, *with charts of excess* ξ_m, *provided* d_i *and* $r - d_i$, $i = 2, \ldots, m+1$, *are sufficiently large.*

Remarks.

1. By "sufficiently large" we mean that there are $r_i(\mathrm{P}, \varepsilon)$ and $d_i(\mathrm{P}, \varepsilon)$, $i = 2, \ldots, m+1$, such that the proposition holds whenever we choose $r - d_i \geq r_i(\mathrm{P}, \varepsilon)$ and $d_i \geq d_i(\mathrm{P}, \varepsilon)$. We will write $r_i(\mathrm{P}) = r_i(\mathrm{P}, \varepsilon)$, and $d_i(\mathrm{P}) = d_i(\mathrm{P}, \varepsilon)$, if the context is clear.
2. The choices of c, ξ and ς do not depend on ε.
3. If we want the smoothed metric on a cone CP^m to be $(\mathrm{B}_a, \varepsilon)$-close to hyperbolic we can choose $\xi = m+1$, $c = e^{4+\xi}$ and $\varsigma < e^{-(12+2\xi)}$. With these choices the method would not work for P of dimension $> m$.
4. The condition $c\varsigma = e^{-(8+2\xi)}$ is stronger than the condition $c\varsigma < e^{-4}$. The latter is used to construct the smoothed metric but it is not strong enough to give us ε-close to hyperbolicity.

Proof. — We assume c, ξ, ς fixed and satisfying (i) and (ii), that is, $c\varsigma < e^{-(8+\xi)}$ and $c \geq e^{4+\xi}$. We will only mention the relevant objects to our argument in the notation for the smoothed metrics. That is, we will write $\mathcal{G}(\mathrm{P}, \mathsf{d}, r, \xi, (c, \varsigma))$ or just $\mathcal{G}(\mathrm{P}, \mathsf{d}, r)$. Our proof is by induction on the dimension m of P^m, with $m+1 < \xi$. Without loss of generality we can assume every ε we take satisfies:

$$\varepsilon < \frac{1}{(1+\xi)^2}. \tag{1}$$

For $m = 1$ we have that the proposition follows from Section 8.1 and Theorem 4.6 by writing $\lambda = r$, choosing $g_r = \sigma_{\mathrm{CP}}$, replacing ξ by $\xi + 1$, and taking $\varepsilon' = \varepsilon$. Also, since $g_r = \sigma_{\mathrm{CP}}$ is ε-close to hyperbolic, for every ε, we can take the ε in Theorem 4.6 to be zero. With all these choices Theorem 4.6 implies that $\mathcal{G}(\mathrm{P}, d_2, r)$ is ε-close to hyperbolic, with charts of excess $\xi = \xi_1$, provided $r - d_2$ and d_2 are large enough.

Let m such that $m+1 \leq \xi$. We write $a_k = r_{k-2} - d_{k+1}$, and note that $\mathcal{G}(\mathrm{P}^k, r, \mathsf{d})$ is, by construction (see P8), radially hyperbolic on the ball of radius a_k. We now assume that the proposition holds for all $k < m$. That is, given $\varepsilon > 0$ and P^k, the smoothed metric $\mathcal{G}(\mathrm{P}^k, r, \mathsf{d})$ is $(\mathrm{B}_{a_k}, \varepsilon)$-close to hyperbolic, with charts of excess ξ_k, provided $r - d_i$ and d_i, $i = 2, \ldots, d_{k+1}$ are large enough. Note that, since we are assuming $k < m$, we get that

$k+1<\xi$. For $0\leq k\leq m-2$ we use the following notation

$$A_k=C(m-k,k+1,\xi_{m-k-1})B=C_2(\xi)\varepsilon_k=\frac{\varepsilon}{3A_kB}, \tag{2}$$

where C is as in Theorem 2.7 and C_2 as in Theorem 4.6. Let $P=P^m$. For $k<m$ write $L_k=\{\mathsf{Link}(\Delta^k,P)\}_{\Delta^k\in P}$. A generic element in L_k will be denoted by $Q=Q^j, j+k=m-1$. By inductive hypothesis, for each Q^j there are $r_i(Q^j)=r_i(Q^j,\varepsilon_k)$ and $d_i(Q^j)=d_i(Q^j,\varepsilon_k)$, $i=2,\dots,j+1$ such that $\mathcal{G}(Q,r,\mathsf{d})$ is (B_{a_j},ε_k)-close to hyperbolic, with charts of excess ξ_j, provided $r-d_i\geq r_i(Q^j)$ and $d_i\geq d_i(Q^j)$. For $2\leq i\leq m$, let $d_i(P)$ be defined by

$$d_i(P)=\max_{Q^j,i\leq j+1}\{d_i(Q^j)\}.$$

We write $\mathsf{d}(P)=\{d_2(P),\dots,d_m(P),\dots\}$ where $d_i(P)$, $i\geq m+1$, is any positive number. This is just for notational purposes and the arguments given below will not depend the $d_i(P)$, $i>m+1$. We do reserve the right to later choose $d_{m+1}(P)$ larger. Also for $2\leq i\leq m$ write

$$r_i(P)=d_i(P)+\max_{Q^j,i\leq j+1}\{4\ln(m),r_i(Q^j),R(\varepsilon_{m-i},m-i+1,\xi_{i-1})\},$$

where R is as in the statement of Theorem 2.7. Therefore we get that (recall $j+k=m-1$)

(**8.4.4**) For every $Q^j\in L_k$, the metric $\mathcal{G}(Q^j,r,\mathsf{d})$ is (B_{a_j},ε_k)-close to hyperbolic, with charts of excess, ξ_j provided $r-d_i\geq r_i(P)$ and $d_i\geq d_i(P)$, $i=2,\dots,k+1$.

By definition we have $r_i(P)\geq 4\ln(m)$. Also, from the definition of r_k (see Section 6.2), we have $r_{j-2}=r_{j-2}(r)>r$. Hence, if $r-d_{j+1}\geq r_{j+1}(P)$ and $0\leq j\leq m-1$ we get that $a_j=r_{j-2}-d_{j+1}>r-d_{j+1}\geq r_{j+1}(P)\geq 4\ln(m)$. Therefore $e^{-(a_j/2)}<\frac{1}{m^2}$, and we get $\xi_j-e^{-(a_j/2)}>\xi_j-\frac{1}{m^2}=\xi-j+\frac{1}{j}-\frac{1}{m^2}>\xi-j+\frac{1}{j+1}\geq\xi-(m-1)+\frac{1}{m}$. Also, from the definition of $r_i(P)$ we get $r_{j+1}(P)\geq R(\varepsilon_k,k+1,\xi_j)$. Therefore $r-d_{j+1}\geq r_{j+1}(P)$ implies $a_j=r_{j-2}-d_{j+1}>r-d_{j+1}\geq r_{j+1}(P)\geq R(\varepsilon_k,k+1,\xi_j)$. We just proved the following two inequalities.

$$\begin{aligned}\xi_j-e^{-a_j/2}&>\xi-(m-1)+\frac{1}{m},\\ a_j&>R(\varepsilon_k,k+1,\xi_j).\end{aligned} \tag{3}$$

Taking the inequalities in (3), together with (8.4.4), Theorem 2.7 and the definitions given in (2) we get that

(**8.4.5**) For every $\mathsf{Link}(\Delta^k,P)\in L_k$, the metric $\mathcal{E}_{k+1}(\mathcal{G}(\mathsf{Link}(\Delta^k,P),r,\mathsf{d}))$, defined on the space $\mathcal{E}_{k+1}(\mathsf{CLink}(\Delta^k,P))$, is $\left(B_{a_j},\frac{\varepsilon}{3B}\right)$-close to hyperbolic, with charts of excess $\xi-(m-1)+\dfrac{1}{m}$, provided

$r-d_i\ge r_i(\mathrm{P})$ and $d_i\ge d_i(\mathrm{P})$, $i=2,\ldots,k+1$.

Lemma **8.4.6.** — *The patched metric* $\wp(\mathrm{P},r,\mathsf{d})$ *is radially* $(\frac{\varepsilon}{3\mathrm{B}})$*-close to hyperbolic on* $\mathrm{CP}-\bar{\mathbf{B}}_{r_{m-2}-1-\xi}$*, with charts of excess* $\xi-(m-1)+\frac{1}{m}$*, provided* $r-d_i\ge r_i(\mathrm{P})$*,* $d_i\ge d_i(\mathrm{P})$*,* $i=2,\ldots,m$*.*

Proof. — The idea of the proof is to apply (8.4.5) on the patches $\mathcal{Y}$. The problem is fitting the ε-close to hyperbolic charts. We need some preliminaries.

For $\Delta=\Delta^k\in\mathrm{P}$ write $\mathrm{Y}_\Delta=\mathcal{Y}(\mathrm{P},\Delta,r,\xi,(c,\varsigma))$ and $\mathrm{Y}=\mathcal{Y}(\mathrm{P},r,\xi,(c,\varsigma))$ (see Section 6.2). For $\Delta=\Delta^k$, $k\le m$ define

$$\mathrm{N}_\Delta=\mathsf{N}_{s_{m,k}}(\mathrm{C}\Delta,\mathrm{CP})-\bigcup_{\Delta^l\in\mathrm{P},l<k}\mathsf{N}_{s_{m,k}}\big(\mathrm{C}\Delta^l,\mathrm{CP}\big)-\bar{\mathbf{B}}_{r_{m-2}-1-\xi}(\mathrm{CP}),\tag{4}$$

$$\mathrm{N}=\mathrm{CP}-\bigcup_{\Delta^l\in\mathrm{P},l\le m-2}\mathsf{N}_{s_{m,k}}\big(\mathrm{C}\Delta^l,\mathrm{CP}\big)-\bar{\mathbf{B}}_{r_{m-2}-1-\xi}(\mathrm{CP}).\tag{5}$$

Write $\mathrm{N}_k=\cup_{\Delta^k\in\mathrm{P}}\mathrm{N}_{\Delta^k}$. It is straightforward to show that $\mathrm{CP}-\bar{\mathbf{B}}_{r_{m-2}-1-\xi}=\mathrm{N}\cup\bigcup_{k\le m-2}\mathrm{N}_k$. Let $c'=e^{4+\xi}c$. From hypothesis (i), that is from $c\varsigma<e^{-10-3\xi}$, we get that $c'\varsigma<e^{-6-2\xi}$, hence we can define the sets $\mathrm{Y}'_\Delta=\mathcal{Y}(\mathrm{P},\Delta,r,\xi,(c',\varsigma))$ and $\mathrm{Y}'=\mathcal{Y}(\mathrm{P},r,\xi,(c',\varsigma))$ (see Section 6.2). That is

$$\mathrm{Y}'_\Delta=\mathring{\mathsf{N}}_{s'_{m,k}}\big(\mathrm{C}\Delta^k,\mathrm{CP}\big)-\bigcup_{\Delta^l\in\mathrm{P},l<k}\mathsf{N}_{r_{m,k}}\big(\mathrm{C}\Delta^l,\mathrm{CP}\big)-\bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP}).$$

Here $s'_{m,k}$ is defined by replacing c by c' in the definition of $s_{m,k}$. Note that if we define Y' in the obvious way, we would just get $\mathrm{Y}'=\mathrm{Y}$. From the definitions we have $\mathrm{N}_\Delta\subset\mathrm{Y}'_\Delta$ and $\mathrm{N}\subset\mathrm{Y}$. Note that if we replace c by 1 in the definition of $s_{m,k}$ we obtain $r_{m,k}$. This together with hypothesis (ii), the definition of c', and Lemma 8.3.2 imply

$$\begin{aligned}&\big(s'_{m,k}-s_{m,k}\big)>3+\xi\\&\big(s_{m,k}-r_{m,k}\big)>3+\xi.\end{aligned}\tag{6}$$

It follows from $c'\varsigma<e^{-6-2\xi}$ and Proposition 8.3.1 that the Riemannian metrics $\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P}),r,\mathsf{d},c)$ and $\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P}),r,\mathsf{d},c')$ coincide. Therefore we have a version of (8.4.5) with $\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P}),r,\mathsf{d},c')$ replacing $\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P}),\mathsf{d},r)=\mathcal{G}(\mathsf{Link}(\Delta^k,\mathrm{P}),r,\mathsf{d},c)$:

(**8.4.7**) For every $\mathsf{Link}\big(\Delta^k,\mathrm{P}\big)\in\mathrm{L}_k$, the metric $\mathcal{E}_{k+1}\Big(\mathcal{G}\big(\mathsf{Link}\big(\Delta^k,\mathrm{P}\big),r,\mathsf{d},c'\big)\Big)$,

defined on the space $\mathcal{E}_{k+1}\big(\mathrm{C}\mathsf{Link}\big(\Delta^k,\mathrm{P}\big)\big)$, is $\left(\mathrm{B}_{a_j},\dfrac{\varepsilon}{3\mathrm{B}}\right)$-close to hyperbolic, with charts of excess $\xi - (m-1) + \dfrac{1}{m}$,

provided $r - d_i \geq r_i(\mathrm{P})$ and $d_i \geq d_i(\mathrm{P})$, $i = 2, \ldots, k+1$.

For $p \in \mathrm{CP}$ denote the ball of radius s centered at p by $\mathbf{B}_{s,p}(\mathrm{CP})$, with respect to the metric σ_{CP}.

Claim **1.** — *For* $\Delta = \Delta^k$, $k \leq m-2$, *we have that* $d_{\wp}(\mathrm{N}_\Delta, \mathrm{CP} - \mathrm{Y}'_\Delta) \geq 3 + \xi$.

Here $d_{\wp}(.,.)$ denotes path distance with respect to the metric $\wp(\mathrm{P}, r)$.

Proof of Claim. — Let $p \in \mathrm{N}_\Delta$ and $q \notin \mathrm{Y}'_\Delta$ and $\alpha : [0,1] \to \mathrm{CP}$ a path joining $\alpha(0) = p$ to $\alpha(1) = q$. After taking a restriction of this path, we can assume that $q \in \partial \mathrm{Y}'_\Delta$ and that $\alpha([0,1)) \subset \mathrm{Y}'_\Delta$. Let $\ell(\alpha)$ be the length of α with respect to the $\wp(\mathrm{P}, r)$ metric. To prove the claim we need to show that $\ell(\alpha) \geq 3 + \xi$. From the definition of Y'_Δ we have that the boundary of Y'_Δ has 3 types of pieces, thus we have three cases.

Case 1. $q \in \partial \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(\mathrm{CP})$. From the definition of N_Δ we have $p \notin \bar{\mathbf{B}}_{r_{m-2}-1-\xi}(\mathrm{CP})$, hence $\ell(\alpha) \geq (r_{m-2} - 1 - \xi) - (r_{m-2} - (4 + 2\xi)) = 3 + \xi$. This concludes case 1.

Case 2. $q \in \partial \mathsf{N}_{s'_{m,k}}(\mathrm{C}\Delta, \mathrm{CP})$. Since $\alpha([0,1)) \in \mathrm{Y}'_\Delta$ and $\mathrm{Y}'_\Delta \subset \mathsf{N}_{s'_{m,k}}(\mathrm{C}\Delta, \mathrm{CP})$ we get that $\alpha([0,1)) \subset \mathsf{N}_{s'_{m,k}}(\mathrm{C}\Delta, \mathrm{CP})$. This last inclusion together with the fact that $p \in \mathrm{N}_\Delta \subset \mathsf{N}_{s_{m,k}}(\mathrm{C}\Delta, \mathrm{CP})$ imply $\ell(\alpha) \geq s'_{m,k} - s_{m,k}$. But by (6) we have $s'_{m,k} - s_{m,k} \geq 3 + \xi$. This concludes case 2.

Case 3. $q \in \partial \mathsf{N}_{r_{m,j}}(\mathrm{C}\Delta^j, \mathrm{CP})$, for some Δ^j, $j < k$. In this case, by restricting α if necessary, we can assume that j is minimum in the following sense: $\alpha([0,1])$ does not intersect any $\mathsf{N}_{r_{m,l}}(\mathrm{C}\Delta^l, \mathrm{CP})$, with $l < j$. Since $p \in \mathrm{N}_\Delta$ we get that $p \notin \mathsf{N}_{s_{m,j}}(\Delta^j, \mathrm{P})$. This together with the fact that (B, A) satisfies **DNP** (see item 4 in (8.0.1)) imply that there is $t \in [0,1]$ such that $\alpha([t,1]) \subset \mathsf{N}_{s_{m,j}}(\Delta^j, \mathrm{P})$ and $\alpha(t) \in \partial \mathsf{N}_{s_{m,j}}(\Delta^j, \mathrm{P})$. Therefore the $\wp$-length of $\alpha|_{[t,1]}$ is at least $s_{m,j} - r_{m,j}$. This with (6) imply $\ell(\alpha) \geq s_{m,j} - r_{m,j} \geq 3 + \xi$. □

Claim **2.** — *We have that* $d_{\wp}(\mathrm{N}, \mathrm{CP} - \mathrm{Y}) > 3 + \xi$.

Proof of Claim. — The proof is similar to Case 3 in the proof of the previous claim. Let $p \in \mathrm{N}$ and $q \notin \mathrm{Y}$ and $\alpha : [0,1] \to \mathrm{CP}$ a path joining $\alpha(0) = p$ to $\alpha(1) = q$. Let $\ell(\alpha)$ be the length of α with respect to the $\wp(\mathrm{P}, r)$ metric. To prove the claim we need to show that $\ell(\alpha) \geq 3 + \xi$. By restricting α if necessary, we can assume that $q \in \partial \mathsf{N}_{r_{m,j}}(\Delta^j, \mathrm{P})$, for some Δ^j, $j < k$. Furthermore, we can assume that j is minimum as before. The rest of the proof is exactly the same as Case 3 in Claim 1. □

Before we prove Lemma 8.4.6 recall that $Y'_\Delta \subset \mathsf{CStar}(\Delta, P)$ (see Proposition 6.2.1(i)). Note that $\mathsf{CStar}(\Delta, P) \subset CP$ but we can (and will) also consider $\mathsf{CStar}(\Delta, P) \subset \mathcal{E}_{k+1}(\mathsf{CLink}(\Delta, P))$ (see 6.1.4).

We are now ready to prove Lemma 8.4.6. By Proposition 8.3.1 it is enough to prove the lemma for $\wp(P, r, \mathsf{d}, c')$. Recall that $CP - \bar{\mathbf{B}}_{r_{m-2}-1-\xi} = N \cup \bigcup_{k\leq m-2} N_k$. First we prove that on each N_k, $0 \leq k \leq m-2$, the patched metric $\wp(P, r, \mathsf{d}, c')$ is radially $(\frac{\varepsilon}{3B})$-close to hyperbolic, with charts of excess $\xi'' = \xi - (m-1) + \frac{1}{m}$. Assume $p \in N_k$. Then $p \in N_{\Delta^k}$ for some Δ^k. Since $N_{\Delta^k} \subset Y'_{\Delta^k} \subset \mathsf{CStar}(\Delta^k, P) \subset \mathcal{E}_{k+1}(\mathsf{CLink}(\Delta^k, P))$, from (8.4.7) we get a radially $(\frac{\varepsilon}{3B})$-close to hyperbolic chart $\phi : \mathbf{T}_{\xi''} \to \mathcal{E}_{k+1}(\mathsf{CLink}(\Delta^k, P))$ with center p. But it follows from Lemma 1.3 and inequality (1) that $d_{\mathcal{E}_{k+1}(\mathsf{CLink}(\Delta^k,P))}(p, \phi(q)) < 3 + \xi$, for every $q \in \mathbf{T}_{\xi''}$. This together with Claim 1 imply that $\phi(\mathbf{T}_{\xi''}) \subset Y'_{\Delta^k}$. Here $Y'_{\Delta^k} \subset \mathsf{CStar}(\Delta^k, P)$, and $\mathsf{CStar}(\Delta^k, P)$ is a subset of the space $\mathcal{E}_{k+1}(\mathsf{CLink}(\Delta^k, P))$. But we can also consider $\mathsf{CStar}(\Delta^k, P)$ is as subset of the space CP, hence we can consider the chart ϕ as a chart with image in $Y'_{\Delta^k} \subset CP - \bar{\mathbf{B}}_{r_{m-2}-1-\xi}$. Therefore, by P3 ϕ is a radially $(\frac{\varepsilon}{3B})$-close to hyperbolic chart with center p on $CP - \bar{\mathbf{B}}_{r_{m-2}-1-\xi}$ with the metric $\wp(P, r)$. This proves the case $p \in N_k$, $0 \leq k \leq m-2$. It remains to prove the case $p \in N$. But this case follows from a similar argument as above (in this case fitting a chart in Y) and using Claim 2 and Property P5. □

We now finish the proof of Proposition 8.4.3. Set $\varepsilon' = \frac{\varepsilon}{3}$. By Lemma 8.4.6 the family $\{\wp(P, r, \mathsf{d})\}_r$ is radially $(\frac{\varepsilon}{3B})$-close to hyperbolic, provided $r - d_i \geq r_i(P)$, $d_i \geq d(P)$, $i = 2, \ldots, m$. Note that $r = r(r_{m-2})$ is large if and only if r_{m-2} is large. We can now apply Theorem 4.6 to the family $\{\wp(P, r(r_{m-2}), \mathsf{d})\}_{r_{m-2}-\frac{1}{2}}$. Notice that we have to use Lemma 8.4.2 to satisfy one of the hypothesis of Theorem 4.6. Since $\varepsilon' + B\frac{\varepsilon}{3B} < \varepsilon$ (recall $B = C_2$, see (2)) from Theorem 4.6 we obtain a number $r_{m+1}(P)$ and a (possibly larger) number $d_{m+1}(P)$ such that $\mathcal{H}_{r_{m-2}-\frac{1}{2}, d_{m+1}-\frac{1}{2}}\wp(P, r(r_{m-2}))$ is (B_a, ε)-close to hyperbolic, provided $r - d_i \geq r_i(P)$ and $d_i \geq d_i(P)$, $i = 2, \ldots, m+1$. Here $a = a_m = (r_{m-2} - \frac{1}{2}) - (d_{m+1} - \frac{1}{2}) = r_{m-2} - d_{m+1}$ (see Proposition 4.5(i)). The excess of the charts given by Theorem 4.6 is $(\xi - (m-1) + \frac{1}{m}) - 1 = \xi_m$. This proves Proposition 8.4.3 because by definition (see (8.2.4)) we have $\mathcal{G}(P, r) = \mathcal{H}_{r_{m-2}-\frac{1}{2}, d_{m+1}-\frac{1}{2}}\wp(P, r(r_{m-2}))$. □

8.5. *Smoothing cones over manifolds*

As in the beginning of Section 8, let P^m be an all-right spherical complex and $\mathcal{S}_P = \mathcal{S}(\mathcal{L}_P)$ a compatible normal smooth structure on P. In the previous sections we have canonically constructed a Riemannian metric $\mathcal{G}(P, \mathcal{L}_P, \phi_P, r, \xi, \mathsf{d}, (c, \varsigma))$ on the cone CP. An important assumption was that $(P, \mathcal{S}_P)$ was diffeomorphic (by ϕ_P) to the sphere $\mathbf{S}^m$. We cannot expect to do the same construction on a general manifold P because CP is not in general a manifold. But we will canonically construct a complete Riemannian metric on $CP - o_{CP}$ that has some of the previous properties.

We consider some of the same data as before: P^m, r, ξ, d, (c, ς) satisfying (8.0.1). We replace the map ϕ_{P} in (8.0.1) by a Riemannian metric h_{P} on the closed smooth manifold $(\mathrm{P}, \mathcal{S}_{\mathrm{P}})$. Here the smooth structure is compatible with the all-right spherical structure of P. Hence, by Theorem 7.1, we can assume that $\mathcal{S}_{\mathrm{P}}$ has a normal atlas $\mathcal{A}(\mathcal{L}_{\mathrm{P}})$ induced by some smooth set of link smoothings $\mathcal{L}_{\mathrm{P}}$. We will assume that P has either dimension ≤ 4 or $\mathrm{W}h(\pi_1 \mathrm{P}) = 0$, so that we can apply Theorem 7.3.1. Therefore we begin with the following data: P^m, $\mathcal{L}_{\mathrm{P}}$, h_{P}, r, ξ, d, (c, ς).

Note that the sets $\mathcal{Y}(\mathrm{P}, \Delta, r)$, $\mathcal{Y}(\mathrm{P}, r)$ are defined for general P (no just for $\mathrm{P} = \mathbf{S}^m$) and satisfy all the properties given in Section 6.2. Now, since all the links of P are spheres, and the patch metric does not depend on ϕ_{P} (see Remark 8.2.5(3)), we can define, as in Sections 8.1 and 8.2 the patch metric $\wp(\mathrm{P}, r) = \wp(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, r, \xi, \mathsf{d}, (c, \varsigma))$ on $C\mathrm{P} - \bar{\mathbf{B}}_{r_{m-2}-(4+2\xi)}(C\mathrm{P})$, and this metric satisfies properties P2, P3, P4 given in Section 8.2.

Recall that in Section 8.2 this construction is completed by applying hyperbolic forcing to the $\odot$-family of metrics $\{\wp(\mathrm{P}, r(r_{m-2}))\}_{r_{m-2}-\frac{1}{2}}$ (see (8.2.4)). This method consists of two parts: warp forcing and then the two variable deformation. In our more general setting here we can still apply warp forcing, but we cannot directly apply hyperbolic forcing (at least not in the way given in Section 3) because we do not have $\mathrm{P} = \mathbf{S}^m$. In our case, to finish our construction we apply first warp forcing and then a version of the two variable deformation for general P; this new version will use the metric h_{P} instead of the canonical metric $\sigma_{\mathbf{S}^m}$ on the sphere $\mathbf{S}^m$.

Consider now the $\odot$-family of metrics $\{\wp(\mathrm{P}, r(r_{m-2}))\}_{r_{m-2}-\frac{1}{2}}$ and apply warp forcing to obtain

$$g_{r_{m-2}} = \mathcal{W}_{r_{m-2}-\frac{1}{2}} \wp\big(\mathrm{P}, r(r_{m-2})\big),$$

and we have that $g_{r_{m-2}}$ is a warp product $\mathbf{B}_{r_{m-2}-\frac{1}{2}}(C\mathrm{P}) - o_{C\mathrm{P}}$, specifically we have $g_{r_{m-2}} = \sinh^2 t g + dt^2$, where g is a Riemannian metric on P (it is the normalized spherical cut of $\wp(\mathrm{P}, r(r_{m-2}))$ at $r_{m-2} - \frac{1}{2}$) and t is the distance-to-the-vertex function on CP. Let $\rho_{a,d}$ be the function in Section 3.1. Define the metric $g_t = h + (\rho_{r_{m-2}-d_{m+1}, d_{m+1}-\frac{1}{2}}(t))(g - h)$. Now define the metric $\mathcal{G}(\mathrm{P}, h, r) = \mathcal{G}(\mathrm{P}, \mathcal{L}_{\mathrm{P}}, h, r, \xi, \mathsf{d}, (c, \varsigma))$ by

$$\mathcal{G}(\mathrm{P}, h, r) = \begin{cases} \sinh^2 t g_t + dt^2 & \text{on } \bar{\mathbf{B}}_{r_{m-2}-\frac{1}{2}}(C\mathrm{P}) - \bar{\mathbf{B}}_{r_{m-2}-d_{m+1}}(C\mathrm{P}) \\ \mu^2(t) h + dt^2 & \text{on } \bar{\mathbf{B}}_{r_{m-2}-d_{m+1}}(C\mathrm{P}), \end{cases}$$

where $\mu(t) = \frac{e^t - e^{\lambda(t)}}{2}$, and $\lambda = \rho_{r_{m-2}-2d_{m+1}, d_{m+1}}$. Also we are assuming $r_{m-2} - 2d_{m+1} > 0$. Note that $\mathcal{G}(\mathrm{P}, h, r) = \frac{1}{2} e^t h + dt^2$ on $\mathbf{B}_{r_{m-2}-2d_{m+1}}(C\mathrm{P}) - o_{C\mathrm{P}}$, that is for $0 < t \leq r_{m-2} - 2d_{m+1}$. We write $C\mathrm{P} - o_{C\mathrm{P}} = \mathrm{P} \times (0, \infty)$ and extend the metric $\mathcal{G}(\mathrm{P}, h, r)$ to $\mathrm{P} \times \mathbf{R}$ by $\frac{1}{2} e^t h + dt^2$ for $-\infty < t \leq r_{m-2} - 2d_{m+1}$.

Corollary **8.5.1.** — *The metrics $\mathcal{G}(\mathrm{P}, h, r)$ and $\wp(\mathrm{P}, r)$ have the following properties*

(i) *$\mathcal{G}(\mathrm{P}, r)$ is a Riemannian metric on $\mathrm{P} \times \mathbf{R}$ that has the same set of rays as $\sigma_{C\mathrm{P}}$ (on $\mathrm{P} \times (0, \infty)$).*

(ii) *Properties* P3 *and* P4.
(iii) $\mathcal{G}(\mathrm{P}, h, r) = \frac{1}{2}e^t h + dt^2$ *for* $-\infty < t \le r_{m-2} - 2d_{m+1}$.
(iv) *Given* $\varepsilon > 0$ *we have that the sectional curvatures of* $\mathcal{G}(\mathrm{P}, h, r)$ *are* ε*-pinched to* -1 *for* $t \ge r_{m-2} - 2d_{m+1}$ *provided* $r - d_i$, d_i, $i = 2, \ldots, m+1$, *and* $r - 2d_{m+1}$ *are large enough.*

Proof. — Item (i) follows the same argument used for P1 in the spherical case. Item (ii) is true by construction (see also Lemma 8.2.2). Item (iii) follows from the discussion above, and (iv) from Proposition 8.4.3 and Bishop-O'Neill warp product curvature formula [3], p. 27. □

9. On Charney-Davis strict hyperbolization process

We use some of the notation in [5]. In particular the canonical n-cube $[0, 1]^n$ will be denoted by $\square^n$ and $\dot{\square}^n = (0, 1)^n$. (This differs with the notation used in Section 7, where an n-cube was denoted by σ^n.) Also B_n is the isometry group of $\square^n$.

A *Charney-Davis strict hyperbolization piece of dimension* n is a compact connected orientable hyperbolic n-manifold with corners $\mathrm{X} = \mathrm{X}^n$ satisfying the properties stated in Lemma 5.1 of [5]. The group B_n acts by isometries on X^n and there is a smooth map $f : \mathrm{X}^n \to \square^n$ constructed in Section 5 of [5] with certain properties. We collect some facts from [5].

(1) For any k-face $\square^k$ of $\square^n$ we have that $f^{-1}(\square^k)$ is totally geodesic in X^n. Moreover X^n is a Charney-Davis hyperbolization piece of dimension k. The submanifold (with corners) $f^{-1}(\square^k)$ is a k*-face* of X^n. Note that the intersection of faces is a face and every k-face is the intersection of exactly $n - k$ distinct $(n-1)$-faces.
(2) The map f is B_n-equivariant.
(3) The faces of X^n intersect orthogonally.
(4) The map f is transversal to the k-faces of $\square^n$, $k < n$.

The k-face $f^{-1}(\square^k)$ of X will be denoted by $\mathrm{X}_{\square^k}$. The interior $f^{-1}(\dot{\square}^k)$ will be denoted by $\dot{\mathrm{X}}_{\square^k}$. The normal neighborhood of a k-face $\mathrm{X}_{\square^k}$ in X of width r is the union of all speed 1 geodesics $\gamma : [0, r) \to \mathrm{X}$ emanating from and perpendicular to $\mathrm{X}_{\square^k}$; it is assumed that this set U is open in X and the exponential map $\mathrm{T}^{\perp}_r \mathrm{X}_{\square^k} \to \mathrm{U}$ is a diffeomorphism. Here $\mathrm{T}^{\perp}_r \mathrm{X}_{\square^k}$ is the subbundle of $\mathrm{TX}|_{\mathrm{X}_{\square^k}}$ formed by vectors of length $< r$ perpendicular to $\mathrm{TX}_{\square^k}$. We say that the width of the normal neighborhood of $\mathrm{X}_{\square^k}$ is larger that r if there is a normal neighborhood of $\mathrm{X}_{\square^k}$ of width $r' > r$. The following is proved in [31] (see Lemma 2.1 in [31]).

Proposition **9.1.** — *For every* n *and* $r > 0$ *there is a Charney-Davis hyperbolization piece of dimension* n *such that the widths of the normal neighborhoods of every* k*-face,* $k = 0, \ldots, n-1$*, are larger that* r.

For a k-face $X_{\square^k}$ and $p \in X_{\square^k}$, the set of inward normal vectors to $X_{\square^k}$ at p can be identified with the canonical all-right $(n-k-1)$-simplex $\Delta_{\mathbf{S}^{n-k-1}}$. In this sense we consider $\Delta_{\mathbf{S}^{n-k-1}} \subset T_p X$. Similarly we can consider $\Delta_{\mathbf{S}^{n-k-1}} \subset T_q \square^n$, for $q \in \square^k$. We make the convention that the two identifications above are done with respect to an ordering of the $(n-1)$-faces $X_{\square^{n-1}}$ of X and the corresponding ordering for $\square^n$. For a proof of the following proposition see [31] (see Lemma 2.5 in [31]).

Lemma **9.2.** — *For $p \in \dot{X}_{\square^k}$, we have that Df_p sends non-zero normal vectors to non-zero normal vectors; thus $Df_p|_{\Delta_{\mathbf{S}^{n-k-1}}} : \Delta_{\mathbf{S}^{n-k-1}} \to \Delta_{\mathbf{S}^{n-k-1}}$. Moreover, $\mathbf{n} \circ (Df_p|_{\Delta_{\mathbf{S}^{n-k-1}}}) : \Delta_{\mathbf{S}^{n-k-1}} \to \Delta_{\mathbf{S}^{n-k-1}}$ is the identity, where $\mathbf{n}(x) = \frac{x}{|x|}$ is the normalization map.*

The strict hyperbolization process of Charney and Davis is done by gluing copies of X^n using the same pattern as the one used to obtain the cube complex K from its cubes. This space is called K_X in [5]. Note that we get a map $F : K_X \to K$, which restricted to each copy of X is just the map $f : X^n \to \square^n$. We will write $X_{\square^k} = F^{-1}(\square^n)$, and $\dot{X}_{\square^k} = X_{\dot{\square}^k} = F^{-1}(\dot{\square}^k)$, for a k-cube $\square^k$ of K.

By Lemma 9.2 we can use the derivative of the map $F : K_X \to K$ (in a piecewise fashion) to identify $\mathsf{Link}(X_{\square^k}, K_X)$ with $\mathsf{Link}(\square^k, K)$, where in both cases we consider the "direction" definition of link, that is, the link $\mathsf{Link}(X_{\square^k}, K_X)$ (at $p \in \dot{X}_{\square^k}$) is the set of normal vectors to $X_{\square^k}$ (at p) and the link $\mathsf{Link}(\square^k, K)$ (at $q \in \dot{\square}^k$) is the set of normal vectors to $\square^k$ (at q). Hence we write $\mathsf{Link}(X_{\square^k}, K_X) = \mathsf{Link}(\square^k, K)$; thus the set of links for K coincides with the set of links for K_X.

In what follows we assume that the width of the normal neighborhoods of all $X_\square$ to be larger than s_0, for some s_0. Also let r such that $s_0 > 2r$. By Proposition 9.1 we can take s_0 and r arbitrarily large.

Let $X_{\square^k} \subset X_{\square^n}$ be a k-face of K_X, contained in the copy $X_{\square^n}$ of X over $\square^n$. For a non-zero vector u normal to $X_{\square^k}$ at $p \in X_{\square^k}$, and pointing inside $X_{\square^n}$, we have that $\exp_p(tu)$ is defined and contained in $X_{\square^n}$, for $0 \le t < s_0/|u|$. Let $h_{\square^k} : \mathbf{S}^{n-k-1} \to \mathsf{Link}(\square^k, K) = \mathsf{Link}(X_{\square^k}, K_X)$ be a link smoothing of the link corresponding to $\square^k \in K$. We define the map

$$\begin{array}{cccc} H_{\square^k} : & \mathbf{D}^{n-k} \times \dot{X}_{\square^k} & \longrightarrow & K_X \\ & (tv, p) & \longmapsto & H_{\square^k}(tv, p) = \exp_p\big(2rth_{\square^k}(v)\big), \end{array}$$

where $\mathbf{D}^{n-k}$ is the open $(n-k)$-disc, $v \in \mathbf{S}^{n-k-1}$ and $t \in [0, 1)$. For $k = n$ we have that $H_{\square^n}$ is the inclusion $\dot{X}_{\square^n} \subset K_X$ (or we can take this as a definition). Note that $H_{\square^k}$ is a topological embedding because we are assuming the width of the normal neighborhood of $X_\square$ to be larger than $s_0 > 2r$. We call a chart of the form of $H_{\square^k}$ (for some link smoothing $h_{\square^k}$) a *normal chart for the k-face* $X_{\square^k}$. A collection $\{H_\square\}_{\square \in K}$ of normal charts is a *normal atlas*, and if this atlas is smooth (or C^k) the induced differentiable structure is called a *normal smooth (or C^k) structure*. The following theorem is proved in [31]; it is the Main Theorem in [31].

Theorem **9.3.** — *Let* $\mathcal{L} = \{h_\Box\}_{\Box\in K}$ *be a set of link smoothings for* K*. If* $\mathcal{L}$ *is smooth then the normal atlas* $\{H_\Box\}_{\Box\in K}$ *on* K_X *is smooth.*

We will write $\mathcal{A}_{K_X} = \{H_\Box\}_{\Box\in K}$. Note that the normal atlas $\mathcal{A}_{K_X}$ depends uniquely on the smooth set of link smoothings $\mathcal{L} = \{h_\Box\}_{\Box\in K}$ for K (hence for K_X). To express this dependence we will sometimes write $\mathcal{A}_{K_X} = \mathcal{A}_{K_X}(\mathcal{L})$. We will denote by $\mathcal{S}_{K_X} = \mathcal{S}_{K_X}(\mathcal{L})$ the smooth structure on K_X induced by the smooth atlas $\mathcal{A}_{K_X}$. The following theorem is proved in [31]; it is the Addendum to the Main Theorem in [31].

Theorem **9.4.** — *The smooth manifold* $(K_X, \mathcal{S}_{K_X})$ *smoothly embeds in* $(K, \mathcal{S}') \times X$*, with trivial normal bundle. Here* $\mathcal{S}'$ *is the normal smooth structure on* K *induced by* $\mathcal{L}$*.*

9.5. *Hyperbolized manifolds with codimension zero singularities*

In this section we treat the case of manifolds with a one point singularity. The case of manifolds with many (isolated) point singularities is similar. We assume the setting and notation of Section 7.3. Let K_X be the Charney-Davis strict hyperbolization of K. Denote also by p the singularity of K_X. Many of the definitions and results given before still hold (with minor changes) in the case of manifolds with a one point singularity (see Section 5 in [31] for more details).

(1) Given a set of link smoothings for K (hence for K_X) we also get a set of charts $H_\Box$. For the vertex p we mean the cone map $H_p = Ch_p : CN \to CL \subset K_X$. We will also denote the restriction of H_p to $CN - \{o_{CN}\}$ by the same notation H_p. As in item (2) of Section 7.3 here we are identifying $CN - \{o_{CN}\}$ with $N \times (0, 1]$ with the product smooth structure obtained from some smooth structure $\tilde{\mathcal{S}}_N$ on N. As before $\{H_\Box\}_{\Box\in K}$ is a *normal atlas* for K_X (or $K_X - \{p\}$). A normal atlas for $K - \{p\}$ induces a *normal smooth structure on* $K_X - \{p\}$.

(2) Again we say that the smooth atlas $\{H_\Box\}$ (or the induced smooth structure, or the set $\{h_\sigma\}$) is *correct with respect to* N if $\mathcal{S}_N$ is diffeomorphic to $\tilde{\mathcal{S}}_N$.

(3) Let the set $\mathcal{L} = \{h_\Box\}_{\Box\in K}$ induce a smooth structure on $K - \{p\}$, that is, $\mathcal{L}$ is smooth. As in Theorem 9.3 we get that $\{H_\Box\}_{\Box\in K}$ is a smooth atlas on $K_X - \{p\}$ and it induces a normal smooth structure $\mathcal{S}_{K_X}$ on $K_X - \{p\}$. Moreover, from Theorem 7.3.1 we get that $\mathcal{S}_{K_X}$ is correct with respect to $\mathcal{S}_N$ when $\dim N \le 4$ (always) or when $\dim N > 4$, provided $Wh(N) = 0$. Note that in this case we can take the domain $CN - \{o_{CN}\} = N \times (0, 1]$ of H_p with smooth product structure $\mathcal{S}_N \times \mathcal{S}_{(0,1]}$.

(4) It can be verified that a version of Theorem 9.4 also holds in this case: $(K_X - \{p\}, \mathcal{S}_{K_X})$ smoothly embeds in $(K - \{p\}, \mathcal{S}') \times X$ with trivial normal bundle.

10. Proof of the Main Theorem

In Section 2 the concept of hyperbolic extension over hyperbolic space was introduced. We next extend, in the obvious way, this concept to hyperbolic extensions over hyperbolic manifolds.

As in Section 2, let (N, h) be a complete Riemannian manifold with center $o = o_{\mathrm{N}}$. Let $(\mathrm{Q}, \sigma_{\mathrm{Q}})$ be a hyperbolic manifold. The *hyperbolic extension of* h *over* Q is the Riemannian metric $g = \cosh^2 r\sigma_{\mathrm{Q}} + h$ on $\mathrm{Q}\times\mathrm{N}$, where $r : \mathrm{N} \to [0, \infty)$ is the distance-to-o function on N. We write $g = \mathcal{E}_{\mathrm{Q}}(h)$ and $(\mathrm{Q}\times\mathrm{N}, g) = \mathcal{E}_{\mathrm{Q}}(\mathrm{N}, h)$ (or simply $\mathcal{E}_{\mathrm{Q}}(\mathrm{N})$) and we call $\mathcal{E}_{\mathrm{Q}}(\mathrm{N})$ the *hyperbolic extension of* N *over* Q.

We now begin the proof of the Main Theorem. Let M^n be a closed smooth manifold. Let K be a smooth cubulation of M. We can take K such that K satisfies the intersection condition (see beginning of Section 5). Let K_{X} be the Charney-Davis strict hyperbolization of M, as in Section 9. We can assume that the Charney-Davis hyperbolization piece X is such that the widths of the normal neighborhoods of every face of X is large (see Proposition 9.1), all larger than a large number $2s_0 > 0$. Let $\mathcal{A}_{\mathrm{K}_{\mathrm{X}}} = \{\mathrm{H}_\square\}_{\square\in\mathrm{K}}$ be a smooth normal atlas for K_{X}, and $\mathcal{S}_{\mathrm{K}_{\mathrm{X}}}$ the induced normal smooth structure on K_{X}. Recall that the $\mathrm{H}_\square$ are constructed from a smooth set of link smoothings $\mathcal{L}_{\mathrm{K}} = \{h_\square\}_{\square\in\mathrm{K}}$ for the links of K (or K_{X}).

The domains of the charts $\mathrm{H}_{\square^j}$ are the sets $\mathbf{D}^{n-j}\times\dot{\mathrm{X}}_{\square^j}$. But in this section, for notational purposes, we will consider the rescaling of $\mathrm{H}_{\square^j}$ given by $\mathrm{H}_{\square^j}(tv, p) = \exp_p(th_{\square^j}(v))$, defined on $\mathbf{D}^{n-j}(s_0)\times\dot{\mathrm{X}}_{\square^j}$. We shall denote this chart also by $\mathrm{H}_{\square^j}$.

In what follows, to simplify our notation, we write $\mathsf{Link}(\mathrm{X}_\square) = \mathsf{Link}(\mathrm{X}_\square, \mathrm{K}_{\mathrm{X}})$. Recall that given $\square\in\mathrm{K}$, the set $\mathcal{L}_{\mathrm{K}}$ of link smoothings for the links $\mathsf{Link}(\mathrm{X}_\square)$ of K_{X} (and of K) induce, by restriction (see Section 7.2), the set of link smoothings $\{h_{\square'}\in\mathcal{L}_{\mathrm{K}}, \square'\subsetneq\square\}$ for the links of $\mathsf{Link}(\mathrm{X}_\square)$. We denote this induced set of smoothings by $\mathcal{L}_{\mathsf{Link}(\mathrm{X}_\square)}$ or just $\mathcal{L}_\square$.

The space K_{X} has a natural piecewise hyperbolic metric which we denote by $\sigma_{\mathrm{K}_{\mathrm{X}}}$. The piecewise hyperbolic metric on the cones $\mathsf{CLink}(\mathrm{X}_\square)$ of the all-right spherical simplices $\mathsf{Link}(\mathrm{X}_\square)$ will be denoted by $\sigma_{\mathsf{CLink}(\mathrm{X}_\square)}$. The restriction of $\sigma_{\mathrm{K}_{\mathrm{X}}}$ to the totally geodesic space $\mathrm{X}_\square$ shall be denoted by $\sigma_{\mathrm{X}_\square}$.

For $\square^j\in\mathrm{K}$, the *(closed) normal neighborhood of* $\dot{\mathrm{X}}_{\square^j}$ *in* K_{X} *of width* $s < s_0$ is the set $\mathsf{N}_s(\dot{\mathrm{X}}_{\square^j}, \mathrm{K}_{\mathrm{X}}) = \mathrm{H}_{\square^j}(\bar{\mathbf{D}}^{n-j}(s)\times\dot{\mathrm{X}}_{\square^j})$. That is, it is the union of the images of all geodesic rays of length $\leq s$ in each copy of X containing $\mathrm{X}_{\square^j}$, that begin at (and are normal to) $\dot{\mathrm{X}}_{\square^j}$. Similarly the *open normal neighborhood of* $\dot{\mathrm{X}}_{\square^j}$ *of width* $s < s_0$ is the set $\mathring{\mathsf{N}}_s(\dot{\mathrm{X}}_{\square^j}, \mathrm{K}_{\mathrm{X}}) = \mathrm{H}_{\square^j}(\mathbf{D}^{n-j}(s)\times\dot{\mathrm{X}}_{\square^j})$. Sometimes we will just write $\mathsf{N}_s(\dot{\mathrm{X}}_{\square^j}) = \mathsf{N}_s(\dot{\mathrm{X}}_{\square^j}, \mathrm{K}_{\mathrm{X}})$ and $\mathring{\mathsf{N}}_s(\dot{\mathrm{X}}_{\square^j}) = \mathring{\mathsf{N}}_s(\dot{\mathrm{X}}_{\square^j}, \mathrm{K}_{\mathrm{X}})$. Note that normal neighborhoods respect faces, that is:

$$\mathsf{N}_s(\dot{\mathrm{X}}_{\square^j})\cap\dot{\mathrm{X}}_{\square^k} = \mathsf{N}_s(\dot{\mathrm{X}}_{\square^j\cap\square^k}, \dot{\mathrm{X}}_{\square^k}). \tag{10.1}$$

Here $\mathsf{N}_s(\dot{\mathrm{X}}_{\square^j\cap\square^k}, \dot{\mathrm{X}}_{\square^k})$ is the union of all geodesics of length $\leq s$ in the hyperbolization piece $\dot{\mathrm{X}}_{\square^k}$ that begin in (and are normal to) $\dot{\mathrm{X}}_{\square^j\cap\square^k}$.

Since the normal bundles of the $X_\square$ are canonically trivial (see construction of X in [5], or Section 2 in [31]) we can make the following canonical identification:

$$(\mathbf{10.2}) \qquad \mathsf{N}_s(\dot{X}_{\square^j}) = \dot{X}_{\square^j} \times C_s\mathsf{Link}(X_{\square^j}),$$

where $C_s\mathsf{Link}(X_{\square^j}) = \bar{\mathbf{B}}_s(\mathsf{CLink}(X_{\square^j}))$ is the closed s-cone of length s, that is, it is the ball of radius s on the (infinite) cone $\mathsf{CLink}(X_{\square^j})$ centered at the vertex, see Section 6.1. Similarly we have the identification $\mathring{\mathsf{N}}_s(\dot{X}_{\square^j}) = \dot{X}_{\square^j} \times \mathring{C}_s\,\mathsf{Link}(X_{\square^j})$, where $\mathring{C}_s\,\mathsf{Link}(X_{\square^j})$ is the open s-cone of length s. Moreover these identifications are also metric identifications, where we consider $\mathsf{N}_s(\dot{X}_{\square^j}, K_X) \subset K_X$ with the (restricted) piecewise hyperbolic metric σ_{K_X} and $\dot{X}_{\square^j} \times C_s\mathsf{Link}(X_{\square^j})$ with the hyperbolic extension metric $\mathcal{E}_{\dot{X}_{\square^j}}(\sigma_{\mathsf{CLink}(X_{\square^j})}) = \cosh^2 t\sigma_{\dot{X}_{\square^j}} + \sigma_{\mathsf{CLink}(X_{\square^j})}$, where t is the distance-to-the-vertex function on the cone $\mathsf{CLink}(X_{\square^j})$. Therefore we have the metric version of (10.2): $\mathsf{N}_s(\dot{X}_{\square^j}) = \mathcal{E}_{\dot{X}_{\square^j}}(C_s\mathsf{Link}(X_{\square^j}))$.

Remarks **10.3.**

1. The metric $\sigma_{\mathsf{CLink}(X_\square)}$ is not smooth but the formula above makes sense, giving a well defined piecewise hyperbolic metric.
2. Since we are identifying $\mathsf{N}_s(\dot{X}_{\square^j})$ with $\dot{X}_{\square^j} \times C_s\mathsf{Link}(X_{\square^j})$ we will consider $\mathsf{N}_s(\dot{X}_{\square^j})$ also as a subset of $\dot{X}_{\square^j} \times \mathsf{CLink}(X_{\square^j})$, where $\mathsf{CLink}(X_{\square^j})$ is the (infinite) cone over $\mathsf{Link}(X_{\square^j})$. Note that the metric $\mathcal{E}_{\dot{X}_{\square^j}}(\sigma_{\mathsf{CLink}(X_{\square^j})})$ is defined on the whole of $\dot{X}_{\square^j} \times \mathsf{CLink}(X_{\square^j})$.

Lemma **10.4.** — *Let* $\square^j = \square^i \cap \square^k$, $j \geq 0$. *Let* $s_1, s_2, s < s_0$ *be positive real numbers such that* $\frac{\sinh s_1}{\sinh s}, \frac{\sinh s_2}{\sinh s} \leq \frac{\sqrt{2}}{2}$. *Then* $\mathsf{N}_{s_1}(\dot{X}_{\square^i}) \cap \mathsf{N}_{s_2}(\dot{X}_{\square^k}) \subset \mathsf{N}_s(\dot{X}_{\square^j})$.

Proof. — Using (10.1) we can reduce the lemma to the case were K_X is just a hyperbolization piece X. This case is proved in [31]; it is Lemma 2.3 in [31]. □

Suppose $\square^j \subset \square^k \in K$. Then $\square^k$ determines the all-right spherical simplex $\Delta_{\mathsf{Link}(\square^j,K)}(\square^k) = \square^k \cap \mathsf{Link}(\square^j, K)$ in $\mathsf{Link}(\square^j, K) = \mathsf{Link}(X_{\square^j})$. We will just write $\Delta(\square^k)$ if there is no ambiguity. (Other definition previously used: $\Delta_{\mathsf{Link}(\square^j,K)}(\square^k) = \mathsf{Link}(\square^j, \square^k)$.) Using this new notation, (10.1) and (10.2) we can write

$$(\mathbf{10.5}) \qquad \mathsf{N}_{s_1}(\dot{X}_{\square^j}) \cap X_{\square^k} = \dot{X}_{\square^j} \times C_{s_1}\Delta\big(\square^k\big) \subset \dot{X}_{\square^j} \times C_{s_1}\mathsf{Link}(X_{\square^j}).$$

Lemma **10.6.** — *Let* $\square^j \subset \square^k$ *and* $s_1, s_2 < s_0$. *Then*

$$\mathsf{N}_{s_1}(\dot{X}_{\square^j}) \cap \mathsf{N}_{s_2}(\dot{X}_{\square^k}) = \dot{X}_{\square^j} \times \mathsf{N}_{s_2}\big(C\Delta\big(\square^k\big), C_{s_1}\mathsf{Link}(X_{\square^j})\big).$$

Note that the last term is a subset of $\dot{X}_{\square^j} \times \mathsf{CLink}(X_{\square^j})$.

Proof. — Using (10.2) we see that both sides of the equality above are contained in $\mathsf{N}_{s_1}(\dot{X}_{\square^j}) = \dot{X}_{\square^j} \times C_{s_1}\mathsf{Link}(X_{\square^j})$. Let $p \in \mathsf{N}_{s_1}(\dot{X}_{\square^j})$. Let $\square^l$ such that $p \in \dot{X}_{\square^l}$. Then $\square^j \subset \square^l$. There is a geodesic segment $[x, p]$, $x \in \dot{X}_{\square^j}$, perpendicular to $X_{\square^j}$ at x, and with length $\leq s_1$. Note that $[x, p]$ is totally contained in $X_{\square^l}$. Using (10.2) and (10.5) we have that $[x, p]$ is a geodesic segment in $\{x\} \times C_{s_1}\Delta(\square^l)$. Now, $p \in \mathsf{N}_{s_2}(\dot{X}_{\square^k})$ implies $\square^k \subset \square^l$ and there is a geodesic segment $[q, p]$, $q \in \dot{X}_{\square^k}$, perpendicular to $X_{\square^k}$ at q, and has length $\leq s_2$. Note that $[q, p]$ is totally contained in $X_{\square^l}$. Since $\{x\} \times C_{s_1}\Delta(\square^l)$ is convex in $\dot{X}_{\square^l}$, and $\{x\} \times C_{s_1}\Delta(\square^k)$ is convex in $\{x\} \times C_{s_1}\Delta(\square^l)$, we have that the segments $[q, p]$ and $[x, q]$ are contained in $\{x\} \times C_{s_1}\Delta(\square^l)$. Moreover, since $[q, p]$ is perpendicular to $X_{\square^k}$ at q, we have that $[q, p]$ is a geodesic segment in $\{x\} \times C_{s_1}\Delta(\square^l)$ perpendicular to $\{x\} \times C\Delta(\square^k)$ at q of length $\leq s_2$. This shows $p \in \dot{X}_{\square^j} \times \mathsf{N}_{s_2}(C\Delta(\square^k), C_{s_1}\mathsf{Link}(X_{\square^j}))$ and the inclusion $\mathsf{N}_{s_1}(\dot{X}_{\square^j}) \cap \mathsf{N}_{s_2}(\dot{X}_{\square^k}) \subset \dot{X}_{\square^j} \times \mathsf{N}_{s_2}(C\Delta(\square^k), C_{s_1}\mathsf{Link}(X_{\square^j}))$. The proof of the other inclusion is similar. □

Remark **10.7.** — Clearly the open version of Lemma 10.6 also holds:

$$\mathring{\mathsf{N}}_{s_1}(\dot{X}_{\square^j}) \cap \mathring{\mathsf{N}}_{s_2}(\dot{X}_{\square^k}) = \dot{X}_{\square^j} \times \mathring{\mathsf{N}}_{s_2}\left(C\Delta\left(\square^j\right), \mathring{C}_{s_1}\mathsf{Link}(X_{\square^k})\right)$$

Now, let d, r, ξ, c and ς be as in items 1, 2, 3, 4 at the beginning of Section 8, and let the numbers $s_{m,k} = s_{m,k}(r)$, $r_{m,k} = r_{m,k}(r)$ be as in Section 6.2. Recall $n = \dim M$. For each $\square^k \in K$ define the sets

$$\mathcal{Z}(X_{\square^k}) = \mathring{\mathsf{N}}_{s_{n,k}}(X_{\square^k}) - \bigcup_{i<k} \mathsf{N}_{r_{n,i}}(X_{\square^i}),$$

$$\mathcal{Z} = K_X - \bigcup_{i<n-1} \mathsf{N}_{r_{n,i}}(X_{\square^i}).$$

Note that these sets depend on r. By Proposition 9.1 we can take s_0 as large as needed, hence we can assume that $\mathcal{Z}(X_{\square^k}) \subset \mathring{\mathsf{N}}_{s_0}(X_{\square^k})$.

We next use the sets $\mathcal{X}(P, \Delta, r)$ and $\mathcal{X}(P, r)$ of Section 6.2. The sets $\mathcal{X}(\mathsf{Link}(X_{\square^j}), \Delta(\square^k), r)$ and $\mathcal{X}(\mathsf{Link}(X_{\square^j}), r)$ are a subsets of the (infinite) cone $C\mathsf{Link}(X_{\square^j})$.

Lemma **10.8.** — *The following properties hold*

(i) *If* $\square^i \cap \square^j = \emptyset$ *then* $\mathcal{Z}(X_{\square^i}) \cap \mathcal{Z}(X_{\square^j}) = \emptyset$.
(ii) *If* $\square^j = \square^i \cap \square^k$, $0 \leq j < i, k$, *then* $\mathsf{N}_{s_{n,i}}(X_{\square^i}) \cap \mathsf{N}_{s_{n,k}}(X_{\square^k}) \subset \mathsf{N}_{r_{n,j}}(X_{\square^j})$.
(iii) *If* $\square^j = \square^i \cap \square^k$, $0 \leq j < i, k$, *then* $\mathcal{Z}(X_{\square^i}) \cap \mathcal{Z}(X_{\square^k}) = \emptyset$.
(iv) *If* $\square^j \subsetneq \square^k$ *then (see Remark* 10.3(2)*)*

$$\mathcal{Z}(X_{\square^j}) \cap \mathcal{Z}(X_{\square^k}) \subset \dot{X}_{\square^j} \times \mathcal{X}\left(C\mathsf{Link}(X_{\square^j}), \Delta\left(\square^k\right), r\right).$$

(v) *For* $k < n-1$ *we have* $\mathcal{Z} \cap \mathcal{Z}(X_{\square^k}) \subset \dot{X}_{\square^k} \times \mathcal{X}(C\mathsf{Link}(X_{\square^k}), r)$.
(vi) $K_X = \mathcal{Z} \cup \bigcup_{i<n-1} \mathcal{Z}(X_{\square^i})$.

Proof. — Let $\square^i \cap \square^j = \emptyset$. Then the distance in K_X from $X_{\square^i}$ to $X_{\square^j}$ is at least $2s_0$. This proves (i). Statement (ii) follows from Lemma 10.4, item (4) at the beginning of Section 8, and the following calculation for $l = i, k$ (see Section 6.2 for the definition of $s_{n,l}$ and $r_{n,l}$)

$$\frac{\sinh s_{n,l}}{\sinh r_{n,j}} = \frac{\left(\frac{\sinh r \sin\beta_l}{\sin\alpha_{n-2}}\right)}{\left(\frac{\sinh r}{\sin\alpha_{n-j-3}}\right)} = c\varsigma^{l-j} \le c\varsigma < e^{-6-2\xi} < \frac{\sqrt{2}}{2}.$$

Statement (iii) follows from (ii) and the definition of the sets $\mathcal{Z}$. We next prove (iv). Write $Z = \mathcal{Z}(X_{\square^j}) \cap \mathcal{Z}(X_{\square^k})$. By the definition of the sets $\mathcal{Z}$ we have

$$\begin{aligned}
Z &= \mathring{\mathsf{N}}_{s_{n,j}}(X_{\square^j}) \cap \mathring{\mathsf{N}}_{s_{n,k}}(X_{\square^k}) - \bigcup_{l<k} \mathsf{N}_{r_{n,l}}(X_{\square^l})\\
&\subset \mathring{\mathsf{N}}_{s_{n,j}}(X_{\square^j}) \cap \mathring{\mathsf{N}}_{s_{n,k}}(X_{\square^k}) - \bigcup_{j\le l<k} \mathsf{N}_{r_{n,l}}(X_{\square^l})\\
&\subset \mathring{\mathsf{N}}_{s_{n,j}}(X_{\square^j}) \cap \mathring{\mathsf{N}}_{s_{n,k}}(X_{\square^k}) - \bigcup_{j<l<k} \mathsf{N}_{r_{n,l}}(X_{\square^l}) - \mathsf{N}_{r_{n,j}}(X_{\square^j})\\
&\subset \mathring{\mathsf{N}}_{s_{n,j}}(X_{\square^j}) \cap \mathring{\mathsf{N}}_{s_{n,k}}(X_{\square^k}) - \bigcup_{j<l<k} \Big(\mathsf{N}_{s_0}(X_{\square^j}) \cap \mathsf{N}_{r_{n,l}}(X_{\square^l})\Big) - \mathsf{N}_{r_{n,j}}(X_{\square^j})
\end{aligned}$$

This together with Lemma 10.6 imply $Z \subset \dot{X}_{\square^j} \times A$ where

$$\begin{aligned}
A = {}& \mathring{\mathsf{N}}_{s_{n,k}}\Big(C\Delta(\square^k), \mathring{C}_{s_{n,j}}\big(\mathsf{Link}(X_{\square^j})\big)\Big)\\
&- \bigcup_{j<l<k} \mathsf{N}_{r_{n,l}}\Big(C\Delta(\square^l), C_{s_0}\big(\mathsf{Link}(X_{\square^j})\big)\Big) - \bar{\mathbf{B}}_{r_{n,j}}\big(C\mathsf{Link}(X_{\square^j})\big),
\end{aligned}$$

hence

$$\begin{aligned}
A \subset {}& \mathring{\mathsf{N}}_{s_{n,k}}\big(C\Delta(\square^k), C\mathsf{Link}(X_{\square^j})\big)\\
&- \bigcup_{j<l<k} \mathsf{N}_{r_{n,l}}\big(C\Delta(\square^l), C\mathsf{Link}(X_{\square^j})\big) - \bar{\mathbf{B}}_{r_{n,j}}\big(C\mathsf{Link}(X_{\square^j})\big).
\end{aligned}$$

But for $i > j$ we have $s_{n,i} = s_{n-j,i-j}$, $r_{n,i} = r_{n-j-1,i-j-1}$ and $r_{n,j} = r_{n-j-3}$ (see definitions in Section 6.2). Therefore $A \subset \mathcal{X}(C\mathsf{Link}(X_{\square^j}), \Delta(\square^k), r)$. This proves (iv). The proof of (v) is similar to the proof of (iv) with minor changes. The proof of (vi) is similar to the proof of (iv) in Proposition 6.2.1. □

We now smooth the metric σ_{K_X}. For each $\square \in K$ using the construction in Section 8 we get a Riemannian metric $\mathcal{G}(\mathsf{Link}(X_\square), \mathcal{L}_\square, h_\square, r, \xi, \mathsf{d}, (c, \varsigma))$ on $C\mathsf{Link}(X_\square)$, which we shall simply denote by $\mathcal{G}(\mathsf{Link}(\mathrm{X}_\square))$. Define the Riemannian metric $\mathcal{G}(\mathrm{X}_\square)$ on $\overset{\circ}{\mathsf{N}}_{s_0}(\dot{\mathrm{X}}_\square)$ by

$$\mathcal{G}(\mathrm{X}_\square) = \mathcal{E}_{\dot{\mathrm{X}}_\square}\big(\mathcal{G}\big(\mathsf{Link}(\mathrm{X}_\square)\big)\big)$$

Remark **10.9.** — Recall that we can also consider $\overset{\circ}{\mathsf{N}}_{s_0}(\dot{\mathrm{X}}_\square)$ contained in $\dot{\mathrm{X}}_\square \times \mathsf{CLink}(\mathrm{X}_\square)$, where $\mathsf{CLink}(\mathrm{X}_\square)$ the infinite cone (see Remark 10.3(2)). In this case note that the definition of $\mathcal{G}(\mathrm{X}_\square)$ makes sense in the whole of $\dot{\mathrm{X}}_\square \times \mathsf{CLink}(\mathrm{X}_\square)$.

Proposition **10.10.** — *The Riemannian metrics $\mathcal{G}(\mathrm{X}_{\square^j})$ and $\mathcal{G}(\mathrm{X}_{\square^k})$ coincide on the intersection $\mathcal{Z}(\mathrm{X}_{\square^j}) \cap \mathcal{Z}(\mathrm{X}_{\square^k})$, $i, j < n-1$. Also the Riemannian metric $\mathcal{G}(\mathrm{X}_{\square^k})$ coincides with $\sigma_{\mathrm{K}_\mathrm{X}}$ on $\mathcal{Z} \cap \mathcal{Z}(\mathrm{X}_{\square^k})$.*

Proof. — For the first statement items (i) and (iii) of Lemma 10.8 imply that we only need to consider the case $\square^j \subset \square^k$, $j < k < n-1$. By item (iv) of Lemma 10.8 it is enough to prove that $\mathcal{G}(\mathrm{X}_{\square^j})$ and $\mathcal{G}(\mathrm{X}_{\square^k})$ coincide on $\dot{\mathrm{X}}_{\square^j} \times \mathcal{X}(\mathsf{CLink}(\mathrm{X}_{\square^j}), \Delta(\square^k), r)$, where we are considering this last set as a subset of $\dot{\mathrm{X}}_{\square^j} \times \mathsf{CLink}(\mathrm{X}_{\square^j})$ (see Remark 10.9). Property P6 in Section 8.2 implies that the metric $\mathcal{G}(\mathrm{X}_{\square^j})$ coincides with the metric

$$\mathcal{E}_{\dot{\mathrm{X}}_{\square^j}}\Big[\mathcal{E}_{\mathrm{C}\Delta(\square^k)}\Big(\mathcal{G}\Big[\mathsf{Link}\big(\Delta\big(\square^k\big), \mathsf{Link}(\mathrm{X}_{\square^j})\big)\Big]\Big)\Big]$$

on $\dot{\mathrm{X}}_{\square^j} \times \mathcal{X}(\mathsf{CLink}(\mathrm{X}_{\square^j}), \Delta(\square^k), r)$. But

$$\begin{aligned}\mathsf{Link}\big(\Delta\big(\square^k\big), \mathsf{Link}(\mathrm{X}_{\square^j})\big) &= \mathsf{Link}\big(\Delta\big(\square^k\big), \mathsf{Link}\big(\square^j, \mathrm{K}\big)\big)\\ &= \mathsf{Link}\big(\square^k, \mathrm{K}\big) = \mathsf{Link}(\mathrm{X}_{\square^k})\end{aligned}$$

Hence we have to prove that $\mathcal{E}_{\dot{\mathrm{X}}_{\square^j}}(\mathcal{E}_{\mathrm{C}\Delta(\square^k)}(g)) = \mathcal{E}_{\dot{\mathrm{X}}_{\square^k}}(g)$, where $g = \mathcal{G}(\mathsf{Link}(\mathrm{X}_{\square^k}))$. This follows from applying Proposition 2.5 locally. To prove the second statement in Proposition 10.10, using a similar argument as above (with P7 instead of P6) we reduce the problem to showing that on $\dot{\mathrm{X}}_{\square^k} \times \mathsf{CLink}(\mathrm{X}_{\square^k})$ we have $\mathcal{E}_{\dot{\mathrm{X}}_{\square^k}}(\sigma_{\mathsf{CLink}(\mathrm{X}_{\square^k})}) = \sigma_{\mathrm{K}_\mathrm{X}}$. And this follows from applying Corollary 6.1.8 locally. □

Finally define the metric $\mathcal{G}(\mathrm{K}_\mathrm{X}) = \mathcal{G}(\mathrm{K}_\mathrm{X}, \mathcal{L}, r, \xi, \mathsf{d}, (c, \varsigma))$ to be equal to $\mathcal{G}(\mathrm{X}_{\square^k})$ on $\mathcal{Z}(\mathrm{X}_{\square^k})$, for $\square^k \in \mathrm{K}$, $k < n-1$. And equal to $\sigma_{\mathrm{K}_\mathrm{X}}$ on $\mathcal{Z}$. By Lemma 10.8(vi) and Proposition 10.10 the metric $\mathcal{G}(\mathrm{K}_\mathrm{X})$ is a well defined Riemannian metric on the smooth manifold $(\mathrm{K}_\mathrm{X}, \mathcal{S}_{\mathrm{K}_\mathrm{X}})$.

Corollary **10.11.** — *Let $\varepsilon > 0$ and M^n closed. Choose ξ, c, ς satisfying* (i) *and* (ii) *in Proposition 8.4.3, and $\xi \geq n$. Then the metric $\mathcal{G}(\mathrm{K}_\mathrm{X})$ has all sectional curvatures ε-pinched to -1, provided d_i, $r - d_i$, $i = 2, \ldots, n$, are sufficiently large.*

Proof. — Choose ε', as in Remarks 1.2(2) and 1.4(3), so that a $(\mathrm{B}_a, \varepsilon')$-close to hyperbolic metric with charts of excess 1 has sectional curvatures ε-pinched to -1. Take A so that $\mathrm{A} \geq \mathrm{C}(n, k, \xi)$ (see Theorem 2.7), for all $k \leq n-1$. Since M is compact we only have finitely many cubes in a cubulation K of M. Hence the set of links of K (hence of K_X) is finite. This together with Proposition 8.4.3 imply that all $\mathcal{G}(\mathsf{Link}(\square^k, \mathrm{K}), r, \mathsf{d})$, are $(\mathrm{B}_{a_j}, \frac{\varepsilon'}{\mathrm{A}})$-close to hyperbolic (here $j = n-k-1$ and $a_j = r_{j-2} - d_{k+1}$), with charts of excess ξ_j. All this provided d_i, $r - d_i$, $i = 2, \ldots, n$, are sufficiently large. We can apply Theorem 2.7 (locally, see remark below) to get that the metrics $\mathcal{G}(\mathrm{X}_{\square^k})$ are $(\mathrm{B}_{a_j}, \varepsilon')$-close to hyperbolic on $\mathcal{Z}(\mathrm{X}_{\square^k})$, with charts of excess $\xi_j - 1$, provided d_i, $r - d_i$, $i = 2, \ldots, n$, are sufficiently large. Here $\xi_j - 1 \geq 1$ and Remark 1.4(4) imply that we can take the excess to be 1. Therefore all the $\mathcal{G}(\mathrm{X}_{\square^k})$ have curvatures ε-close to -1. □

The corollary proves (i) of the Main Theorem. Items (ii), (iii) follow from [5]. Item (iv) follows from Theorem 9.4. This proves the Main Theorem. □

Remark **10.12.** — Note that it does not make sense to say that $\mathcal{G}(\mathrm{X}_{\square^k})$ is ε'-close to hyperbolic because neither $\dot{\mathrm{X}}_{\square^k}$ nor $\dot{\mathrm{X}}_{\square^k} \times \mathsf{CLink}(\mathrm{X}_{\square^k})$ have a center. What we mean by the "local application of Theorem 2.7" mentioned in the proof above is the following. Take $p \in \mathcal{Z}(\mathrm{X}_{\square^k})$ and let $\mathrm{B} \subset \dot{\mathrm{X}}_{\square^k}$ be an open ball centered at p. Note that we can also consider $\mathrm{B} \times \mathsf{CLink}(\mathrm{X}_{\square^k}) \subset \mathbf{H}^{n-k} \times \mathsf{CLink}(\mathrm{X}_{\square^k}) = \mathcal{E}_k(\mathsf{CLink}(\mathrm{X}_{\square^k}))$ and we can now apply Theorem 2.7 to $\mathcal{E}_k(\mathsf{CLink}(\mathrm{X}_{\square^k}))$, where we are considering p as the center.

11. Proof of Theorem A

Let N be a closed smooth manifold that bounds a compact smooth manifold M^m. Denote the given smooth structure of N by $\mathcal{S}_\mathrm{N}$. Let Q be the smooth m-manifold with one point singularity formed by gluing the cone $\mathrm{C}_1\mathrm{N}$ to M along $\mathrm{N} \subset \mathrm{M}$. Let q be the singularity of Q and note that it is modeled on CN (see Section 7.3). A triangulation of Q is obtained by coning a smooth triangulation of the manifold with boundary M, and let $f : \mathrm{K} \to \mathrm{Q}$ be the induced cubulation. Write $f^{-1}(q) = p$. Note that (K, f) is a smooth cubulation of Q in the sense of Section 7.3. By item (2) of Section 7.3 we have that $\mathrm{Q} - \{q\}$ has a normal smooth structure $\mathcal{S}'$ for K, induced by a set of links smoothings $\mathcal{L}$.

Let K_X be the Charney-Davis strict hyperbolization of K. Also denote by p the singularity of K_X. By item (1) of Section 9.5, the space $\mathrm{K}_\mathrm{X} - \{p\}$ has a normal smooth atlas $\{\mathrm{H}_\square\}_{\square \in \mathrm{K}}$ and normal smooth structure $\mathcal{S}_{\mathrm{K}_\mathrm{X}}$. Moreover, since we are assuming $\mathrm{W}h(\mathrm{N}) = 0$ (if $\dim \mathrm{N} > 4$) we have that we can take the domain $\mathrm{CN} - \{o_{\mathrm{CN}}\} = \mathrm{N} \times (0, 1]$ of H_p with product smooth structure $\mathcal{S}_\mathrm{N} \times \mathcal{S}_{(0,1]}$ (see Theorem 9.4).

We can now proceed exactly as in Section 10 and define the sets $\mathcal{Z}(\mathrm{X}_\square)$, $\mathcal{Z}$, and the metrics $\mathcal{G}(\mathrm{X}_\square)$ depending on $\mathcal{L}, r, \xi, \mathsf{d}, (c, \varsigma)$. For the special case $\square^0 = p$ we use the results in Section 8.5. We obtain in this way a Riemannian metric $\mathcal{G}(\mathrm{K}_\mathrm{X}) = \mathcal{G}(\mathrm{K}_\mathrm{X}, \mathcal{L}, r, \xi, \mathsf{d}, (c, \varsigma))$ on $\mathrm{K}_\mathrm{X} - \{p\}$. Theorem A and its addendum now follow from

Corollary 8.5.1(iii), (iv) and the result of Belegradek and Kapovitch [2] mentioned in the introduction (before the addendum to Theorem A). To be able to apply Corollary 8.5.1 we need to satisfy the hypothesis made at the beginning of Section 8.5: that the Whitehead group $Wh(\pi_1 N)$ vanishes. But this follows from [11]. □

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Publisher's Note

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