

CATEGORICAL ACTIONS ON UNIPOTENT REPRESENTATIONS OF FINITE UNITARY GROUPS

by O. DUDAS, M. VARAGNOLO, and E. VASSEROT

To Gerhard Hiss for his sixtieth birthday

ABSTRACT

Using Harish-Chandra induction and restriction, we construct a categorical action of a Kac-Moody algebra on the category of unipotent representations of finite unitary groups in non-defining characteristic. We show that the decategorified representation is naturally isomorphic to a direct sum of level 2 Fock spaces. From our construction we deduce that the Harish-Chandra branching graph coincides with the crystal graph of these Fock spaces, solving a recent conjecture of Gerber-Hiss-Jacon. We also obtain derived equivalences between blocks, yielding Broué's abelian defect group conjecture for unipotent ℓ -blocks at linear primes ℓ .

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CONTENTS

Introduction

Let \mathbf{G} be a connected algebraic group defined over a finite field \mathbf{F}_q . The finite group $G = \mathbf{G}(\mathbf{F}_q)$ of its \mathbf{F}_q -rational points is a *finite reductive group*. The irreducible representations of G over fields of characteristic $\ell \nmid q$ fall into Harish-Chandra series, which are defined in terms of Harish-Chandra induction R_L^G and restriction ${}^*R_L^G$ from proper Levi subgroups $L \subset G$. Therefore the classification of the irreducible representations (up to isomorphism) can be reduced to the following two problems.

- (a) Classification of the cuspidal irreducible representations.
- (b) Determination of the endomorphism algebra of representations obtained by Harish-Chandra induction of cuspidal representations.

This was achieved by Lusztig in [37] when $\ell = 0$ but it remains open for representations in positive characteristic for most of the finite reductive groups. By results of Geck-Hiss-Malle [21], we know however that the algebras $\mathrm{End}_G(R_L^G(V))$ occurring in (b) have a structure of (extended) Hecke algebras of finite type, only the parameters of the deformation are unknown in general.

When G is a classical group, e.g., $GL_n(q)$, $GU_n(q)$, $Sp_{2n}(q)$... it turns out that most of the structure of $\text{End}_G(R_L^G(V))$ does not depend on the representation V . This suggests to study the endomorphism algebra $\text{End}(R_L^G)$ of the Harish-Chandra induction functor R_L^G rather than the endomorphism algebra of the induced representation. This was already achieved in [7] for $G = GL_n(q)$. In this series of papers we aim at extending Chuang-Rouquier's framework to other classical groups.

In this paper we will focus on the case of finite unitary groups $GU_n(q)$. We will work with both ordinary representations (characteristic zero) and modular representations in non-defining characteristic (characteristic $\ell \nmid q$). More precisely, the field of coefficients R of the representations will be an extension of either \mathbf{Q}_ℓ or \mathbf{F}_ℓ . Using the tower of inclusion of groups $\cdots \subset GU_n(q) \subset GU_{n+2}(q) \subset \cdots$ one can form the abelian category

$$\mathcal{U}_R = \bigoplus_{n \geq 0} GU_n(q)\text{-umod}$$

of unipotent representations of the various finite unitary groups. Furthermore, under mild assumptions on ℓ , we can modify the Harish-Chandra induction and restriction functors to obtain an adjoint pair (E, F) of functors on \mathcal{U}_R . The functor F corresponds to a Harish-Chandra induction from $GU_n(q)$ to $GU_{n+2}(q)$ whereas E corresponds to the restriction. Note that only specific Levi subgroups are considered, and we must work with a variation of the usual Harish-Chandra theory (the *weak Harish-Chandra theory*) introduced in [24].

In this framework, problem (a) amounts to finding the modules V such that $EV = 0$ and problem (b) is about the structure of $\text{End}_G(F^m V)$ for such cuspidal modules V . As mentioned before, most of the structure of this endomorphism algebra is already contained in $\text{End}(F^m)$. In Section 4.2, we construct natural transformations X of F and T of F^2 where X should be thought of as a Jucys-Murphy element and T satisfies a quadratic relation with eigenvalues q^2 and -1 . This endows $\text{End}(F^m)$ with a morphism from an affine Hecke algebra $\mathbf{H}_m^{q^2}$ of type $A_{m-1}^{(1)}$ with parameter q^2 . Back to our original problem, the evaluation at a cuspidal module V provides a natural map $\mathbf{H}_m^{q^2} \longrightarrow \text{End}_G(F^m V)$. Then, we prove that this map induces a natural isomorphism between $\text{End}_G(F^m V)$ and the quotient of $\mathbf{H}_m^{q^2}$ by the ideal generated by the relation of X on FV if V is a unipotent representation in characteristic zero (see Theorem 4.12). In that case, V is the unique cuspidal representation of $GU_n(q)$ with $n = t(t+1)/2$ for some $t \geq 0$ and the eigenvalues of X on FV are $\mathbf{Q}_t = \{(-q)^t, (-q)^{-1-t}\}$. The result was already proved in [36], for V any unipotent cuspidal representation of a classical group, and in [29] for V any cuspidal representation of a finite reductive group, but with some ambiguity on the signs of the eigenvalues of X . We remove that ambiguity by using the structure of unipotent blocks with cyclic defects.

Having proved that the eigenvalues of X are powers of $-q$, we can form a Lie algebra \mathfrak{g} corresponding to the quiver with vertices $(-q)^{\mathbf{Z}}$ and arrows given by multiplication by q^2 . When working in characteristic zero, \mathfrak{g} is isomorphic to two copies of $\mathfrak{sl}_{\mathbf{Z}}$, whereas

in positive characteristic ℓ it will depend on the parity of e , the order of $(-q)$ modulo ℓ . When e is even (*linear prime case*), it is a subalgebra of $(\widehat{\mathfrak{sl}}_{e/2})^{\oplus 2}$ whereas it is isomorphic to $\widehat{\mathfrak{sl}}_e$ when e is odd (*unitary prime case*). Our main result is that E and F induce a categorical action of \mathfrak{g} on \mathcal{U}_R (see Theorems 4.14 and 4.21, and Section 1.3.2 for the definition of categorical actions).

Theorem A. — The representation datum (E, F, X, T) given by Harish-Chandra induction and restriction endows \mathcal{U}_R with a structure of categorical \mathfrak{g} -module.

Let $E = \bigoplus E_i$ and $F = \bigoplus F_i$ be the decomposition of the functors into generalized i -eigenspaces for X . Then $\{[E_i], [F_i]\}_{i \in (-q)\mathbf{z}}$ act as the Chevalley generators of \mathfrak{g} on the Grothendieck group $[\mathcal{U}_R]$ of \mathcal{U}_R and many problems on \mathcal{U}_R have a Lie-theoretic counterpart. For example,

- weakly cuspidal modules correspond to highest weight vectors;
- the decomposition of \mathcal{U}_R into Harish-Chandra series corresponds to the decomposition of the \mathfrak{g} -module $[\mathcal{U}_R]$ into a direct sum of irreducible highest weight modules,
- the parameters of the ramified Hecke algebras $\text{End}_G(F^m V)$ are given by the weight of $[V]$,
- the blocks of \mathcal{U}_R , or equivalently the unipotent ℓ -blocks, correspond to the weight spaces for the action of \mathfrak{g} (inside a Harish-Chandra series if e is even).

Such observations were already used in other situations (for symmetric groups, cyclotomic rational double affine Hecke algebras or cyclotomic q -Schur algebras, etc).

For this dictionary to be efficient one needs to determine the \mathfrak{g} -module structure on $[\mathcal{U}_R]$. This is done in Section 4.5 by looking at the action of $[E_i]$ and $[F_i]$ on the basis of $[\mathcal{U}_R]$ formed by unipotent characters (if $\text{char}(\mathbf{R}) = 0$) or their ℓ -reduction (if $\text{char}(\mathbf{R}) = \ell$). On this basis the action can be made explicit, and we prove that there is a natural \mathfrak{g} -module isomorphism

$$[\mathcal{U}_R] \xrightarrow{\sim} \bigoplus_{t \geq 0} \mathbf{F}(Q_t)$$

between the Grothendieck group of \mathcal{U}_R and a direct sum of level 2 Fock spaces $\mathbf{F}(Q_t)$, each of which corresponds to an ordinary Harish-Chandra series (see Corollary 4.17). Through this isomorphism, the basis of unipotent characters (or their ℓ -reduction) is sent to the standard monomial basis.

Our original motivation for constructing a categorical action of \mathfrak{g} on \mathcal{U}_R comes from a conjecture of Gerber-Hiss-Jacon [24], which predicts an explicit relation between the Harish-Chandra branching graph and the crystal graph of the Fock spaces $\bigoplus_{t \geq 0} \mathbf{F}(Q_t)$. See also [25]. Using our categorical methods and the unitriangularity of the decomposition matrix we obtain a complete proof of the conjecture (see Theorem 4.37).

Theorem B. — Assume $e > 1$ is odd. Then the Harish-Chandra branching graph coincides with the union of the crystal graphs of the Fock spaces $\mathbf{F}(Q_t)$.

Note that the construction of these crystal graphs depends on the choice of a charge, which is made explicit in Section 4.7 and which indeed differs slightly from the charge used in [24]. Note also that the proof is based on the following basic ingredients:

- to any categorical representation is associated a perfect basis as in [43, Proposition 6.2],
- a unicity statement for crystals of perfect bases which seems to be new (Proposition 1.14),
- a particular choice of partial order on the basis elements of the Fock space which comes from the representation theory of rational double affine Hecke algebras and uses the main theorem in [42].

A similar result can be deduced when e is even. However, in that case, the situation is already well-understood by work of Gruber-Hiss [27] on classical groups. The case where e is odd (unitary primes) is considered as more challenging and Theorem B is the first major result in that direction since the case of $\mathrm{GL}_n(q)$ was solved by Dipper-Du [9]. This solves completely the problem of classification of irreducible unipotent modules for unitary groups mentioned at the beginning of the introduction. More precisely, there are two notions of Harish-Chandra series for unipotent modules in non defining positive characteristic. Our work describes the weak Harish-Chandra series. Another categorical construction can be used in order to get the usual (non weak) Harish-Chandra series, by adapting some technics from [44]. We mention it very briefly in a conjectural form in Section 5 in order that the paper remains of a reasonable length. We will come back to this elsewhere.

By the work of Chuang-Rouquier, categorical actions also provide derived equivalences between weight spaces. In our situation, these weight spaces are exactly the unipotent ℓ -blocks and we obtain many derived equivalences between blocks with the same local structure. Together with Livesey's construction of good blocks in the linear prime case, we deduce a proof of Broué's abelian defect group conjecture (see Theorem 4.34).

Theorem C. — Assume $e > 2$ is even. Then any unipotent ℓ -block of a unitary group with abelian defect group is derived equivalent to its Brauer correspondent.

Many results and construction can be applied to other classical groups. This will be the purpose of a subsequent paper.

The paper is organized as follows. In Section 1 we set our notations and recall the definition of categorical actions. We also record several applications for later use, including the existence of perfect bases and the construction of derived equivalences. In Section 2 we introduce the Fock spaces, which are certain level l representations of Kac-Moody algebras. They have a basis given by charged l -partitions, and a crystal graph which can be defined combinatorially. In Section 3 we recall standard results on unipotent representations of finite reductive groups in non-defining characteristic. Section 4 is the core of our paper. We define a representation datum on the category of representations, which is then shown to induce a categorical action on unipotent modules. We

give two main applications of our construction, solving the recent conjecture of Gerber-Hiss-Jacon, and Broué's abelian defect group conjecture for unipotent ℓ -blocks of unitary groups at linear primes ℓ . In the last section, we sketch a strategy towards the determination of the usual (non weak) Harish-Chandra series using the action of a Heisenberg algebra.

1. Categorical representations

Throughout this section, R will denote a noetherian commutative domain (with unit).

1.1. Rings and categories. — An R -category \mathcal{C} is an additive category enriched over the tensor category of R -modules. All the functors F on \mathcal{C} will be assumed to be R -linear. Given such a functor, we denote by 1_F or sometimes F the identity element in the endomorphism ring $\text{End}(F)$. The identity functor on \mathcal{C} will be denoted by $1_{\mathcal{C}}$. A composition of functors E and F is written as EF , while a composition of morphisms of functors (or natural transformations) ψ and ϕ is written as $\psi \circ \phi$. We say that \mathcal{C} is *Hom-finite* if the Hom spaces are finitely generated over R . Since \mathcal{C} is additive, it is an exact category with split exact sequences. If the category \mathcal{C} is abelian or exact, we denote by $[\mathcal{C}]$ the complexified Grothendieck group and by $\text{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects of \mathcal{C} . The class of an object M of \mathcal{C} in the Grothendieck group is denoted by $[M]$. An exact endofunctor F of \mathcal{C} induces a linear map on $[\mathcal{C}]$ which we will denote by $[F]$.

Assume that \mathcal{C} is Hom-finite. Given an object $M \in \mathcal{C}$ we set $\mathcal{H}(M) = \text{End}_{\mathcal{C}}(M)^{\text{op}}$. It is an R -algebra which is finitely generated as an R -module. Consider the adjoint pair $(\mathfrak{E}_M, \mathfrak{F}_M)$ of functors given by

$$\begin{aligned}\mathfrak{F}_M &= \text{Hom}_{\mathcal{C}}(M, -) : \mathcal{C} \longrightarrow \mathcal{H}(M)\text{-mod}, \\ \mathfrak{E}_M &= M \otimes_{\mathcal{H}(M)} - : \mathcal{H}(M)\text{-mod} \longrightarrow \mathcal{C}.\end{aligned}$$

Let $\text{add}(M) \subset \mathcal{C}$ be the smallest R -subcategory containing M which is closed under direct summands. Then the functors $\mathfrak{E}_M, \mathfrak{F}_M$ satisfy the following properties:

- $\mathfrak{E}_M, \mathfrak{F}_M$ are equivalences of R -categories between $\text{add}(M)$ and $\mathcal{H}(M)\text{-proj}$,
- $\mathfrak{F}_M \mathfrak{E}_M = 1_{\mathcal{H}(M)\text{-mod}}$,
- if R is a field, M is projective and \mathcal{C} is abelian and has enough projectives, then \mathfrak{F}_M yields a bijection

$$\{L \in \text{Irr}(\mathcal{C}) \mid M \twoheadrightarrow L\} \xleftarrow{1:1} \text{Irr}(\mathcal{H}(M)).$$

Assume now that $\mathcal{C} = H\text{-mod}$, where H is an R -algebra with 1 which is finitely generated and free as an R -module. We abbreviate $\text{Irr}(H) = \text{Irr}(\mathcal{C})$. Given an homomorphism $R \rightarrow S$, we can form the S -category $S\mathcal{C} = SH\text{-mod}$ where $SH = S \otimes_R H$.

Given another \mathbf{R} -category \mathcal{C}' as above and an exact (\mathbf{R} -linear) functor $F : \mathcal{C} \rightarrow \mathcal{C}'$, then F is represented by a projective object $P \in \mathcal{C}$. We set $SF = \text{Hom}_{S\mathcal{C}}(SP, -) : S\mathcal{C} \rightarrow S\mathcal{C}'$. Let \mathbf{K} be the field of fractions of \mathbf{R} , $A \subset \mathbf{R}$ be a subring which is integrally closed in \mathbf{K} and $\theta : \mathbf{R} \rightarrow \mathbf{k}$ be a ring homomorphism into a field \mathbf{k} such that \mathbf{k} is the field of fractions of $\theta(A)$. If \mathbf{kH} is split, then there is a *decomposition map* $d_\theta : [\mathbf{KH}\text{-mod}] \rightarrow [\mathbf{kH}\text{-mod}]$, see e.g. [22, Section 3.1] for more details.

1.2. Kac-Moody algebras of type A and their representations. — The Lie algebras which will act on the categories we are interested in will always be finite sums of Kac-Moody algebras of type A_∞ or $A_{e-1}^{(1)}$. They will arise from quivers of the same type.

1.2.1. Lie algebra associated with a quiver. — Let $v \in \mathbf{R}^\times$ and $I \subset \mathbf{R}^\times$. We assume that $v \neq 1$ and that I is stable by multiplication by v and v^{-1} with finitely many orbits. To the pair (I, v) we associate a quiver $I(v)$ (also denoted by I) as follows:

- the vertices of $I(v)$ are the elements of I ;
- the arrows of $I(v)$ are $i \rightarrow iv$ for $i \in I$.

Since I is assumed to be stable by multiplication by v and v^{-1} , such a quiver is the disjoint union of quivers of type A_∞ if v is not a root of unity, or of cyclic quivers of type $A_{e-1}^{(1)}$ if v is a primitive e -th root of 1.

The quiver $I(v)$ defines a symmetric generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ with $a_{ii} = 2$, $a_{ij} = -1$ when $i \rightarrow j$ or $j \rightarrow i$ and $a_{ij} = 0$ otherwise. To this Cartan matrix one can associate the (derived) Kac-Moody algebra \mathfrak{g}'_I over \mathbf{C} , which has Chevalley generators e_i, f_i for $i \in I$, subject to the usual relations.

More generally, let $(X_I, X_I^\vee, \langle \bullet, \bullet \rangle_I, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$ be a *Cartan datum* associated with A . For each $i \in I$ let $\Lambda_i \in X_I$ be the fundamental weight satisfying $\langle \alpha_j^\vee, \Lambda_i \rangle_I = \delta_{ij}$ for all $j \in I$. Let $Q_I^\vee = \bigoplus \mathbf{Z}\alpha_i^\vee$ be the coroot lattice and $P_I = \bigoplus \mathbf{Z}\Lambda_i$ be the weight lattice.

Let \mathfrak{g}_I be the Kac-Moody algebra corresponding to the datum above, which is a Lie algebra defined over \mathbf{C} generated by the Chevalley generators e_i, f_i for $i \in I$ and the Cartan algebra $\mathfrak{h} = \mathbf{C} \otimes X_I^\vee$. An element $h \in \mathfrak{h}$ acts by $[h, e_i] = \langle h, \alpha_i \rangle e_i$. Write $h_i = 1 \otimes \alpha_i^\vee$ in \mathfrak{h} . The Lie algebra \mathfrak{g}'_I is the derived subalgebra $[\mathfrak{g}_I, \mathfrak{g}_I]$.

When I is infinite, it will be sometimes useful to consider a completion of \mathfrak{g}_I denoted by $\overline{\mathfrak{g}}_I$, which has $\prod \mathbf{C}h_i \simeq \mathbf{C}^I$ as a Cartan subalgebra. This allows to consider some infinite sums of the generators, such as $c = \sum h_i$ which is a central element in $\overline{\mathfrak{g}}_I$. This will not affect the representation theory of \mathfrak{g}_I as we will be working with integrable representations only (see the following section).

Let S be another commutative domain with unit, and $\theta : \mathbf{R} \rightarrow S$ be a ring homomorphism. Then there is a Lie algebra homomorphism $\overline{\mathfrak{g}}'_{\theta(I)} \rightarrow \overline{\mathfrak{g}}'_I$ defined on the Chevalley generators by

$$e_i \mapsto \sum_{\theta(j)=i} e_j \quad \text{and} \quad f_i \mapsto \sum_{\theta(j)=i} f_j.$$

To avoid cumbersome notation, we may write $\mathfrak{g} = \mathfrak{g}_I$, $P = P_I$, $Q^\vee = Q_I^\vee$, etc. when there is no risk of confusion.

1.2.2. Examples. — Let first observe that if $I = v^{\mathbf{Z}}$, then two cases arise:

- (a) If I is infinite, then \mathfrak{g}'_I is isomorphic to $\mathfrak{sl}_{\mathbf{Z}}$, the Lie algebra of traceless matrices with finitely many non-zero entries.
- (b) If v has finite order e , then I is isomorphic to a cyclic quiver of type $A_{e-1}^{(1)}$. We can form $X^\vee = Q^\vee \oplus \mathbf{Z}\partial$ and $X = P \oplus \mathbf{Z}\delta$ with $\langle \partial, \Lambda_i \rangle = 0$, $\langle \partial, \alpha_i \rangle = \delta_{i1}$ and $\delta = \sum_{i \in I} \alpha_i$. The pairing is non-degenerate, and

$$\mathfrak{g}_I = \widehat{\mathfrak{sl}}_e = \mathfrak{sl}_e(\mathbf{C}) \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}\partial.$$

An explicit isomorphism sends e_{vi} (resp. f_{vi}) to the matrix $E_{i,i+1} \otimes 1$ (resp. $E_{i+1,i} \otimes 1$) if $i \neq e$ and e_1 (resp. f_1) to $E_{e,1} \otimes t$ (resp. $E_{1,e} \otimes t^{-1}$). Via this isomorphism the central element c corresponds to $\sum_{i \in I} h_i$, and

$$\mathfrak{g}'_I = \widetilde{\mathfrak{sl}}_e = \mathfrak{sl}_e(\mathbf{C}) \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c.$$

Now, take $R = \mathbf{Z}_\ell$ and $v \in \mathbf{Z}_\ell^\times$ which is not a root of unity. Assume however that the image of v in $S = \mathbf{Z}_\ell / \ell \mathbf{Z}_\ell$ is an e -th primitive root of unity (with $e \neq 1$). Then the canonical map $\theta : \mathbf{Z}_\ell \rightarrow \mathbf{Z}_\ell / \ell \mathbf{Z}_\ell$ restricts to a map $I = v^{\mathbf{Z}} \rightarrow I / \Gamma$ where $\Gamma = v^{\mathbf{Z}} \subset \mathbf{Z}_\ell^\times$ acts by multiplication on I . Then $\widetilde{\mathfrak{sl}}_e$ is the Γ -fixed points sub-Lie-algebra in $\mathfrak{sl}_{\mathbf{Z}}$ where Γ acts by automorphisms of the Dynkin diagram by the *folding construction*, yielding a homomorphism of Lie algebras $\mathfrak{sl}_e \rightarrow \widetilde{\mathfrak{sl}}_{\mathbf{Z}}$ which sends e_i to $\sum_{j=i} e_j$, as in [31, Proposition 7.9].

Now, let us consider some particular notations, to be used below.

Let $R = K$ be a field of characteristic zero, $q \in K^\times$ be an element of infinite order, and let I_∞ denote the subset $(-q)^{\mathbf{Z}}$ of K^\times . We define \mathfrak{g}_∞ to be the (derived) Kac-Moody algebra associated to the quiver $I_\infty(q^2)$. To avoid cumbersome notation, we will write for short $I_\infty = I_\infty(q^2)$, and $(-)_\infty = (-)_{I_\infty}$. We denote by $\{\Lambda_i\}$, $\{\alpha_i\}$ and $\{\alpha_i^\vee\}$ the fundamental weights, simple roots and simple coroots of \mathfrak{g}_∞ . Here X_∞ coincides with $P_\infty = \bigoplus \mathbf{Z}\Lambda_i$. Consequently, there is a Lie algebra isomorphism $(\mathfrak{sl}_{\mathbf{Z}})^{\oplus 2} \xrightarrow{\sim} \mathfrak{g}_\infty$ such that $(\alpha_d^\vee, 0) \mapsto \alpha_{-q^{2d-1}}^\vee$ and $(0, \alpha_d^\vee) \mapsto \alpha_{q^{2d}}^\vee$.

Next, let k be a field of positive characteristic, $q \in k^\times$ such that $(-q)$ has order e , and let I_e be the subset $(-q)^{\mathbf{Z}}$ of k^\times . We denote by I_e and $I_{e,\circ}$ the finite quivers $I_e(q^2)$ and $I_e(-q)$. The quivers I_e and $I_{e,\circ}$ have the same set of vertices, but the arrows in I_e are the composition of two consecutive arrows in $I_{e,\circ}$. The quiver $I_{e,\circ}$ is cyclic, whereas the quiver I_e is cyclic if e is odd, and is a union of two cyclic quivers if e is even. Therefore the corresponding Kac-Moody algebras are isomorphic to $\widehat{\mathfrak{sl}}_e$ or $(\widehat{\mathfrak{sl}}_{e/2})^{\oplus 2}$. To avoid cumbersome notation, we will write $(\bullet)_{e,\circ} = (\bullet)_{I_{e,\circ}}$ and $(\bullet)_e = (\bullet)_{I_e}$.

Now, for a future use, we introduce Lie algebras $\mathfrak{g}_{e,\circ}$ and \mathfrak{g}_e such that $\mathfrak{g}'_{e,\circ} = [\mathfrak{g}_{e,\circ}, \mathfrak{g}_{e,\circ}]$ and $\mathfrak{g}'_e = [\mathfrak{g}_e, \mathfrak{g}_e]$. The Chevalley generators of $\mathfrak{g}'_{e,\circ}$ and \mathfrak{g}'_e are $e_{i,\circ}, f_{i,\circ}$ and e_i ,

f_i respectively, for $i \in (-q)^{\mathbf{Z}}$. It is easy to see that there exists a morphism of Lie algebras $\kappa : \mathfrak{g}'_e \longrightarrow \mathfrak{g}'_{e,\circ}$ defined by

$$\kappa(e_i) = [e_{-qi,\circ}, e_{i,\circ}] \quad \text{and} \quad \kappa(f_i) = [f_{-qi,\circ}, f_{i,\circ}].$$

It restricts to a map between the coroot lattices sending α_i^\vee to $\alpha_{i,\circ}^\vee + \alpha_{-qi,\circ}^\vee$.

We denote by $\mathfrak{g}_{e,\circ}$ the Kac-Moody algebra associated with the lattices $X_{e,\circ} = P_{e,\circ} \oplus \mathbf{Z}\delta_\circ$ and $X_{e,\circ}^\vee = Q_{e,\circ}^\vee \oplus \mathbf{Z}\partial_\circ$, where $\delta_\circ = \sum \alpha_{i,\circ}$, $\partial_\circ = \Lambda_{1,\circ}^\vee$ and the pairing $X_{e,\circ}^\vee \times X_{e,\circ} \longrightarrow \mathbf{Z}$ is given by

$$\langle \alpha_{j,\circ}^\vee, \Lambda_{i,\circ} \rangle_{e,\circ} = \delta_{ij}, \quad \langle \partial_\circ, \Lambda_{i,\circ} \rangle_{e,\circ} = \langle \alpha_{j,\circ}^\vee, \delta_\circ \rangle_{e,\circ} = 0, \quad \langle \partial_\circ, \delta_\circ \rangle_{e,\circ} = 1.$$

Then $\mathfrak{g}_{e,\circ}$ and $\mathfrak{g}'_{e,\circ}$ are isomorphic to $\widehat{\mathfrak{sl}}_e$ and $\widetilde{\mathfrak{sl}}_e$ respectively.

Let $\widetilde{\mathfrak{g}}_e$ be the usual Kac-Moody algebra associated with I_e . Its derived Lie subalgebra is equal to \mathfrak{g}'_e . Let \widetilde{X}_e and \widetilde{X}_e^\vee be the lattices corresponding to $\widetilde{\mathfrak{g}}_e$. If e is odd, then I_e is a cyclic quiver and $\widetilde{\mathfrak{g}}_e$ is isomorphic to $\widehat{\mathfrak{sl}}_e$. Let α_1 be the affine root, then we have $\widetilde{X}_e = P_e \oplus \mathbf{Z}\delta$ and $\widetilde{X}_e^\vee = Q_e^\vee \oplus \mathbf{Z}\partial$ with $\delta = \sum \alpha_i$ and $\partial = \Lambda_1^\vee$. If e is even, then I_e is the disjoint union of two cyclic quivers and $\widetilde{\mathfrak{g}}_e$ is isomorphic to $(\widehat{\mathfrak{sl}}_{e/2})^{\oplus 2}$. Let α_1 and $\alpha_{-q^{-1}}$ be the affine roots, then we have $\widetilde{X}_e = P_e \oplus \mathbf{Z}\delta_1 \oplus \mathbf{Z}\delta_2$ and $\widetilde{X}_e^\vee = Q_e^\vee \oplus \mathbf{Z}\partial_1 \oplus \mathbf{Z}\partial_2$ with $\delta_1 = \sum_{j \text{ odd}} \alpha_{-q^j}$, $\delta_2 = \sum_{j \text{ even}} \alpha_{q^j}$, $\partial_1 = \Lambda_{-q^{-1}}^\vee$ and $\partial_2 = \Lambda_1^\vee$. We abbreviate $\widetilde{\partial} = \partial_1 + \partial_2$ and $\widetilde{\delta} = \delta_1 + \delta_2$.

The map $\kappa : \mathfrak{g}'_e \longrightarrow \mathfrak{g}'_{e,\circ}$ may not extend to a morphism of Lie algebras $\widetilde{\mathfrak{g}}_e \longrightarrow \mathfrak{g}_{e,\circ}$. For this reason we will define \mathfrak{g}_e to be a Lie subalgebra of $\widetilde{\mathfrak{g}}_e$ containing \mathfrak{g}'_e . More precisely, we define \mathfrak{g}_e to be Lie subalgebra $\mathfrak{g}'_e \oplus \mathbf{C}\partial$ of $\widetilde{\mathfrak{g}}_e$, where $\partial := \Lambda_1^\vee + \Lambda_{-q^{-1}}^\vee$. If e is odd, then $\partial = 2\widetilde{\partial} + h$ for some coweight h of \mathfrak{sl}_e , hence we have $\mathfrak{g}_e = \widetilde{\mathfrak{g}}_e$. If e is even, then $\partial = \widetilde{\partial}$ and \mathfrak{g}_e is strictly smaller than $\widetilde{\mathfrak{g}}_e$.

In both cases we can view \mathfrak{g}_e as the Kac-Moody algebra associated with the lattice $X_e^\vee = Q_e^\vee \oplus \mathbf{Z}\partial$ above and a lattice $X_e = P_e \oplus \mathbf{Z}\delta$ that we now define. If e is odd we set $\delta = \widetilde{\delta}/2$, if e is even we set $X_e = \widetilde{X}_e/(\delta_1 - \delta_2)$ with $\delta = \widetilde{\delta}/2$. The perfect pairing $X_e^\vee \times X_e \longrightarrow \mathbf{Z}$ is induced in the obvious way by the pairing $\widetilde{X}_e^\vee \times \widetilde{X}_e \longrightarrow \mathbf{Z}$. We have

$$\begin{aligned} \langle \alpha_j^\vee, \Lambda_i \rangle_e &= \delta_{ij}, & \langle \partial, \alpha_j \rangle_e &= \delta_{j1} + \delta_{j,-q^{-1}}, \\ \langle \partial, \Lambda_1 \rangle_e &= \langle \alpha_j^\vee, \delta \rangle_e = 0, & \langle \partial, \delta \rangle_e &= 1. \end{aligned}$$

This definition of \mathfrak{g}_e ensures that κ extends to a Lie algebra homomorphism $\mathfrak{g}_e \longrightarrow \mathfrak{g}_{e,\circ}$.

Lemma 1.1. — *There is a morphism of Lie algebras $\kappa : \mathfrak{g}_e \longrightarrow \mathfrak{g}_{e,\circ}$ which extends the map κ above. The restriction of κ to the Cartan subalgebra \mathfrak{h}_e is given by*

$$\kappa(h_i) = h_{i,\circ} + h_{-qi,\circ}, \quad \kappa(\partial) = \partial_\circ.$$

The map $\kappa : X_e^\vee \longrightarrow X_{e,\circ}^\vee$ has an adjoint $\kappa^ : X_{e,\circ} \longrightarrow X_e$ such that*

$$\kappa^*(\Lambda_{i,\circ}) \equiv \Lambda_i + \Lambda_{-q^{-1}i} \pmod{\delta}, \quad \kappa^*(\delta_\circ) = \delta.$$

Proof. — Recall that $i \in (-q)^{\mathbf{Z}}$ and that we have already defined the Lie algebra homomorphism $\kappa : \mathfrak{g}'_e \rightarrow \mathfrak{g}'_{e,\circ}$ sending e_i to $\kappa(e_i) = [e_{-qi,\circ}, e_{i,\circ}]$. We set $\kappa(\partial) = \partial_\circ$ and $\kappa(e_i) = [e_{-qi,\circ}, e_{i,\circ}]$ in $\mathfrak{g}_{e,\circ}$. By definition of \mathfrak{g}_e we have

$$[\partial, e_i] = \langle \partial, \alpha_i \rangle_e e_i = (\delta_{i1} + \delta_{i,-q^{-1}}) e_i.$$

Thus, we have

$$[\kappa(\partial), \kappa(e_i)] = \langle \partial_\circ, \alpha_{i,\circ} + \alpha_{-qi,\circ} \rangle_{e,\circ} \kappa(e_i) = (\delta_{i1} + \delta_{i,-q^{-1}}) \kappa(e_i) = \kappa([\partial, e_i]),$$

because the weight of $\kappa(e_i)$ in $\mathfrak{g}_{e,\circ}$ is $\alpha_{i,\circ} + \alpha_{-qi,\circ}$. The same holds for f_i instead of e_i . The second claim is an easy computation using the relation $\langle \kappa(h), \alpha_\circ \rangle_{e,\circ} = \langle h, \kappa^*(\alpha_\circ) \rangle_e$ for all $h \in X_e^\vee$ and $\alpha_\circ \in X_{e,\circ}$. \square

Remark 1.2. — If e is even then $\langle \partial, \Lambda_i \rangle_e = 0$, hence $\kappa^*(\Lambda_{i,\circ}) = \Lambda_i + \Lambda_{-q^{-1}i}$ for all i .

1.2.3. Integrable representations. — Let V be a \mathfrak{g} -module. Given $\omega \in X$, the ω -weight space of V is

$$V_\omega = \{v \in V \mid h_i \cdot v = \langle \alpha_i^\vee, \omega \rangle v, \forall i \in I\}.$$

An *integrable \mathfrak{g} -module* V is a \mathfrak{g} -module on which the action of the Chevalley generators is locally nilpotent, and which has a weight decomposition $V = \bigoplus_{\omega \in X} V_\omega$. The set $\text{wt}(V) = \{\omega \in X \mid V_\omega \neq 0\}$ is the set of weights in V .

We denote by \mathcal{O}^{int} the category of *integrable highest weight modules*, i.e. \mathfrak{g} -modules V satisfying

- $V = \bigoplus_{\omega \in X} V_\omega$ and $\dim V_\omega < \infty$ for all $\omega \in X$,
- the action of e_i and f_i is locally nilpotent for all $i \in I$,
- there exists a finite set $F \subset X$ such that $\text{wt}(V) \subset F + \sum_{i \in I} \mathbf{Z}_{\leq 0} \alpha_i$.

Let $X^+ = \{\omega \in X \mid \langle \alpha_i^\vee, \omega \rangle \in \mathbf{N} \text{ for all } i \in I\}$ be the set of *integral dominant weights*. Given $\Lambda \in X^+$, we denote by $\mathbf{L}(\Lambda)$ the simple integrable highest weight module with highest weight Λ . Then \mathcal{O}^{int} is semisimple, and any object in \mathcal{O}^{int} is a direct sum of such simple modules.

1.2.4. Quantized enveloping algebras. — Let u be a formal variable and $A = \mathbf{C}[u, u^{-1}]$. Let $U_u(\mathfrak{g})$ be the quantized enveloping algebra of \mathfrak{g} over $\mathbf{C}(u)$. Let $U_A(\mathfrak{g}) \subset U_u(\mathfrak{g})$ be Lusztig's divided power version of $U_u(\mathfrak{g})$. For each integral weight Λ the module $\mathbf{L}(\Lambda)$ admits a deformed version $\mathbf{L}_u(\Lambda)$ over $U_u(\mathfrak{g})$ and an integral form $\mathbf{L}_A(\Lambda)$ which is the $U_A(\mathfrak{g})$ -submodule of $\mathbf{L}_u(\Lambda)$ generated by the highest weight vector $|\Lambda\rangle$. Let $\mathcal{O}_u^{\text{int}}$ be the category consisting of the $U_u(\mathfrak{g})$ -modules which are (possibly infinite) direct sums of $\mathbf{L}_u(\Lambda)$'s. If $V_u \in \mathcal{O}_u^{\text{int}}$, then its integral form V_A is the corresponding sum of the modules $\mathbf{L}_A(\Lambda)$. It depends on the choice of a family of highest weight vectors of the constituents of V_u .

1.3. Categorical representations on abelian categories. — In this section we recall from [7, 41] the notion of a categorical action of \mathfrak{g} . It consists of the data of functors E_i, F_i lifting the Chevalley generators e_i, f_i of \mathfrak{g} , together with an action of an affine Hecke algebra on $(\bigoplus_{i \in I} F_i)^m$.

1.3.1. Affine Hecke algebras and representation data. — Let \mathcal{C} be an abelian \mathbf{R} -category and $v \in \mathbf{R}^\times$.

Definition 1.3. — A representation datum on \mathcal{C} is a tuple (E, F, X, T) where E, F are bi-adjoint functors $\mathcal{C} \rightarrow \mathcal{C}$ and $X \in \text{End}(F)^\times$, $T \in \text{End}(F^2)$ are endomorphisms of functors satisfying the following conditions:

- (a) $1_F T \circ T 1_F \circ 1_F T = T 1_F \circ 1_F T \circ T 1_F$,
- (b) $(T + 1_{F^2}) \circ (T - v 1_{F^2}) = 0$,
- (c) $T \circ (1_F X) \circ T = v X 1_F$.

This definition can also be formulated in terms of actions of affine Hecke algebras. For $m \geq 1$, the affine Hecke algebra $\mathbf{H}_{\mathbf{R},m}^v$ is the \mathbf{R} -algebra generated by T_1, \dots, T_{m-1} , $X_1^{\pm 1}, \dots, X_m^{\pm 1}$ subject to the relations

- Type A_{m-1} Hecke relations for T_1, \dots, T_{m-1} :

$$(T_i + 1)(T_i - v) = 0,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{and} \quad T_i T_j = T_j T_i \quad \text{if } |i - j| > 1,$$

- Laurent polynomial ring relations for $X_1^{\pm 1}, \dots, X_m^{\pm 1}$:

$$X_i X_j = X_j X_i \quad \text{and} \quad X_i X_i^{-1} = X_i^{-1} X_i = 1,$$

- Mixed relations:

$$T_i X_i T_i = v X_{i+1} \quad \text{and} \quad X_i T_j = T_j X_i \quad \text{if } i - j \neq 0, 1.$$

Therefore $\mathbf{H}_{\mathbf{R},m}^v$ contains both a finite dimensional Hecke algebra of finite type A_{m-1} and a Laurent polynomial ring in m variables. We will also set $\mathbf{H}_{\mathbf{R},0}^v = \mathbf{R}$.

Given (E, F) a pair of biadjoint functors, and $X \in \text{End}(F)$, $T \in \text{End}(F^2)$, the tuple (E, F, X, T) is a representation datum if and only if for each $m \in \mathbf{N}$, the map

$$\begin{aligned} \phi_{F^m} : \mathbf{H}_{\mathbf{R},m}^v &\longrightarrow \text{End}(F^m) \\ X_k &\longmapsto 1_{F^{m-k}} X 1_{F^{k-1}} \\ T_l &\longmapsto 1_{F^{m-l-1}} T 1_{F^{l-1}} \end{aligned}$$

is a well-defined \mathbf{R} -algebra homomorphism.

Remark 1.4. — Transposing an endomorphism of F^m relatively to the adjunction (F, E) yields a canonical R -algebra isomorphism $\text{End}(E^m) = \text{End}(F^m)^{\text{op}}$, see e.g., [7, Section 4.1.2]. Therefore, if (E, F, X, T) is a representation datum, the morphisms X, T yield morphisms $X \in \text{End}(E)$, $T \in \text{End}(E^2)$ which induce an R -algebra homomorphism $\phi_{E^m} : \mathbf{H}_{R,m}^v \longrightarrow \text{End}(E^m)^{\text{op}}$.

1.3.2. Categorical representations. — We assume now that R is a field and that \mathcal{C} is Hom-finite. We fix a pair (I, v) as in Section 1.2 and we denote by $\mathfrak{g} = \mathfrak{g}_I$ the Lie algebra associated to that pair.

Definition 1.5 ([41]). — A \mathfrak{g} -representation on \mathcal{C} consists of a representation datum (E, F, X, T) on \mathcal{C} and of a decomposition $\mathcal{C} = \bigoplus_{\omega \in X} \mathcal{C}_\omega$. For each $i \in I$, let F_i, E_i be the generalized i -eigenspaces of X acting on F, E respectively. We assume in addition that

- (a) $F = \bigoplus_{i \in I} F_i$ and $E = \bigoplus_{i \in I} E_i$,
- (b) the action of $[E_i]$ and $[F_i]$ for $i \in I$ endow $[\mathcal{C}]$ with a structure of integrable \mathfrak{g} -module such that $[\mathcal{C}]_\omega = [\mathcal{C}_\omega]$ for all $\omega \in X$,
- (c) $E_i(\mathcal{C}_\omega) \subset \mathcal{C}_{\omega+\alpha_i}$ and $F_i(\mathcal{C}_\omega) \subset \mathcal{C}_{\omega-\alpha_i}$ for all $\omega \in X$ and $i \in I$.

We say that the tuple (E, F, X, T) and the decomposition $\mathcal{C} = \bigoplus_{\omega \in X} \mathcal{C}_\omega$ is a \mathfrak{g} -categorification of the integrable \mathfrak{g} -module $[\mathcal{C}]$.

1.4. Outcomes. — In Section 4 we will endow the category of unipotent representations of finite unitary groups with a structure of \mathfrak{g} -representation. We recall here three main applications of the existence of a categorical action, which we will use in Sections 4.6 and 4.7 to determine:

- (1) the Hecke algebras associated to cuspidal representations,
- (2) the branching graph for the parabolic induction and restriction,
- (3) derived equivalences between blocks.

Note that for most of the results in this section we will assume that R is a field and that I is finite. In particular $v \in R^\times$ will be a root of unity.

1.4.1. Minimal categorical representations. — Let $m \geq 0$, $v \in R^\times$ and $\mathbf{H}_{R,m}^v$ be the affine Hecke algebra as defined in Section 1.3.1. We fix a tuple $Q = (Q_1, \dots, Q_l)$ in $(R^\times)^l$. The cyclotomic Hecke algebra $\mathbf{H}_{R,m}^{Q,v}$ is the quotient of $\mathbf{H}_{R,m}^v$ by the two-sided ideal generated by $\prod_{i=1}^l (X_i - Q_i)$.

Example 1.6. — (a) If $l = 2$, then $\mathbf{H}_{R,m}^{Q,v}$ is generated by X_1 and $\{T_i\}_{i=1, \dots, m-1}$. Set $T_0 = -Q_2^{-1}X_1$ and $u = -Q_1Q_2^{-1}$. Then, we have $(T_i + 1)(T_i - v) = 0$ and $(T_0 + 1)(T_0 - u) = 0$. Therefore $\mathbf{H}_{R,m}^{Q,v}$ is isomorphic to an Iwahori-Hecke algebra of type B_m with parameters (u, v) .

(b) If $v = 1$ and $Q_\rho = \zeta^{\rho-1}$, where ζ is a primitive l -th root of unity, then $\mathbf{H}_{R,m}^{Q,v} = \text{RG}(l, 1, m)$ is the group algebra of the complex reflection group $(\mathbf{Z}/l\mathbf{Z})^m \rtimes \mathfrak{S}_m$.

Assume now that R is a field. Any finite dimensional $\mathbf{H}_{R,m}^{Q,v}$ -module M is the direct sum of the weight subspaces

$$M_v = \{v \in M \mid (X_r - i_r)^d v = 0, \ r \in [1, m], \ d \gg 0\},$$

$$v = (i_1, \dots, i_m) \in R^m.$$

Decomposing the regular module, we get a system of orthogonal idempotents $\{e_v \mid v \in R^m\}$ in $\mathbf{H}_{R,m}^{Q,v}$ such that $e_v M = M_v$ for each M . By [10], [26, Lemma 4.7] the eigenvalues of X_r are always of the form $Q_i v^j$ for some $i \in \{1, \dots, l\}$ and $j \in \mathbf{Z}$. As a consequence, if we set $I = \bigcup Q_i v^{\mathbf{Z}}$, then $e_v = 0$ unless $v \in I$. The pair (I, v) satisfies the assumptions of Section 1.2 and we can consider a corresponding Kac-Moody algebra \mathfrak{g}_I and its root lattice Q_I . Given $\alpha \in Q_I^+$ of height m , let $e_\alpha = \sum_v e_v$ where the sum runs over the set of all tuples such that $\sum_{r=1}^m \alpha_{i_r} = \alpha$. The nonzero e_α 's are the primitive central idempotents in $\mathbf{H}_{R,m}^{Q,v}$.

To the dominant weight $\Lambda_Q = \sum_{i=1}^l \Lambda_{Q_i}$ of \mathfrak{g}_I and any $\alpha \in Q_I^+$ we associate the following abelian categories:

$$\mathcal{L}(\Lambda_Q) = \bigoplus_{m \in \mathbf{N}} \mathbf{H}_{R,m}^{Q,v}\text{-mod} \quad \text{and} \quad \mathcal{L}(\Lambda_Q)_{\Lambda_Q - \alpha} = e_\alpha \mathbf{H}_{R,m}^{Q,v}\text{-mod}.$$

For any $m < n$, the R -algebra embedding of the affine Hecke algebras $\mathbf{H}_{R,m}^v \hookrightarrow \mathbf{H}_{R,n}^v$ given by $T_i \mapsto T_i$ and $X_j \mapsto X_j$ induces an embedding $\mathbf{H}_{R,m}^{Q,v} \hookrightarrow \mathbf{H}_{R,n}^{Q,v}$. The R -algebra $\mathbf{H}_{R,n}^{Q,v}$ is free as a left and as a right $\mathbf{H}_{R,m}^{Q,v}$ -module. This yields a pair of exact adjoint functors $(\text{Ind}_m^n, \text{Res}_m^n)$ between $\mathbf{H}_{R,n}^{Q,v}\text{-mod}$ and $\mathbf{H}_{R,m}^{Q,v}\text{-mod}$. They induce endofunctors E and F of $\mathcal{L}(\Lambda_Q)$ by $E = \bigoplus_{m \in \mathbf{N}} \text{Res}_m^{m+1}$ and $F = \bigoplus_{m \in \mathbf{N}} \text{Ind}_m^{m+1}$. The right multiplication on $\mathbf{H}_{R,m+1}^{Q,v}$ by X_{m+1} yields an endomorphism of the functor Ind_m^{m+1} . The right multiplication by T_{m+1} yields an endomorphism of Ind_m^{m+2} . We define $X \in \text{End}(F)$ and $T \in \text{End}(F^2)$ by $X = \bigoplus_m X_{m+1}$ and $T = \bigoplus_m T_{m+1}$.

This construction yields a categorification of the simple highest module $\mathbf{L}(\Lambda_Q)$ of \mathfrak{g}_I . The following result is well known and it goes back essentially to [1, 26].

Theorem 1.7.

- (a) The endofunctors E and F of $\mathcal{L}(\Lambda_Q)$ are biadjoint.
- (b) The tuple (E, F, X, T) and the decomposition $\mathcal{L}(\Lambda_Q) = \bigoplus_{\omega \in X} \mathcal{L}(\Lambda_Q)_\omega$ is a \mathfrak{g}_I -categorification of $\mathbf{L}(\Lambda_Q)$. \square

This categorical representation is called the *minimal categorical \mathfrak{g}_I -representation* of highest weight Λ_Q , see [7].

The \mathfrak{g}_I -modules we are interested in are direct sums of various irreducible highest weight modules $\mathbf{L}(\Lambda_Q)$. This decomposition admits the following categorical counterpart. Let (I, v) as in Section 1.2, and $\mathfrak{g} = \mathfrak{g}_I$ be a corresponding Kac-Moody algebra. Let

(E, F, X, T) be a \mathfrak{g} -representation on an abelian R -category \mathcal{C} . Recall that for any $m \geq 0$ we have an R -algebra homomorphism $\phi_{F^m} : \mathbf{H}_{R,m}^{Q,v} \longrightarrow \text{End}(F^m)^{\text{op}}$. Given an object M in \mathcal{C} , it specializes to an R -algebra homomorphism

$$\mathbf{H}_{R,m}^{Q,v} \longrightarrow \text{End}(F^m M)^{\text{op}} =: \mathcal{H}(F^m M).$$

Proposition 1.8 ([41]). — Assume that the simple roots are linearly independent in X . Let (E, F, X, T) be a representation of \mathfrak{g} in a abelian R -category \mathcal{C} , and $M \in \mathcal{C}_\omega$ for some $\omega \in X$. Assume that $EM = 0$ and $\text{End}_{\mathcal{C}}(M) = R$. Then

- (a) $\omega \in X^+$ is an integral dominant weight,
- (b) if we write $\Lambda_Q = \sum_{i \in I} \langle \alpha_i^\vee, \omega \rangle \Lambda_i = \sum_{p=1}^l \Lambda_{Q_p}$ for some $Q = (Q_1, \dots, Q_l) \in I^l$ and $l \geq 1$, then for all $m \geq 0$ the map ϕ_{F^m} factors to an R -algebra isomorphism

$$\mathbf{H}_{R,m}^{Q,v} \xrightarrow{\sim} \mathcal{H}(F^m M). \quad \square$$

In the framework of Section 4, the functors E and F are the parabolic induction and restriction functors. Therefore from Proposition 1.8 we deduce that the endomorphism algebra of the induction of a cuspidal module is a cyclotomic Hecke algebra whose parameters are given by the weight of the cuspidal module.

1.4.2. Perfect bases, crystals and branching graphs. — We start by a review of Kashiwara's theory of perfect bases and crystals. A good reference is [33], or [32] for a short review. We will be working with the Kac-Moody algebra \mathfrak{g} coming from a pair (I, v) as in Section 1.2.

Definition 1.9. — An abstract crystal is a set B together with maps $\text{wt} : B \rightarrow P$, $\varepsilon_i, \varphi_i : B \rightarrow \mathbf{Z} \sqcup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \sqcup \{0\}$ for all $i \in I$ satisfying the following properties:

- (a) $\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \text{wt}(b) \rangle$,
- (b) $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$ and $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$,
- (c) $b = \tilde{e}_i b'$ if and only if $\tilde{f}_i b = b'$, where $b, b' \in B$, $i \in I$,
- (d) if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$,
- (e) if $b \in B$ and $\tilde{e}_i b \in B$, then $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$,
- (f) if $b \in B$ and $\tilde{f}_i b \in B$, then $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.

Note that by (a), the map φ_i is entirely determined by ε_i and wt . We may therefore omit φ_i in the data of an abstract crystal and denote it by $(B, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \text{wt})$.

An isomorphism between crystals B_1, B_2 is a bijection $\psi : B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ such that $\psi(0) = 0$ which commutes with wt , ε_i , $\varphi_i, \tilde{f}_i, \tilde{e}_i$.

To an abstract crystal we can associate a *crystal graph* as follows: the vertices of the graph are indexed by the elements of B and the arrows are $b \xrightarrow{i} b'$ for each $b, b' \in B$ and $i \in I$ with $b = \tilde{e}_i b'$ (or equivalently $b' = \tilde{f}_i b$). The operators \tilde{e}_i and \tilde{f}_i are thus entirely determined by the graph.

Let V_u be an integrable $U_u(\mathfrak{g})$ -module in $\mathcal{O}_u^{\text{int}}$. Let V_A be an integral form of V_u . A *lower crystal lattice* in V_u is a free $\mathbf{C}[u]$ -submodule \mathcal{L} of V_A such that $V_A = A\mathcal{L}$, $\mathcal{L} = \bigoplus_{\lambda \in X} \mathcal{L}_\lambda$ with $\mathcal{L}_\lambda = \mathcal{L} \cap (V_A)_\lambda$ and \mathcal{L} is preserved by the *lower Kashiwara crystal operators* $\tilde{e}_i^{\text{lw}}, \tilde{f}_i^{\text{lw}}$ on V_u . A *lower crystal basis* of V_u is a pair (\mathcal{L}, B) where \mathcal{L} is a lower crystal lattice of V_u and B is a basis of $\mathcal{L}/u\mathcal{L}$ such that we have $B = \bigsqcup_{\lambda \in X} B_\lambda$ where $B_\lambda = B \cap (\mathcal{L}_\lambda/u\mathcal{L}_\lambda)$, $\tilde{e}_i^{\text{lw}}(B), \tilde{f}_i^{\text{lw}}(B) \subset B \sqcup \{0\}$ and $b' = \tilde{f}_i^{\text{lw}}b$ if and only if $b = \tilde{e}_i^{\text{lw}}b'$ for each $b, b' \in B$. A *lower global basis* of V_u (or, in Lusztig terminology, a *canonical basis*) is an A -basis \mathbf{B} of V_A such that the lattice $\mathcal{L} = \bigoplus_{b \in \mathbf{B}} \mathbf{C}[u]b$ and the basis $B = \{b \bmod u\mathcal{L} \mid b \in \mathbf{B}\}$ of $\mathcal{L}/u\mathcal{L}$ form a lower crystal basis.

One defines in a similar way an *upper crystal lattice*, an *upper crystal basis* and an *upper global basis* (or a *dual canonical basis*) using the *upper Kashiwara crystal operators* $\tilde{e}_i^{\text{up}}, \tilde{f}_i^{\text{up}}$ on V_u , see, e.g., [32, Definition 4.1, 4.2].

If $(\mathcal{L}, B), (\mathcal{L}^\vee, B^\vee)$ are lower, upper crystal bases, then $(B, \tilde{e}_i^{\text{lw}}, \tilde{f}_i^{\text{lw}}), (B^\vee, \tilde{e}_i^{\text{up}}, \tilde{f}_i^{\text{up}})$ are abstract crystals.

Let E_i, F_i, u^h with $i \in I, h \in X^\vee$, be the standard generators of $U_u(\mathfrak{g})$. There exists a unique non-degenerate symmetric bilinear form (\bullet, \bullet) on the module $\mathbf{L}_u(\Lambda)$ with highest weight vector $|\Lambda\rangle$ satisfying

- $(|\Lambda\rangle, |\Lambda\rangle) = 1$,
- $(E_i x, y) = (x, F_i y), (F_i x, y) = (x, E_i y), (u^h x, y) = (x, u^h y)$,
- $(\mathbf{L}_u(\Lambda)_\lambda, \mathbf{L}_u(\Lambda)_\mu) = 0$ if $\lambda \neq \mu$.

Any $U_u(\mathfrak{g})$ -module in $\mathcal{O}_u^{\text{int}}$ admits a lower and an upper crystal and global basis. If (\mathcal{L}, B) is a lower crystal basis of $\mathbf{L}_u(\Lambda)$ then the pair $(\mathcal{L}^\vee, B^\vee)$ such that $\mathcal{L}^\vee = \{x \in \mathbf{L}_u(\Lambda) \mid (x, \mathcal{L}) \subset \mathbf{C}[u]\}$ and B^\vee is the basis of $\mathcal{L}^\vee/u\mathcal{L}^\vee$ which is dual to B with respect to the non-degenerate bilinear form $\mathcal{L}^\vee/u\mathcal{L}^\vee \times \mathcal{L}/u\mathcal{L} \rightarrow \mathbf{C}$ induced by (\bullet, \bullet) , is an upper crystal basis, see, e.g., [32, Proposition 4.4]. Finally, taking a basis element in B to the dual basis element in B^\vee is a crystal isomorphism $(B, \tilde{e}_i^{\text{lw}}, \tilde{f}_i^{\text{lw}}) \rightarrow (B^\vee, \tilde{e}_i^{\text{up}}, \tilde{f}_i^{\text{up}})$ by [32, Lemma 4.3].

The crystals that we will consider in this paper all come from particular bases of \mathfrak{g}_I -modules called *perfect bases*, introduced in [2]. Let us recall their definition. Let $V \in \mathcal{O}^{\text{int}}$ be an integrable highest weight \mathfrak{g}_I -module. Under this assumption we define, for $i \in I$ and $x \in V$

$$\ell_i(x) = \max\{k \in \mathbf{N} \mid e_i^k x \neq 0\} = \min\{k \in \mathbf{N} \mid e_i^{k+1} x = 0\}$$

with the convention that $\ell_i(0) = -\infty$. For each integer k , we also consider the vector spaces

$$V_i^{\leq k} = \{x \in V \mid \ell_i(x) \leq k\} \quad \text{and} \quad V^{\leq k} = \bigcap_{i \in I} V_i^{\leq k}.$$

Note that $V_i^{\leq k} = \ker e_i^{k+1}$ when $k \geq 0$. Finally we set $V_i^k = V_i^{\leq k} / V_i^{< k}$.

Definition 1.10. — A basis B of V is perfect if

- (a) $B = \bigsqcup_{\mu \in X_1} B_\mu$ where $B_\mu = B \cap V_\mu$,
- (b) for any $i \in I$, there exists a map $\tilde{\mathbf{e}}_i : B \rightarrow B \sqcup \{0\}$ such that for any $b \in B$, we have
 - (i) if $\ell_i(b) = 0$, then $\tilde{\mathbf{e}}_i b = 0$,
 - (ii) if $\ell_i(b) > 0$, then $\tilde{\mathbf{e}}_i b \in B$ and $e_i b \in \mathbf{C}^\times \tilde{\mathbf{e}}_i b + V_i^{< \ell_i(b)-1}$,
- (c) if $\tilde{\mathbf{e}}_i b = \tilde{\mathbf{e}}_i b' \neq 0$ for $b, b' \in B$, then $b = b'$.

Any \mathfrak{g} -module in \mathcal{O}^{int} admits a perfect basis. More precisely, we have the following, see, e.g., [32, Section 4] for a proof.

Proposition 1.11. — *If V is an integrable \mathfrak{g} -module in \mathcal{O}^{int} with a quantum deformation V_u , then the specialization at $u = 1$ of an upper global basis of V_u is a perfect basis of V .* \square

To any categorical representation we associate a perfect basis as in [43, Proposition 6.2]. More precisely, let R be a field (of any characteristic) and consider a \mathfrak{g} -representation on an abelian artinian R -category \mathcal{C} . Then, for each $i \in I$ we define the maps

$$\begin{aligned} \tilde{E}_i : \text{Irr}(\mathcal{C}) &\rightarrow \text{Irr}(\mathcal{C}) \sqcup \{0\}, & [L] &\mapsto [\text{soc}(E_i L)], \\ \tilde{F}_i : \text{Irr}(\mathcal{C}) &\rightarrow \text{Irr}(\mathcal{C}) \sqcup \{0\}, & [L] &\mapsto [\text{hd}(F_i L)]. \end{aligned}$$

Proposition 1.12. — *The tuple $(\text{Irr}(\mathcal{C}), \tilde{E}_i)$ defines a perfect basis of $[\mathcal{C}]$.* \square

We now recall how to construct an abstract crystal from a perfect basis B , following [2]. We set $\tilde{e}_i = \tilde{\mathbf{e}}_i$. For all $b \in B$ we set $\tilde{f}_i b = b'$ if $\tilde{\mathbf{e}}_i b' = b$ for some $b' \in B$, and 0 otherwise. Then it follows easily from the definition that $(B, \tilde{e}_i, \tilde{f}_i, \ell_i, \text{wt})$ is an abstract crystal. In the case where the perfect basis comes from a categorical \mathfrak{g} -representation, the corresponding crystal graph is the *branching graph* of the exact functors E_i, F_i . More precisely, it is the colored graph with vertices labelled by $\text{Irr}(\mathcal{C})$ and arrows $[L] \xrightarrow{i} [L']$ whenever L' appears in the head of $F_i L$, or equivalently when L appears in the socle of $E_i L'$.

We finish this section with two results which will be important to identify the crystal graph obtained by the categorification with the crystal graph of some Fock space (see Section 2.3.2 for the definition of the crystal of a charged Fock space). For each $i \in I$ and $k \in \mathbf{N}$, we set $B^{\leq k} = V^{\leq k} \cap B$ and $B_i^{\leq k} = V_i^{\leq k} \cap B$. For a given $i \in I$ and for $b \in B$, let $[b]_i$ be the image of b in $V_i^{\ell_i(b)}$. We have the following well-known facts.

Lemma 1.13. — *Let B be a perfect basis of $V \in \mathcal{O}^{\text{int}}$. Let $i \in I$ and $b, b' \in B$.*

- (a) $b = b'$ if and only if $\ell_i(b) = \ell_i(b')$ and $[b]_i = [b']_i$.
- (b) $B^{\leq k}$ and $B_i^{\leq k}$ are bases of $V^{\leq k}$ and $V_i^{\leq k}$.

Proof. — Part (b) is [2, rem. 5.52]. Let us prove (a). Let $i \in I$. For each $b \in B$, we set $\tilde{\mathbf{e}}_i^+ b = \tilde{\mathbf{e}}_i^{\ell_i(b)} b$ and $e_i^+ b = e_i^{\ell_i(b)} b$. Note that $e_i V_i^{\leq k} \subset V_i^{< k}$. Applying successively the axiom (b)(ii) of perfect bases, we get

$$(1.1) \quad e_i^k b \in \mathbf{C}^\times \tilde{\mathbf{e}}_i^k b + V_i^{< \ell_i(b)-k}, \quad \text{for all } k = 1, 2, \dots, \ell_i(b).$$

In particular we have $e_i^+ b \in \mathbf{C}^\times \tilde{\mathbf{e}}_i^+ b$ and $\tilde{\mathbf{e}}_i^+ b \in B_i^{\leq 0}$. Furthermore, the axiom (c) of perfect bases implies that $\tilde{\mathbf{e}}_i^+ b \neq \tilde{\mathbf{e}}_i^+ b'$ whenever $b \neq b'$. Now, if $b, b' \in B$ are such that $\ell_i(b) = \ell_i(b') = k$ and $[b]_i = [b']_i$ in V_i^k , then, by (1.1), we have $e_i^+[b]_i = e_i^+ b \in \mathbf{C}^\times \tilde{\mathbf{e}}_i^+ b$ and $e_i^+[b']_i = e_i^+ b' \in \mathbf{C}^\times \tilde{\mathbf{e}}_i^+ b'$. We deduce that $\tilde{\mathbf{e}}_i^+ b = \tilde{\mathbf{e}}_i^+ b'$, from which we get $b = b'$. \square

We deduce the following proposition.

Proposition 1.14. — *Let B and B' be perfect bases of V . Assume that there is a bijection $\varphi : B \rightarrow B'$ and a partial order \leq on B such that $\varphi(b) \in b + \sum_{c>b} \mathbf{C} c$ for each $b \in B$. Then the map φ is a crystal isomorphism $B \xrightarrow{\sim} B'$.*

Proof. — For convenience we shall write ℓ'_i for the restriction of the map ℓ_i to the basis B' of V .

First observe that $\ell'_i \circ \varphi = \ell_i$ for all $i \in I$. Indeed, by Lemma 1.13(b), the sets $B_i^{\leq k} = \{b \mid b \in B, \ell_i(b) \leq k\}$ and $(B')_i^{\leq k} = \{\varphi(b) \mid b \in B, \ell'_i(\varphi(b)) \leq k\}$ are both bases of $V_i^{\leq k}$. Set $r = \dim V_i^{\leq k}$ and write

$$B_i^{\leq k} = \{b_1, b_2, \dots, b_r\}, \quad (B')_i^{\leq k} = \{\varphi(b'_1), \varphi(b'_2), \dots, \varphi(b'_r)\}.$$

We claim that $\{b_1, b_2, \dots, b_r\} = \{b'_1, b'_2, \dots, b'_r\}$, from which we deduce that $(B')_i^{\leq k} = \varphi(B_i^{\leq k})$, hence that

$$\ell'_i(\varphi(b)) \leq k \iff \ell_i(b) \leq k,$$

or, equivalently, $\ell_i(b) = \ell'_i(\varphi(b))$. To prove the claim, if $\{b_1, b_2, \dots, b_r\} \neq \{b'_1, b'_2, \dots, b'_r\}$ we may fix $s \in [1, r]$ such that $\ell_i(b'_s) > k$. By definition of φ , we have $\varphi(b'_s) \in b'_s + \sum_{b \in S} \mathbf{C} b$ for some subset $S \subset B \setminus \{b'_s\}$. Then, by Lemma 1.13(b), we have $\ell_i(\varphi(b'_s)) > k$, which contradicts $\varphi(b'_s) \in (B')_i^{\leq k}$.

We also deduce that $\varphi(b) \in b + \sum_c \mathbf{C} c$ where c runs over elements of B satisfying

$$(1.2) \quad \ell_i(c) \leq \ell_i(b) = \ell'_i(\varphi(b)), \quad c > b \quad \text{and} \quad \text{wt}(b) = \text{wt}(\varphi(b)) = \text{wt}(c).$$

In particular, the map φ yields a weight preserving bijection $B^{\leq 0} \rightarrow (B')^{\leq 0}$. Therefore it extends to an automorphism of the \mathfrak{g} -module V , and in turn, to a crystal isomorphism $\psi : B \rightarrow \sim B'$ such that $\psi(b) = \varphi(b)$ for all $b \in B^{\leq 0}$ (see [2, Theorem 5.37]).

We claim that $\varphi = \psi$. We will prove it by induction with respect to ℓ_i . Fix an element $b \in B$ with $k := \ell_i(b) > 0$ for some $i \in I$, and assume that $\varphi(c) = \psi(c)$ for all $c \in B_i^{\leq k}$. We have $\ell'_i(\varphi(b)) = \ell_i(b) = k > 0$, hence $\tilde{\mathbf{e}}_i(b), \tilde{\mathbf{e}}'_i(\varphi(b))$ are both non-zero. Since ψ is a crystal isomorphism, the induction hypothesis applied to $c = \tilde{\mathbf{e}}_i(b)$ yields $\varphi(\tilde{\mathbf{e}}_i(b)) = \psi(\tilde{\mathbf{e}}_i(b)) = \tilde{\mathbf{e}}'_i(\psi(b))$. The axiom (c) of perfect bases would imply that $\varphi(b) = \psi(b)$ if we can show that

$$(1.3) \quad \varphi(\tilde{\mathbf{e}}_i(b)) = \tilde{\mathbf{e}}'_i(\varphi(b)).$$

Let us prove this equality. We have $\ell'_i(\varphi(\tilde{\mathbf{e}}_i(b))) = \ell_i(\tilde{\mathbf{e}}_i(b)) = k - 1$ and $\ell'_i(\tilde{\mathbf{e}}'_i(\varphi(b))) = \ell'_i(\varphi(b)) - 1 = k - 1$. Therefore by Lemma 1.13(a) it is enough to check that

$$(1.4) \quad [\varphi(\tilde{\mathbf{e}}_i(b))]_i = [\tilde{\mathbf{e}}'_i(\varphi(b))]_i \quad \text{in } V_i^{k-1}$$

which we shall prove by induction with respect to the order \geq .

Recall that the map φ is unitriangular in the basis \mathbf{B} . Therefore the same holds for φ^{-1} in the basis \mathbf{B}' and we have, by projection to V_i^k and using (1.2)

$$(1.5) \quad [b]_i \in [\varphi(b)]_i + \sum_{\substack{\ell_i(c)=k \\ c > b}} \mathbf{C}[\varphi(c)]_i.$$

Since $e_i V_i^{\leq k} \subset V_i^{< k}$, the map e_i factors through a linear map $e_i : V_i^k \longrightarrow V_i^{k-1}$. The axiom (b)(ii) of perfect bases implies that $e_i([b]_i) \in \mathbf{C}^\times [\tilde{\mathbf{e}}_i(b)]_i$ in V_i^{k-1} . Now, applying e_i to (1.5) yields the following relation in V_i^{k-1}

$$(1.6) \quad [\tilde{\mathbf{e}}_i(b)]_i \in \mathbf{C}^\times [\tilde{\mathbf{e}}'_i(\varphi(b))]_i + \sum_{\substack{\ell_i(c)=k \\ c > b}} \mathbf{C}[\tilde{\mathbf{e}}'_i(\varphi(c))]_i.$$

Assume now by induction that (1.4) holds for any $c > b$. Then we can rewrite (1.6) as

$$(1.7) \quad [\tilde{\mathbf{e}}_i(b)]_i \in \mathbf{C}^\times [\tilde{\mathbf{e}}'_i(\varphi(b))]_i + \sum_{\substack{\ell_i(c)=k \\ c > b}} \mathbf{C}[\varphi(\tilde{\mathbf{e}}_i(c))]_i.$$

On the other hand, applying (1.5) to $\tilde{\mathbf{e}}_i(b)$ instead of b , we also get the following relation in V_i^{k-1}

$$(1.8) \quad [\tilde{\mathbf{e}}_i(b)]_i \in [\varphi(\tilde{\mathbf{e}}_i(b))]_i + \sum_{\substack{\ell_i(c)=k-1 \\ c > \tilde{\mathbf{e}}_i(b)}} \mathbf{C}[\varphi(c)]_i.$$

Now, observe that $[\varphi(\tilde{\mathbf{e}}_i(b))]_i \notin \mathbf{C}[\varphi(\tilde{\mathbf{e}}_i(c))]_i$ whenever $\ell_i(c) = k$ and $c \neq b$, by Lemma 1.13(a) and the axiom (c) of perfect bases. Therefore, comparing (1.7) and (1.8), we get the identity (1.4).

Finally, in the case where b is maximal with respect to \geq , the relation (1.6) becomes $[\tilde{\mathbf{e}}_i(b)]_i \in \mathbf{C}^\times [\tilde{\mathbf{e}}'_i(\varphi(b))]_i$ and therefore (1.7) still holds. Hence (1.4) holds in that case as well. \square

1.4.3. Derived equivalences. — Given V an integrable \mathfrak{g} -module, and $i \in I$ one can consider the action of the simple reflection $s_i = \exp(-f_i) \exp(e_i) \exp(-f_i)$ on V . For each weight $\omega \in X$, this action maps a weight space V_ω to $V_{s_i(\omega)}$ with $s_i(\omega) = \omega - \langle \alpha_i^\vee, \omega \rangle \alpha_i$. If \mathcal{C} is a categorification of V , then it restricts to an $\mathfrak{sl}_2(\mathbf{C})$ -categorification in the sense of

Chuang-Rouquier. In particular, the simple objects are weight vectors for the categorical $\mathfrak{sl}_2(\mathbf{C})$ -action. Thus, the theory of Chuang-Rouquier can be applied and [7, Theorem 6.6] implies that s_i can be lifted to a derived equivalence Θ_i of \mathcal{C} .

Theorem 1.15. — *Assume that R is a field. Let (E, F, X, T) be a representation of \mathfrak{g} in an abelian R -category \mathcal{C} , and $i \in I$. Then there exists a derived self-equivalence Θ_i of \mathcal{C} which restricts to derived equivalences*

$$\Theta_i : D^b(\mathcal{C}_\omega) \xrightarrow{\sim} D^b(\mathcal{C}_{s_i(\omega)})$$

for all weights $\omega \in X$. Furthermore, $[\Theta_i] = s_i$ as a linear map of $[\mathcal{C}]$. \square

In the context of Section 4, each weight space \mathcal{C}_ω will be a unipotent block of a finite unitary group. As a consequence of this theorem we will obtain many derived equivalences between unipotent blocks, in the spirit of Broué's abelian defect group conjecture (see Section 4.6.3).

2. Representations on Fock spaces

Let R be a noetherian commutative domain with unit. As in Section 1.2, we fix an element $v \in R^\times$ and a subset I of R^\times which is stable by multiplication by v and v^{-1} . We explained in Section 1.2.1 how one can associate a Lie algebra $\mathfrak{g} = \mathfrak{g}_I$ to this data. In this section we recall the construction of (charged) Fock spaces which are particular integrable representations of \mathfrak{g} . These will be the representations that we shall categorify using unipotent representations of finite unitary groups (see Section 4.4 and Section 4.5).

2.1. Combinatorics of l -partitions.

2.1.1. Partitions and l -partitions. — A *partition* of n is a non-increasing sequence of non-negative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ whose terms add up to n . We denote by \mathcal{P}_n the set of partitions of n and by $\mathcal{P} = \bigsqcup_n \mathcal{P}_n$ the set of all partitions. Given a partition λ , we write $|\lambda|$ for the *weight* of λ , $l(\lambda)$ for the number of non-zero parts in λ and ${}^t\lambda$ for the transposed partition. We associate to $\lambda = (\lambda_1, \lambda_2, \dots)$ the *Young diagram* $Y(\lambda)$ defined by $Y(\lambda) = \{(x, y) \in \mathbf{Z}_{>0} \times \mathbf{Z}_{>0} \mid y \leq \lambda_x\}$. It may be visualised by an array of boxes in left justified rows with λ_x boxes in the x -th row. If λ, μ are partitions of n then we write $\lambda \geq \mu$ if for all $n \geq i \geq 1$ we have $\sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i \mu_j$. This relation defines a partial order on \mathcal{P} called the *dominance order*.

An *l -partition* of n is an l -tuple of partitions whose weights add up to n . We denote by \mathcal{P}_n^l the set of l -partitions of n and by $\mathcal{P}^l = \bigsqcup_n \mathcal{P}_n^l$ the set of all l -partitions. The Young diagram of the l -partition $\lambda = (\lambda^1, \dots, \lambda^l)$ is the set $Y(\lambda) = \bigsqcup_{p=1}^l Y(\lambda^p) \times \{p\}$. Its weight is the integer $|\lambda| = \sum_p |\lambda^p|$.

2.1.2. Residues and content. — We fix $Q = (Q_1, \dots, Q_l) \in \mathbf{I}^l$. Let λ be an l -partition and $A = (x, y, p)$ be a node in $Y(\lambda)$. The (Q, v) -shifted residue of the node A is the element of \mathbf{I} given by $\text{res}(A, Q)_I = v^{p-x} Q_p$. Let $n_i(\lambda, Q)_I$ be the number of nodes of (Q, v) -shifted residue i in $Y(\lambda)$. If λ, μ are l -partitions such that $|\mu| = |\lambda| + 1$ we write $\text{res}(\mu - \lambda, Q)_I = i$ if $Y(\mu)$ is obtained by adding a node of (Q, v) -shifted residue i to $Y(\lambda)$. We denote by $\text{add}_i(\lambda, Q)_I$ (resp. $\text{rem}_i(\lambda, Q)_I$) the set of *addable nodes* (resp. *removable nodes*) of (Q, v) -shifted residue i . With $n = |\lambda|$ we have

$$\begin{aligned} \text{add}_i(\lambda, Q)_I &= \{A = Y(\mu) \setminus Y(\lambda) \mid \mu \in \mathcal{P}_{n+1}^l \text{ s.t. } \text{res}(\mu - \lambda, Q)_I = i\}, \\ \text{rem}_i(\lambda, Q)_I &= \{A = Y(\lambda) \setminus Y(\mu) \mid \mu \in \mathcal{P}_{n-1}^l \text{ s.t. } \text{res}(\lambda - \mu, Q)_I = i\}. \end{aligned}$$

A *charge* of the tuple $Q = (Q_1, \dots, Q_l)$ is an l -tuple of integers $s = (s_1, \dots, s_l)$ such that $Q_p = v^{s_p}$ for all $p = 1, \dots, l$. Conversely, given $\mathbf{I} \subset \mathbf{R}^\times$ and $v \in \mathbf{R}^\times$ as in Section 1.2, any l -tuple of integers $s \in \mathbf{Z}^l$ defines a tuple $Q = (v^{s_1}, \dots, v^{s_l})$ with charge s . The s -shifted content of the box $A = (x, y, p)$ is the integer $\text{ct}^s(A) = s_p + y - x$. It is related to the residue of A by the formula $\text{res}(A, Q)_I = v^{\text{ct}^s(A)}$. We will also write $p(A) = p$. We will call *charged l -partition* a pair (μ, s) in $\mathcal{P}^l \times \mathbf{Z}^l$.

2.1.3. l -cores and l -quotients. — The set of β -numbers of a charged partition $(\lambda, d) \in \mathcal{P} \times \mathbf{Z}$ is the set given by $\beta_d(\lambda) = \{\lambda_u + d + 1 - u \mid u \geq 1\}$. The charged partition (λ, d) is uniquely determined by the set $\beta_d(\lambda)$.

If $Q = v^d$, then the set $\text{rem}_i(\lambda, Q)$ of removable nodes of (Q, v) -shifted residue i is the set of integers $j \notin \beta_d(\lambda)$ such that $i = v^j$ and $j + 1 \in \beta_d(\lambda)$. Removing such a node has the effect of replacing $j + 1$ by j . We have an analogous description for addable nodes. More generally, for any positive integer e , an e -hook of (λ, d) is a pair $(x, x + e)$ such that $x + e \in \beta_d(\lambda)$ and $x \notin \beta_d(\lambda)$. Removing the e -hook $(x, x + e)$ corresponds to replacing $x + e$ with x in $\beta_d(\lambda)$. We say that the charged partition (λ, d) is an e -core if it does not have any e -hook. This does not depend on d .

Next, we construct a bijection $\tau_l : \mathcal{P} \times \mathbf{Z} \rightarrow \mathcal{P}^l \times \mathbf{Z}^l$. It takes the pair (λ, d) to (μ, s) , where $\mu = (\mu^1, \dots, \mu^l)$ is an l -partition and $s = (s_1, \dots, s_l)$ is an l -tuple in

$$\mathbf{Z}^l(d) = \{s \in \mathbf{Z}^l \mid s_1 + \dots + s_l = d\}.$$

The bijection is uniquely determined by the following relation

$$\beta_d(\lambda) = \bigsqcup_{p=1}^l (p - l + l\beta_{s_p}(\mu^p)).$$

See [46] for details.

The bijection τ_l takes the pair $(\lambda, 0)$ to $(\lambda^{[l]}, s)$, where the l -partition $\lambda^{[l]}$ is the l -quotient of λ and s lies in $\mathbf{Z}^l(0)$. Moreover the partition λ is an l -core if and only if $\lambda^{[l]} = \emptyset$. Thus, the bijection τ_l identifies the set of l -cores and $\mathbf{Z}^l(0)$. In particular, for any partition

λ , we can view the tuple s as the partition $\lambda_{[l]}$ such that $\tau_l(\lambda_{[l]}, 0) = (0, s)$. This partition is an l -core, and we'll call it the l -core of λ .

We define the l -weight $w_l(\lambda)$ of the partition λ to be the weight of the l -partition $\lambda_{[l]}$. Equivalently, it is the number of l -hooks that can be successively removed from λ to get its l -core $\lambda_{[l]}$, and we get

$$w_l(\lambda) = |\lambda^{[l]}| = (|\lambda| - |\lambda_{[l]}|)/l.$$

We will mostly consider the bijection τ_l for $l = 2$. In particular, a 2-core is either $\Delta_0 = \emptyset$ or a triangular partition $\Delta_t = (t, t-1, \dots, 1)$ with $t \in \mathbf{N}$. We abbreviate $\sigma_t = (\Delta_t)_{[2]}$, and we write $\sigma_t = (\sigma_1, \sigma_2)$. We have

$$(2.1) \quad \sigma_t = \begin{cases} (-t/2, t/2) & \text{if } t \text{ is even,} \\ ((1+t)/2, -(1+t)/2) & \text{if } t \text{ is odd.} \end{cases}$$

For each bipartition μ , let $\varpi_t(\mu)$ denote the unique partition with 2-quotient μ and 2-core Δ_t . Thus, the bijection τ_2 maps $(\varpi_t(\mu), 0)$ to the pair (μ, σ_t) .

2.2. Fock spaces. — For a reference for the results presented in this section, see for example [45], [46]. Let $\mathbf{Q} = (Q_1, \dots, Q_l) \in \mathbf{I}^l$. It defines an integral dominant weight $\Lambda_{\mathbf{Q}} = \sum_{p=1}^l \Lambda_{Q_p} \in \mathbf{P}^+$. The *Fock space* $\mathbf{F}(\mathbf{Q})_{\mathbf{I}}$ is the \mathbf{C} -vector space with basis $\{|\lambda, \mathbf{Q}\rangle_{\mathbf{I}} \mid \lambda \in \mathcal{P}^l\}$ called the *standard monomial basis*, and action of e_i, f_i for all $i \in \mathbf{I}$ given by

$$(2.2) \quad f_i(|\lambda, \mathbf{Q}\rangle_{\mathbf{I}}) = \sum_{\mu} |\mu, \mathbf{Q}\rangle_{\mathbf{I}}, \quad e_i(|\mu, \mathbf{Q}\rangle_{\mathbf{I}}) = \sum_{\lambda} |\lambda, \mathbf{Q}\rangle_{\mathbf{I}},$$

where the sums run over all partitions such that $\text{res}(\mu - \lambda, \mathbf{Q})_{\mathbf{I}} = i$. This endows $\mathbf{F}(\mathbf{Q})_{\mathbf{I}}$ with a structure of \mathfrak{g}' -module. The Fock space $\mathbf{F}(\mathbf{Q})_{\mathbf{I}}$ can also be equipped with a symmetric non-degenerate bilinear form $\langle \bullet, \bullet \rangle_{\mathbf{I}}$ for which the standard monomial basis is orthonormal. To avoid cumbersome notation, we shall omit the subscript \mathbf{I} when not necessary.

It is easy to see that each element of the standard monomial basis is a weight vector whose weight can be explicitly computed by the formula

$$\alpha_i^\vee(|\lambda, \mathbf{Q}\rangle) = (|\text{add}_i(\lambda, \mathbf{Q})| - |\text{rem}_i(\lambda, \mathbf{Q})|) |\lambda, \mathbf{Q}\rangle$$

for $i \in \mathbf{I}$. In particular, the vector $|\emptyset, \mathbf{Q}\rangle$ has weight $\Lambda_{\mathbf{Q}}$.

Proposition 2.1. — *The \mathfrak{g}' -submodule of $\mathbf{F}(\mathbf{Q})$ generated by $|\emptyset, \mathbf{Q}\rangle$ is isomorphic to $\mathbf{L}(\Lambda_{\mathbf{Q}})$. Furthermore, if $\mathbf{I} = \Lambda_\infty$, then $\mathbf{F}(\mathbf{Q}) = \mathbf{L}(\Lambda_{\mathbf{Q}})$. \square*

Using the minimal categorification $\mathcal{L}(\Lambda_{\mathbf{Q}})$ of $\mathbf{L}(\Lambda_{\mathbf{Q}})$, the map $\mathbf{L}(\Lambda_{\mathbf{Q}}) \rightarrow \mathbf{F}(\mathbf{Q})$ can be made more explicit. To explain this, let us first recall briefly the definition of the

Specht modules. Assume that \mathbf{R} has characteristic 0 and contains a primitive l -th root ζ of 1, so \mathbf{R} is a splitting field of the complex reflection group $G(l, 1, m)$. Let $\text{Irr}(\mathbf{R}\mathfrak{S}_m) = \{\phi_\lambda \mid \lambda \in \mathcal{P}_m\}$ be the standard labelling of the characters of the symmetric group. Then

$$\text{Irr}(\mathbf{R}G(l, 1, m)) = \{\mathcal{X}_\lambda \mid \lambda \in \mathcal{P}_m^l\}$$

is the labelling of the simple modules such that \mathcal{X}_λ is induced from the $G(l, 1, |\lambda_1|) \times \dots \times G(l, 1, |\lambda_l|)$ -module

$$\phi_{\lambda^{(1)}} \chi^0 \otimes \phi_{\lambda^{(2)}} \chi^1 \otimes \dots \otimes \phi_{\lambda^{(l)}} \chi^{l-1}.$$

Here, we denote by χ^p the one dimensional module of the $|\lambda_p|$ -th cartesian power of the cyclic group $G(l, 1, 1)$ given by the p -th power of the determinant, see, e.g., [22, Section 5.1.3]. Recall that (for every field \mathbf{R}) the \mathbf{R} -algebra $\mathbf{H}_{\mathbf{R}, m}^{\mathcal{Q}, v}$ is split and that it is semisimple if and only if we have, see, e.g., [40, Section 3.2],

$$\prod_{i=1}^m (1 + v + \dots + v^{i-1}) \prod_{a < b} \prod_{-m < r < m} (v^r Q_a - Q_b) \neq 0.$$

Thus, by Tits' deformation theorem, under the specialization $v \mapsto 1$ and $Q_b \mapsto \zeta^{b-1}$, the labelling of $\text{Irr}(\mathbf{R}G(l, 1, m))$ yields a canonical labelling

$$\text{Irr}(\mathbf{H}_{\mathbf{R}, m}^{\mathcal{Q}, v}) = \{S(\lambda)_{\mathbf{R}}^{\mathcal{Q}, v} \mid \lambda \in \mathcal{P}_m^l\}.$$

Now, if \mathbf{R} is a commutative domain with fraction field \mathbf{K} of characteristic 0 as above, we define the $\mathbf{H}_{\mathbf{R}, m}^{\mathcal{Q}, v}$ -module $S(\lambda)_{\mathbf{R}}^{\mathcal{Q}, v}$ as in [42, Section 2.4.3] or [22, Section 5.3], using $S(\lambda)_{\mathbf{K}}^{\mathcal{Q}, v}$ and the dominance order on \mathcal{P}_m^l , and if $\theta : \mathbf{R} \rightarrow \mathbf{k}$ is a ring homomorphism such that \mathbf{k} is the fraction field of $\theta(\mathbf{R})$ we set $S(\lambda)_{\mathbf{k}}^{\mathcal{Q}, v} = \mathbf{k} S(\lambda)_{\mathbf{R}}^{\mathcal{Q}, v}$. Then, we have the following, see, e.g., [42].

Proposition 2.2. — *Let \mathbf{R} be a field of characteristic 0 which contains a primitive l -th root of 1. The composition $[\mathcal{L}(\Lambda_Q)] \xrightarrow{\sim} \mathbf{L}(\Lambda_Q) \rightarrow \mathbf{F}(Q)$ obtained from Theorem 1.7 and Proposition 2.1 sends the class of $S(\lambda)_{\mathbf{R}}^{\mathcal{Q}, v}$ to the standard monomial $|\lambda, Q\rangle$.* \square

For each $p = 1, \dots, l$, let \mathbf{I}_p be the subquiver of \mathbf{I} corresponding to the subset $v^{\mathbf{Z}} Q_p$ of \mathbf{I} . We define a relation on $\{1, \dots, l\}$ by $i \sim j \iff \mathbf{I}_i = \mathbf{I}_j$. Let $\Omega = \{1, \dots, l\} / \sim$ be the set of equivalence classes for this relation. Given $p \in \Omega$, we denote by Q_p the tuple of $(Q_{i_1}, \dots, Q_{i_r})$ where (i_1, \dots, i_r) is the ordered set of elements in p . The decomposition $\mathbf{I} = \bigsqcup_{p \in \Omega} \mathbf{I}_p$ yields a canonical decomposition of Lie algebras $\mathfrak{g}'_{\mathbf{I}} = \bigoplus_{p \in \Omega} \mathfrak{g}'_{\mathbf{I}_p}$. The corresponding decomposition of Fock spaces is given in the following proposition.

Proposition 2.3. — *The map $|\lambda, Q\rangle_{\mathbf{I}} \mapsto \bigotimes_{p \in \Omega} |\lambda^p, Q_p\rangle_{\mathbf{I}_p}$ yields an isomorphism of $\mathfrak{g}'_{\mathbf{I}}$ -modules*

$$\mathbf{F}(Q)_{\mathbf{I}} \xrightarrow{\sim} \bigotimes_{p \in \Omega} \mathbf{F}(Q_p)_{\mathbf{I}_p}. \quad \square$$

2.3. Charged Fock spaces. — A *charged Fock space* is a pair $\mathbf{F}(s) = (\mathbf{F}(\mathbf{Q}), s)$ such that $s \in \mathbf{Z}^I$ is a charge of \mathbf{Q} , that is $\mathbf{Q} = (v^{s_1}, \dots, v^{s_l})$. Throughout this section, we will always assume that I is either of type A_∞ or a cyclic quiver. For more general quivers we can invoke Proposition 2.3 to reduce to that case.

2.3.1. The \mathfrak{g} -action on the Fock space. — The action of \mathfrak{g}' on $\mathbf{F}(\mathbf{Q})$ can be extended to an action of \mathfrak{g} when \mathbf{Q} admits a charge s . We describe this action in the case where v has finite order e , and $l = 1$. In that case $I = \mathbf{Z}$ is isomorphic to the cyclic quiver $A_{e-1}^{(1)}$ and the charge s is just an integer $d \in \mathbf{Z}$ such that $\mathbf{Q} = v^d$. If we fix the affine simple root to be α_1 , then $\mathbf{X} = \mathbf{P} \oplus \mathbf{Z}\delta$ and $\mathbf{X}^\vee = \mathbf{Q}^\vee \oplus \mathbf{Z}\delta$ with $\delta = \sum_{i \in I} \alpha_i$ and $\partial = \Lambda_1^\vee$, see Section 1.2.2.

For each tuple $s \in \mathbf{Z}^I$, we define the integers

$$(2.3) \quad \Delta(d, e) = (\bar{d}(1 - \bar{d}/e) + d(d/e - 1))/2, \quad \Delta(s, 1) = \sum_{i \in I} s_i(s_i - 1)/2,$$

where \bar{d} is the residue of d modulo e in $[0, e - 1]$. Then, we define the action of the derivation ∂ on $\mathbf{F}(d) = (\mathbf{F}(\mathbf{Q}), d)$ by

$$\partial(|\lambda, \mathbf{Q}\rangle) = -(n_1(\lambda, \mathbf{Q})_I + \Delta(d, e))|\lambda, \mathbf{Q}\rangle,$$

so that the highest weight of $\mathbf{F}(d)$ is $\Lambda_{\mathbf{Q}} - \Delta(d, e)\delta$. For this action the weight of a standard basis element is

$$(2.4) \quad \text{wt}(|\lambda, \mathbf{Q}\rangle) = \Lambda_{\mathbf{Q}} - \sum_{i \in I} n_i(\lambda, \mathbf{Q})_I \alpha_i - \Delta(d, e)\delta,$$

compare with Uglov's formula [46, Proposition 3.7(W_1)].

We now describe the action of the affine Weyl group of \mathfrak{g} on $\mathbf{F}(d)$. For $i \in I \setminus \{1\}$, we denote by $\alpha_i^{\text{cl}} = 2\Lambda_i - \Lambda_{iw} - \Lambda_{i^{-1}}$ and $\Lambda_i^{\text{cl}} = \Lambda_i - \Lambda_1$ the i -th simple root and fundamental weight of \mathfrak{sl}_e . These (classical) simple roots span the lattice of classical roots \mathbf{Q}^{cl} . It is a sublattice of \mathbf{P} of rank $e - 1$. The affine Weyl group of \mathfrak{g} is $W = \mathfrak{S}_I \ltimes \mathbf{Q}^{\text{cl}}$. It acts linearly on \mathbf{X} . We will denote by $t_\gamma \in \text{End}(\mathbf{X})$ the action of an element $\gamma \in \mathbf{Q}^{\text{cl}}$ as defined in [31, Chapter 6], i.e., for each $\alpha \in \mathbf{X}$ we set

$$(2.5) \quad t_\gamma(\alpha) = \alpha + (\alpha : \delta)\gamma - (\alpha : \gamma)\delta - \frac{1}{2}(\alpha : \delta)(\gamma : \gamma)\delta$$

where $(\bullet : \bullet)$ is the standard symmetric non-degenerate bilinear form on $\mathbf{X} \times \mathbf{X}$. For each tuple $s \in \mathbf{Z}^I$ we consider the element $\pi_s = \sum_{i \in I} (s_i - s_{iw})\Lambda_i$. If $s \in \mathbf{Z}^I(d)$, then $\pi_s - \Lambda_{\mathbf{Q}}^{\text{cl}} \in \mathbf{Q}^{\text{cl}}$ and we can consider the corresponding operator $t_{\pi_s - \Lambda_{\mathbf{Q}}^{\text{cl}}} \in \text{End}(\mathbf{X})$, from which we can compute the weight of $|\lambda, \mathbf{Q}\rangle$, and the action of W as follows.

Proposition 2.4. — *Let λ, v be two partitions. Let $\lambda_{[e]}$ and $w_e(\lambda)$ be the e -core and the e -weight of λ .*

(a) Let $s \in \mathbf{Z}^1(d)$ be such that $(\lambda^{[e]}, s) = \tau_e(\lambda, d)$. The weight of $|\lambda, Q\rangle$ is

$$\text{wt}(|\lambda, Q\rangle) = t_{\pi_s - \Lambda_Q^{\text{cl}}}(\Lambda_Q) - w_e(\lambda) \delta.$$

(b) The weights of $|\lambda, Q\rangle$ and $|\nu, Q\rangle$ are W -conjugate if and only if $w_e(\lambda) = w_e(\nu)$.

Proof. — By Uglov's formula [46, Proposition 3.7(W₄)], the weight of $|\lambda_{[e]}, Q\rangle$ in $\mathbf{F}(d)$ is

$$(2.6) \quad \text{wt}(|\lambda, Q\rangle) = \Lambda_1 + \pi_s - \Delta(s, 1) \delta - w_e(\lambda) \delta.$$

Using (2.5), we deduce that

$$t_{\pi_s - \Lambda_Q^{\text{cl}}}(\Lambda_Q) = \Lambda_1 + \pi_s - \frac{1}{2}(\pi_s + \Lambda_Q^{\text{cl}} : \pi_s - \Lambda_Q^{\text{cl}}) \delta,$$

hence

$$\text{wt}(|\lambda, Q\rangle) = t_{\pi_s - \Lambda_Q^{\text{cl}}}(\Lambda_Q) - w_e(\lambda) \delta$$

proving part (a). Part (b) is a direct consequence of (a), using the fact that δ is fixed by W and that the set of weights of the $\widehat{\mathfrak{sl}}_e$ -module $\mathbf{F}(d)$ is contained in

$$\{t_\gamma \Lambda_Q - k\delta \mid k \in \mathbf{N}, \gamma \in Q^{\text{cl}}\},$$

see, e.g., [31, Lemma 12.6]. □

In the particular case where the charge is $d = 0$ (forcing Q to be 1), the weight of $|\lambda, 1\rangle$ is given by

$$(2.7) \quad \text{wt}(|\lambda, 1\rangle) = t_{\pi_s}(\Lambda_1) - w_e(\lambda) \delta,$$

where the tuple s is such that $(\emptyset, s) = \tau_e(\lambda_{[e]}, 0)$. Therefore weight spaces are parametrized by pairs (ν, w) where ν is an e -core and w is a non-negative integer. The basis element $|\lambda, 1\rangle$ is in the weight space corresponding to $(\lambda_{[e]}, w_e(\lambda))$. We will see later that these weight spaces correspond to the unipotent ℓ -blocks of finite unitary groups $\text{GU}_n(q)$ when e is the order of $-q$ modulo ℓ .

Remark 2.5. — For a future use, note that if $\tau_e(\lambda, d) = (\emptyset, s)$ with $s \in \mathbf{Z}^e(d)$, then the formulas (2.4), (2.6) imply that $n_1(\lambda, Q)_I = \Delta(s, 1) - \Delta(d, e)$.

2.3.2. The crystal of the Fock space. — We explain here how to associate an abstract crystal to a charged Fock space $\mathbf{F}(s)$. By Proposition 2.3, we can assume that I is either cyclic or of type A_∞ . Then, the abstract crystal of $\mathbf{F}(s)$ is the abstract crystal associated with Uglov's canonical basis of $\mathbf{F}(s)$. We assume that the reader is familiar with [45]. Another good reference is [46].

When I has type A_∞ , Uglov's basis coincides with the standard monomial basis and the discussion is trivial in that case. We will therefore assume that v has finite order e and $I = v^{\mathbf{Z}}$, so that I has type $A_{e-1}^{(1)}$. Let u be a formal parameter and $A = \mathbf{C}[u, u^{-1}]$. The \mathfrak{g} -module $\mathbf{F}(\mathbf{Q})$ admits a quantum deformation $\mathbf{F}_u(s)$ with an A -lattice $\mathbf{F}_A(s)$, which is a free A -module with basis $\{|\mu, s\rangle \mid \mu \in \mathcal{P}^l\}$. It is equipped with an integrable representation of $U_A(\mathfrak{g})$ which is given by the formulas (29), (35), (36) in [45]. Note that the action of the Chevalley generators e_i and f_i depends on the choice of the charge s . The representation of \mathfrak{g} on $\mathbf{F}(s)$ given in Section 2.3.1 is recovered by specializing the parameter u to 1.

Uglov has constructed a remarkable A -basis of $\mathbf{F}_A(s)$ in [45, p. 283]

$$\mathbf{B}_u^+(s) = \{\mathbf{b}_u^+(\mu, s) \mid \mu \in \mathcal{P}^l\}.$$

It depends on s and it is a lower global basis for the representation of $U_u(\mathfrak{g})$ on $\mathbf{F}_u(s)$. In [45], this basis is denoted by the symbol

$$\mathcal{G}^+(s_l) = \{\mathcal{G}^+(\lambda_l, s_l) \mid \lambda_l \in \mathcal{P}^l\}.$$

In order to match Uglov's parameters with ours, we set $q, l, n, s_l = u, l, e, s$ in the definition of $\mathcal{G}^+(s_l)$ to get our basis $\mathbf{B}_u^+(s)$.

Next, we consider the pairing (\bullet, \bullet) on $\mathbf{F}_u(s)$ defined in [46, Section 4.3]. It is a modified version of the pairing (\bullet, \bullet) introduced in Section 2.2. Then, let $\mathbf{B}_u^\vee(s) = \{\mathbf{b}_u^\vee(\mu, s) \mid \mu \in \mathcal{P}^l\}$ be the $\mathbf{C}(u)$ -basis of $\mathbf{F}_u(s)$ dual to $\mathbf{B}_u^+(s)$ with respect to the bilinear form (\bullet, \bullet) . By Kashiwara's theory of global bases, we deduce that $\mathbf{B}_u^\vee(s)$ is an upper global basis of $\mathbf{F}_u(s)$, compare [46, Lemma 4.13]. Let $\mathbf{B}^\vee(s) = \{\mathbf{b}^\vee(\mu, s) \mid \mu \in \mathcal{P}^l\}$ be the specialization at $u = 1$ of $\mathbf{B}_u^\vee(s)$, with the obvious labeling of its elements. It is a perfect basis of the usual Fock space $\mathbf{F}(s) = \mathbf{F}_u(s)|_{u=1}$ by Proposition 1.11.

Next, we equip the set of l -partitions with the abstract crystal structure $B(s) = (\mathcal{P}^l, \tilde{e}_i, \tilde{f}_i)$ defined in [30]. See [12] for a reformulation closer to our notations. Let $B(s) = \{b(\mu, s) \mid \mu \in \mathcal{P}^l\}$ be the obvious labeling. The operators \tilde{e}_i, \tilde{f}_i are described in a combinatorial way: we have $\tilde{f}_i(b(\mu, s)) = b(\gamma, s)$ if and only if γ is obtained from μ by adding a *good i -node*. The definition of a good i -node depends on the charge s . See [46, Section 3.4.2] for more details.

We will need the following well-known result.

Proposition 2.6. — *The map $\mathbf{b}^\vee(\mu, s) \in \mathbf{B}^\vee(s) \mapsto b(\mu, s) \in B(s)$ is a crystal isomorphism.*

Proof. — The proposition follows from [30]. More precisely, it is proved there that the formulae (27), (33), (34) in [45] for the action of the quantum group on $\mathbf{F}_\Lambda(s)$ imply that the pair formed by $\mathcal{L}(s) = \bigoplus_{\mu \in \mathcal{P}^l} \mathbf{C}[u]|\mu, s\rangle$ and the basis $\{|\mu, s\rangle \bmod u \mathcal{L}(s) \mid \mu \in \mathcal{P}^l\}$ of $\mathcal{L}(s)/u\mathcal{L}(s)$ is a lower crystal basis of $\mathbf{F}_u(s)$ and that the assignment $|\mu, s\rangle \mapsto b(\mu, s)$ is a crystal isomorphism onto $\mathbf{B}(s)$. In other words, the abstract crystal associated with the lower global basis $\mathbf{B}_\Lambda^+(s)$ is canonically isomorphic to $\mathbf{B}(s)$, *i.e.*, the map $\mathbf{b}_u^+(\mu, s) \in \mathbf{B}_\Lambda^+(s) \mapsto b(\mu, s) \in \mathbf{B}(s)$ realizes this isomorphism. To conclude, we use the canonical bijections $\mathbf{B}_u^+(s) \rightarrow \mathbf{B}_u^\vee(s) \rightarrow \mathbf{B}^\vee(s)$ whose composition is an isomorphism of crystals as well. \square

3. Unipotent representations

In this section we record standard results on unipotent representations of finite reductive groups in non-defining characteristic. A good reference is [6].

3.1. Basics. — By an ℓ -modular system we will mean a triple $(\mathbf{K}, \mathcal{O}, \mathbf{k})$ where \mathbf{K} is a field of characteristic zero, \mathcal{O} is a complete discrete valuation ring with fraction field \mathbf{K} , and \mathbf{k} is the residue field of \mathcal{O} with $\text{char}(\mathbf{k}) = \ell$. When working with representations of a finite group Γ , we will always assume that $(\mathbf{K}, \mathcal{O}, \mathbf{k})$ is a *splitting ℓ -modular system* for Γ , which means that \mathbf{K} and \mathbf{k} are splitting fields for all subgroups of Γ . When Γ comes from an algebraic group in characteristic p , we will in addition assume that $\ell \neq p$. This case is usually referred to as the *non-defining characteristic* case.

Let \mathbf{R} be any commutative domain (with 1) and Γ be a finite group. We will assume that p is invertible in \mathbf{R} (and \mathcal{O}). Let $\mathbf{R}\Gamma$ denote the group ring of Γ over \mathbf{R} . For any subset $S \subseteq \Gamma$ such that $|S|$ is invertible in \mathbf{R} , let e_S be the element $e_S = |S|^{-1} \sum_{g \in S} g$ in $\mathbf{R}\Gamma$. If S is a subgroup of Γ then e_S is an idempotent. If \mathbf{R} is not a field, an $\mathbf{R}\Gamma$ -module which is free as an \mathbf{R} -module will be called an $\mathbf{R}\Gamma$ -lattice.

The \mathbf{R} -module of class functions $\Gamma \rightarrow \mathbf{R}$ is denoted by $\mathbf{R}\text{Irr}(\mathbf{K}\Gamma)$ or $\mathbf{R}\text{Irr}(\Gamma)$. If \mathbf{R} is a field, this vector space is endowed with the canonical scalar product, $\langle -, - \rangle_\Gamma$, for which the set of irreducible characters $\text{Irr}(\mathbf{K}\Gamma)$ of $\mathbf{K}\Gamma$ is an orthonormal basis.

3.2. Unipotent $\mathbf{K}\mathbf{G}$ -modules. — Let \mathbf{G} be a connected reductive group over $\overline{\mathbf{F}}_q$ with a Frobenius endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ defining an \mathbf{F}_q -structure on \mathbf{G} . Fix a parabolic subgroup \mathbf{P} of \mathbf{G} and an F -stable Levi complement \mathbf{L} of \mathbf{P} . We *do not* assume \mathbf{P} to be F -stable. Write $\mathbf{L} = \mathbf{L}^F$ and $\mathbf{G} = \mathbf{G}^F$.

From now on, we assume that \mathbf{K} contains the field of ℓ -adic numbers \mathbf{Q}_ℓ . Let $\mathbf{R}_{\mathbf{L}\mathbf{CP}}^\mathbf{G}$ and ${}^*\mathbf{R}_{\mathbf{L}\mathbf{CP}}^\mathbf{G}$ denote respectively the Lusztig induction and restriction maps from $\mathbf{Z}\text{Irr}(\mathbf{K}\mathbf{L})$ to $\mathbf{Z}\text{Irr}(\mathbf{K}\mathbf{G})$. We will assume that the Mackey formula holds for $\mathbf{R}_{\mathbf{L}\mathbf{CP}}^\mathbf{G}$ and ${}^*\mathbf{R}_{\mathbf{L}\mathbf{CP}}^\mathbf{G}$, which we know for the groups we will focus on later (see [3] for more details). Under this condition, the Lusztig induction and restriction do not depend on the choice

of the parabolic subgroup \mathbf{P} , see [10, Chapter 6]. We abbreviate $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} = R_{\mathbf{L}}^{\mathbf{G}}$ and $*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} = *R_{\mathbf{L}}^{\mathbf{G}}$.

Let \mathbf{T} be an F -stable maximal torus of \mathbf{G} and let \mathbf{N} be the normalizer of \mathbf{T} in \mathbf{G} . Fix an F -stable Borel subgroup \mathbf{B} of \mathbf{G} containing \mathbf{T} . Write $\mathbf{B} = \mathbf{B}^F$, $\mathbf{T} = \mathbf{T}^F$ and $\mathbf{N} = \mathbf{N}^F$. The groups \mathbf{B} , \mathbf{N} form a reductive BN-pair of \mathbf{G} with Weyl group $\mathbf{W} = \mathbf{W}_{\mathbf{G}}$ given by $\mathbf{W} = \mathbf{N}/\mathbf{T}$. Since \mathbf{B} , \mathbf{N} are stable by F and \mathbf{G} is connected, the finite groups \mathbf{B} , \mathbf{N} form a split BN-pair of \mathbf{G} whose Weyl group $W = W(\mathbf{T})$ is given by $W = \mathbf{W}^F = \mathbf{N}/\mathbf{T}$ (see [15, Section 4] for more details).

The G -conjugacy classes of F -stable maximal tori of \mathbf{G} are parametrized by the F -conjugacy classes in \mathbf{W} . For each $w \in \mathbf{W}$ let \mathbf{T}_w be an F -stable maximal torus in the G -conjugacy class parametrized by w . Under conjugation by some element of \mathbf{G} , the pair (\mathbf{T}_w, F) is identified with the pair (\mathbf{T}, wF) . In particular, we have $\mathbf{T}_w := \mathbf{T}_w^F \simeq \mathbf{T}^{wF}$ and $W(\mathbf{T}_w) \simeq \mathbf{W}^{wF}$. The virtual characters $R_{\mathbf{T}_w}^{\mathbf{G}}(1)$ obtained by induction of the trivial representation of the tori \mathbf{T}_w are called the *Deligne-Lusztig characters* of \mathbf{G} . They satisfy the following orthogonality relations:

$$\langle R_{\mathbf{T}_w}^{\mathbf{G}}(1), R_{\mathbf{T}_{w'}}^{\mathbf{G}}(1) \rangle_{\mathbf{G}} = \begin{cases} |\mathbf{W}^{wF}| & \text{if } w \text{ and } w' \text{ are } F\text{-conjugate in } \mathbf{W}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.1. — *An irreducible \mathbf{KG} -module is unipotent if its character, say χ , occurs as a constituent of a Deligne-Lusztig character $R_{\mathbf{T}_w}^{\mathbf{G}}(1)$ for some element $w \in W$, i.e., if we have $\langle \chi, R_{\mathbf{T}_w}^{\mathbf{G}}(1) \rangle_{\mathbf{G}} \neq 0$.*

We denote by $\mathbf{KG}\text{-umod}$ the full subcategory of $\mathbf{KG}\text{-mod}$ consisting of the modules which are sums of irreducible unipotent modules. The objects of this category are the *unipotent \mathbf{KG} -modules*.

3.3. Unipotent \mathbf{kG} -modules and ℓ -blocks. — As a result of the lifting of idempotents, the blocks of $\mathcal{O}\mathbf{G}$ and \mathbf{kG} correspond by reduction. Both are usually called the *ℓ -blocks* of \mathbf{G} . For $\mathbf{R} = \mathcal{O}$ or \mathbf{k} , any block \mathbf{B} of \mathbf{RG} is of the form $\mathbf{B} = \mathbf{RG} \cdot b$, where b is a central primitive idempotent of \mathbf{RG} . The unit b of \mathbf{B} is called the *block idempotent* of \mathbf{B} . We will also call *block of $\mathbf{RG}\text{-mod}$ associated with \mathbf{B}* the Serre subcategory generated by the simple modules which are not killed by b . The ℓ -blocks of \mathbf{G} induce a partition of $\text{Irr}(\mathbf{KG})$ such that the piece associated with \mathbf{B} is the set of all irreducible characters χ of \mathbf{KG} with $\chi(b) = \chi(1)$. If $\chi \in \text{Irr}(\mathbf{KG})$, we will write $\mathbf{B}(\chi) \subseteq \text{Irr}(\mathbf{KG})$ for the piece containing χ . When there is no risk of confusion, we will also call $\mathbf{B}(\chi)$ an ℓ -block of \mathbf{G} .

Definition 3.2. — *An ℓ -block of $\mathcal{O}\mathbf{G}$ is unipotent if it contains at least one unipotent \mathbf{KG} -module. A simple \mathbf{kG} -module is unipotent if it lies in a unipotent block of \mathbf{kG} .*

We denote by $\mathbf{kG}\text{-umod}$ the Serre subcategory of $\mathbf{kG}\text{-mod}$ generated by the simple unipotent \mathbf{kG} -modules. It corresponds to the sum of unipotent blocks of $\mathbf{kG}\text{-mod}$. The *unipotent \mathbf{kG} -modules* are by definition the objects of this category.

Recall that $(\mathbf{K}, \mathcal{O}, \mathbf{k})$ is a splitting ℓ -modular system for \mathbf{G} . To this system one can associate a decomposition map $d_{\mathcal{O}\mathbf{G}} : [\mathbf{KG}\text{-mod}] \rightarrow [\mathbf{kG}\text{-mod}]$. It follows from the definition above that the constituents of the ℓ -reduction of a unipotent \mathbf{KG} -module are all unipotent \mathbf{kG} -modules. In other words, the restriction of the decomposition map to the unipotent characters induces a map $d_{\mathcal{U}} : [\mathbf{KG}\text{-umod}] \rightarrow [\mathbf{kG}\text{-umod}]$.

From now on we will assume that the centre $\mathbf{Z}(\mathbf{G})$ of \mathbf{G} is connected. The following proposition ensures that under some mild assumption on ℓ , the irreducible unipotent characters form a so-called *basic set* for the unipotent blocks.

Proposition 3.3 ([17], [19]). — *Assume ℓ is good for \mathbf{G} . Then the map $d_{\mathcal{U}}$ is a linear isomorphism $[\mathbf{KG}\text{-umod}] \xrightarrow{\sim} [\mathbf{kG}\text{-umod}]$.* \square

Given a positive integer f , let Φ_f be the f th cyclotomic polynomial. If $\mathbf{T} \subset \mathbf{G}$ is an \mathbf{F} -stable torus, then the order of $\mathbf{T} := \mathbf{T}^{\mathbf{F}}$ is a polynomial in q with coefficients in \mathbf{Z} . We say that \mathbf{T} (and by extension \mathbf{T}) is a Φ_f -torus if this polynomial is a power of Φ_f . An \mathbf{F} -stable Levi subgroup $\mathbf{L} \subseteq \mathbf{G}$ is *f -split* if it is the centralizer in \mathbf{G} of a Φ_f -torus. A *unipotent f -pair* is a pair (\mathbf{L}, χ) where \mathbf{L} is an f -split Levi subgroup and χ is an irreducible unipotent \mathbf{KL} -module. The pair (\mathbf{L}, χ) is *f -cuspidal* if for every proper f -split Levi subgroup $\mathbf{M} \subseteq \mathbf{L}$ we have ${}^*\mathbf{R}_{\mathbf{M}}^{\mathbf{L}}(\chi) = 0$.

Now assume that f is the smallest positive integer such that ℓ divides $q^f - 1$. In other words, f is the order of the class of q in \mathbf{k} . Under the assumption that ℓ is good, unipotent ℓ -blocks correspond to f -cuspidal f -pairs (see for example [6, Theorem 22.9]).

Theorem 3.4. — *Assume ℓ is good for \mathbf{G} , and $\ell \neq 3$ if \mathbf{G} has a component of type ${}^3\mathbf{D}_4$. Then there is a bijection between the \mathbf{G} -conjugacy classes of unipotent f -cuspidal f -pairs and the set of unipotent ℓ -blocks of \mathbf{G} which takes the class of (\mathbf{L}, χ) to the ℓ -block $\mathbf{B}_{\mathbf{L}, \chi}$ such that the irreducible unipotent characters in $\mathbf{B}_{\mathbf{L}, \chi}$ are exactly the irreducible constituents of $\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}(\chi)$.* \square

3.4. Harish-Chandra series. — Assume now that the parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ considered in Section 3.2 is \mathbf{F} -stable. In that case the group \mathbf{L} is \mathbf{G} -conjugate to a standard Levi subgroup of \mathbf{G} . Let $\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}$ be the corresponding Harish-Chandra induction and restriction functors between $\mathbf{RL}\text{-mod}$ and $\mathbf{RG}\text{-mod}$. Let $\mathbf{P} = \mathbf{P}^{\mathbf{F}}$ and $\mathbf{U} = \mathbf{U}^{\mathbf{F}}$, where $\mathbf{U} \subset \mathbf{P}$ is the unipotent radical of \mathbf{P} . Notice that the Harish-Chandra induction is the special case of Lusztig induction for 1-split Levi subgroups.

The order of \mathbf{U} is a power of q , hence it is invertible in \mathbf{R} . Thus, for all $\mathbf{M} \in \mathbf{RL}\text{-mod}$, $\mathbf{N} \in \mathbf{RG}\text{-mod}$ we have

$$\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}(\mathbf{M}) = \mathbf{RG} \cdot e_{\mathbf{U}} \otimes_{\mathbf{RL}} \mathbf{M} \quad \text{and} \quad {}^*\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}(\mathbf{N}) = e_{\mathbf{U}} \cdot \mathbf{RG} \otimes_{\mathbf{RG}} \mathbf{N}.$$

We will say that the functors R_L^G and $*R_L^G$ are *represented* by the (RG, RL) -bimodule $RG \cdot e_U$ and the (RL, RG) -bimodule $e_U \cdot RG$ respectively.

Here are some well-known basic properties of the functors $R_L^G, *R_L^G$, see for example [6, Proposition 1.5].

- (a) $R_L^G, *R_L^G$ do not depend on P ,
- (b) $R_L^G, *R_L^G$ are exact and left and right adjoint to one another,
- (c) if $L \subseteq M \subseteq G$ there are isomorphisms of functors $R_L^G = R_M^G R_L^M$ and $*R_L^G = *R_L^M *R_M^G$.

Let $R = K$ or k . An irreducible RG -module E is *cuspidal* if $*R_L^G(E) = 0$ for all standard Levi subgroups $L \subsetneq G$. A *cuspidal pair* of RG is a pair (L, E) where L is as above and $E \in \text{Irr}(RL)$ is cuspidal. Since the group L is uniquely recovered from E , from now on we may omit it from the notation. Then, the set $\text{Irr}(RG, E) \subseteq \text{Irr}(RG)$ consisting of the constituents of the head of $R_L^G(E)$ is equal to the set of constituents of the socle of $R_L^G(E)$ and is called the *Harish-Chandra series* of (L, E) . The R -algebra $\mathcal{H}(RG, E) = \text{End}_{RG}(R_L^G(E))^{\text{op}}$ is its *ramified Hecke algebra*. We have the following facts:

- (d) the Harish-Chandra series form a partition of $\text{Irr}(RG)$,
- (e) the functor $\mathfrak{F}_{R_L^G(E)}$ as defined in Section 1.1 yields a bijection $\text{Irr}(RG, E) \longleftrightarrow^{1:1} \text{Irr}(\mathcal{H}(RG, E))$.

See [28], [21] and [22, Theorems 4.2.6, 4.2.9] for details.

The following result is well-known. It follows from the fact that the ℓ -rational series are stable by Harish-Chandra induction (see for example [4, Theorem 10.3] for a more general statement).

Proposition 3.5. — *If $R = K$ or k , then the Harish-Chandra induction and restriction functors preserve the category of unipotent RG -modules.*

4. Finite unitary groups

This section is devoted to the construction of categorical actions on the category of unipotent representations of finite unitary groups $GU_n(q)$. It contains the main results of this paper.

Let R be a commutative domain with unit. Under mild assumptions on R , we construct in Section 4.2 a representation datum on the abelian category

$$RG\text{-mod} := \bigoplus_{n \geq 0} RGU_n(q)\text{-mod}$$

given by Harish-Chandra induction and restriction. It consists of the adjoint pair (E, F) of the functors themselves, together with natural transformations X and T of F and F^2 . The construction of the latter are similar to the case of $GL_n(q)$ given in [7], and X should be thought of as a Jucys-Murphy element, whereas T satisfies a Hecke relation with parameter q^2 . When R is an extension of \mathbf{Q}_ℓ or \mathbf{F}_ℓ , the categorical datum restricts to the category

\mathcal{U}_R of unipotent representations of $\mathbf{RG}\text{-mod}$. On this smaller category, the eigenvalues of \mathbf{X} are powers of $-q$, which hints that the Lie algebra \mathfrak{g} that should act on $[\mathcal{U}_R]$ corresponds to the quiver with vertices $(-q)^{\mathbf{Z}}$ and arrows given by multiplication by q^2 .

We prove that the representation datum lifts indeed to a categorical action of \mathfrak{g} on \mathcal{U}_R . We start in Section 4.5 with the case where $\mathbf{Q}_\ell \subset \mathbf{R}$. Then $\mathfrak{g} \simeq (\mathfrak{sl}_{\mathbf{Z}})^{\oplus 2}$ and $[\mathcal{U}_R]$ is isomorphic to a direct sum of level 2 Fock spaces, each of which corresponds to an ordinary Harish-Chandra series. When $\mathbf{F}_\ell \subset \mathbf{R}$, the category \mathcal{U}_R is no longer semisimple but we show in Section 4.5 a compatibility between weights for the action of \mathfrak{g} and unipotent ℓ -blocks which yields our second categorification result. The situation depends on the parity of e , the order of $-q$ modulo ℓ . When e is even (linear prime case), \mathfrak{g} is a subalgebra of $(\widehat{\mathfrak{sl}_{e/2}})^{\oplus 2}$ and each ordinary Harish-Chandra series categorifies a level 2 Fock space for \mathfrak{g} . When e is odd (unitary prime case), $\mathfrak{g} \simeq \widehat{\mathfrak{sl}_e}$ and weight spaces correspond to unipotent ℓ -blocks, which are now transverse to the ordinary Harish-Chandra series.

For studying the weight space decomposition of $[\mathcal{U}_R]$ as well as the action of the affine Weyl group we use the action of a bigger Lie algebra \mathfrak{g}_\circ , which comes from Harish-Chandra induction and restriction for general linear groups. Going from linear to unitary groups by Ennola duality introduces signs for this action (see Lemma 4.18). It is to be expected that the action of \mathfrak{g}_\circ on $[\mathcal{U}_R]$ lifts to an action by triangulated functors on $D^b(\mathcal{U}_R)$ coming from Lusztig induction, although we will not use it.

We give two main applications of our categorical construction. In Section 4.6 we use Chuang-Rouquier's framework to produce derived equivalences between blocks, from which we deduce Broué's abelian defect group conjecture when e is even. Finally, we show in Section 4.7 that the crystal graph of the level 2 Fock spaces that we categorified coincide with the Harish-Chandra branching graph. This solves a recent conjecture of Gerber-Hiss-Jacon [24] and gives a combinatorial way to compute the modular Harish-Chandra series and the parameters of the various ramified Hecke algebras.

4.1. Definition. — Fix a positive integer n . We equip the reductive algebraic group $\mathbf{GL}_n = \mathbf{GL}_n(\overline{\mathbf{F}}_q)$ with the standard Frobenius map $F_q : \mathbf{GL}_n \rightarrow \mathbf{GL}_n$, $(a_{ij}) \mapsto (a_{ij}^q)$, given by raising every coefficient to the q th power. The finite general linear group $\mathbf{GL}_n(q)$ is given by the fixed points of \mathbf{GL}_n under F_q . In this section we will work with a twisted version of this group obtained by twisting the Frobenius map. Let \mathbf{J}_n be the $n \times n$ matrix with entry 1 in $(i, n - i + 1)$ and zero elsewhere. We will often write $\mathbf{J} = \mathbf{J}_n$ when there is no risk of confusion on the size of the matrices. We define a new Frobenius map \mathbf{F} on \mathbf{GL}_n , called the *twisted* Frobenius map, by setting $\mathbf{F} = F_q \circ j$ where $j(g) = \mathbf{J} \cdot {}^t g^{-1} \cdot \mathbf{J}$ for each $g \in \mathbf{GL}_n$. The *finite unitary group* $\mathbf{G}_n = \mathbf{GU}_n(q)$ is then given by

$$\mathbf{G}_n = (\mathbf{GL}_n)^{\mathbf{F}} = \{g \in \mathbf{GL}_n \mid \mathbf{F}(g) = g\}.$$

Since $\mathbf{F}^2 = (F_q)^2$ we have $\mathbf{G}_n \subset \mathbf{GL}_n$, where we abbreviate $\mathbf{GL}_n = \mathbf{GL}_n(q^2) := (\mathbf{GL}_n)^{\mathbf{F}^2}$. By convention we also define $\mathbf{G}_0 = \{1\}$ to be the trivial group.

We equip \mathbf{GL}_n with the standard split BN-pair such that \mathbf{B} is the subgroup of upper triangular matrices and \mathbf{N} is the subgroup of all monomial matrices. Since \mathbf{B}, \mathbf{N} are stable by F and \mathbf{GL}_n is connected, the groups $\mathbf{B} = \mathbf{B}^F$, $\mathbf{N} = \mathbf{N}^F$ form a split BN-pair of the finite group G_n . Let \mathbf{T} be the diagonal torus in \mathbf{GL}_n and $\mathbf{T} = \mathbf{T}^F$. Let $\mathbf{W} = \mathbf{W}_n$ be the Weyl group of \mathbf{GL}_n and $W = W_n$ be the Weyl group of G_n . We have $\mathbf{W} \simeq \mathfrak{S}_n$, and F induces on \mathbf{W} the automorphism given by conjugation with the longest element w_0 . We will embed \mathbf{W} in G_n using permutation matrices, so that w_0 corresponds to J . We have $W = \mathbf{W}^F = C_{\mathbf{W}}(w_0)$. It is a Weyl group of type B_m if $n = 2m$ or $2m + 1$.

Let $\varepsilon_1, \dots, \varepsilon_n$ be the characters of \mathbf{T} such that $t = \text{diag}(\varepsilon_1(t), \varepsilon_2(t), \dots, \varepsilon_n(t))$. The roots (resp. simple roots) of \mathbf{GL}_n are given by $\{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ (resp. $\{\varepsilon_r - \varepsilon_{r+1}\}$). Let $s_r = (r, r+1)$ be the simple reflection in \mathbf{W} associated with the simple root $\alpha_r = \varepsilon_r - \varepsilon_{r+1}$. The action of F on the roots induces an automorphism σ of the Dynkin diagram of \mathbf{GL}_n such that

$$F \circ \alpha_r^\vee = q\sigma(\alpha_r)^\vee \quad \text{with } \sigma(\alpha_r) = \alpha_{n-r}.$$

For each root α , let \mathbf{U}_α and $\alpha^\vee \in \text{Hom}(\mathbf{G}_m, \mathbf{T})$ be the corresponding root subgroup and cocharacter. We also choose an isomorphism $u_\alpha : \mathbf{G}_a \xrightarrow{\sim} \mathbf{U}_\alpha$ such that $F(u_{\sigma(\alpha)}(t)) = u_\alpha(-t^q)$. Note that a one-parameter subgroup of \mathbf{T} has either $q^2 - 1$ or $q + 1$ elements, and a root subgroup of G_n has either q or q^2 elements.

The standard Levi subgroups $L_{r,m}$ of G_n are parametrized by pairs (r, m) where r is a non-negative integer and $m = (m_1, m_2, \dots, m_s)$ is a tuple of positive integers such that $n = r + 2 \sum_{u=1}^s m_u$. The group $L_{r,m}$ consists of all matrices of \mathbf{GL}_n which belong to G_n and are of block-diagonal type

$$\prod_{u=1}^s \mathbf{GL}_{m_u} \times \mathbf{GL}_r \times \prod_{u=s}^1 \mathbf{GL}_{m_u}.$$

Consequently we have a group isomorphism $L_{r,m} \simeq G_r \times \prod_u \mathbf{GL}_{m_u}$. If m is a positive integer we abbreviate $L_{r,1^m} = L_{r,(1^m)}$ and $L_{r,m} = L_{r,(m)}$.

4.2. The representation datum on RG-mod . — Let \mathbf{R} be a commutative domain with unit. We assume that $q(q^2 - 1)$ is invertible in \mathbf{R} . Using parabolic induction and restriction, we show in this section how to construct a representation datum on

$$\text{RG-mod} := \bigoplus_{n \in \mathbf{N}} \text{RG}_n\text{-mod}.$$

Fix a positive integer n . Parabolic (or Harish-Chandra) induction provides functors between $L\text{-mod}$ and $G_n\text{-mod}$ for any standard Levi subgroup $L = L_{r,m} \subset G_n$. Since we want functors between $G_r\text{-mod}$ and $G_n\text{-mod}$ we will consider a slight variation of the usual parabolic induction.

Let $0 \leq r < n$. We denote by V_r the unipotent radical of the standard parabolic subgroup $P_{r,1} \subset G_{r+2}$ with Levi complement $L_{r,1}$. Let $U_r \subset G_{r+2}$ be the subgroup given by

$$(4.1) \quad U_r = V_r \rtimes \mathbf{F}_{q^2}^\times = \begin{pmatrix} * & * & \cdots & \cdots & * \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & * \\ & & & & * \end{pmatrix}$$

so that $P_{r,1} = V_r \rtimes L_{r,1} \cong U_r \rtimes G_r$. If $n - r$ is even, under this identification, we may write $U_{n,r} = U_{n-2} \rtimes \cdots \rtimes U_r$ as a subgroup in G_n and we define $e_{n,r} = e_{U_{n,r}}$ to be the corresponding idempotent. In particular, we have $e_{r+2,r} = e_{U_r}$. We embed G_r into the Levi subgroup $L_{r,1^m} = G_r \times \mathrm{GL}_1^m$ in the obvious way. This yields an embedding $G_r \subset G_n$ and functors

$$F_{n,r} = \mathrm{RG}_n \cdot e_{n,r} \otimes_{\mathrm{RG}_r} - : \mathrm{RG}_r\text{-mod} \longrightarrow \mathrm{RG}_n\text{-mod},$$

$$E_{n,r} = e_{n,r} \cdot \mathrm{RG}_n \otimes_{\mathrm{RG}_n} - : \mathrm{RG}_n\text{-mod} \longrightarrow \mathrm{RG}_r\text{-mod}.$$

Note that $F_{n,r}$ can be seen as the composition of the inflation $G_r\text{-mod} \longrightarrow L_{r,1^m}\text{-mod}$ with the parabolic induction from $L_{r,1^m}$ to G_n .

An endomorphism of the functor $F_{n,r}$ can be represented by an $(\mathrm{RG}_n, \mathrm{RG}_r)$ -bimodule endomorphism of $\mathrm{RG}_n \cdot e_{n,r}$, or equivalently by an element of $e_{n,r} \cdot \mathrm{RG}_n \cdot e_{n,r}$ centralizing RG_m . Thus, the elements

$$X_{r+2,r} = (-q)^r e_{r+2,r}(1, r+2) e_{r+2,r},$$

$$T_{r+4,r} = q^2 e_{r+4,r}(1, 2)(r+3, r+4) e_{r+4,r}$$

define respectively natural transformations of the functors $F_{r+2,r}$ and $F_{r+4,r}$. We set

$$E = \bigoplus_{r \geq 0} E_{r+2,r}, \quad F = \bigoplus_{r \geq 0} F_{r+2,r},$$

$$X = \bigoplus_{r \geq 0} X_{r+2,r}, \quad T = \bigoplus_{r \geq 0} T_{r+4,r}.$$

Proposition 4.1. — *The endomorphisms $X \in \mathrm{End}(F)$ and $T \in \mathrm{End}(F^2)$ satisfy the following relations:*

$$(a) \quad 1_F T \circ T 1_F \circ 1_F T = T 1_F \circ 1_F T \circ T 1_F,$$

$$(b) \quad (T + 1_{F^2}) \circ (T - q^2 1_{F^2}) = 0,$$

$$(c) \quad T \circ (1_F X) \circ T = q^2 X 1_F.$$

Proof. — The relation (a) comes from the usual braid relations. For (b), we compute

$$\begin{aligned} (T_{r+4,r})^2 &= q^4 e_{r+4,r}(1, 2)(r+3, r+4) e_{r+4,r}(1, 2)(r+3, r+4) e_{r+4,r} \\ &= q^4 e_{r+4,r} e_V e_{r+4,r}, \end{aligned}$$

where $V \subset G_{r+4}$ is the subgroup consisting of the matrices with diagonal entries equal to 1 and off-diagonal entries equal to zero, except for the entries $(2, 1)$ and $(r+4, r+3)$, *i.e.*

$$V = \begin{pmatrix} 1 & & & \\ * & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & * & 1 \end{pmatrix}.$$

The group V is the root subgroup $u_{-\alpha}(\mathbf{F}_{q^2})$ of G_{r+4} associated with the negative root say $-\alpha$. The corresponding simple reflexion s_α is given by the permutation $(1, 2)(r+3, r+4)$. Let B_α be the (finite) Borel subgroup of upper triangular matrices in the copy of GL_2 in G_{r+4} associated with α . We have $B_\alpha u_{-\alpha}(t) B_\alpha = B_\alpha s_\alpha B_\alpha$ if and only if $t \neq 0$, so that

$$q^2 e_{B_\alpha} e_{U_{-\alpha}} e_{B_\alpha} = e_{B_\alpha} + (q^2 - 1) e_{B_\alpha s_\alpha B_\alpha}$$

in RGL_2 . Now, the image of B_α through the embedding $GL_2 \subset G_{r+4}$ lies in $U_{r+4,r}$. As a consequence,

$$U_{r+4,r} e_V U_{r+4,r} = q^{-2} e_{r+4,r} + (1 - q^{-2}) e_{r+4,r}(1, 2)(r+3, r+4) e_{r+4,r},$$

which yields the expected formula

$$(T_{r+4,r})^2 = (q^2 - 1) T_{r-4} + q^2 e_{r+4,r}.$$

Finally, to prove (c) we must compute $T_{r+4,r} X_{r+2,r} T_{r+4,r}$. Using the group V introduced above, it equals

$$(-q)^{r+4} e_{r+4,r} e_V(1, 2)(r+3, r+4)(2, r+3)(1, 2)(r+3, r+4) e_V e_{r+4,r},$$

which simplifies to $(-q)^{r+4} e_{r+4,r} e_V(1, r+4) e_V e_{r+4,r}$. Let $V' = {}^{(1,r+4)}V \subset U_{r+4,r}$. The only off-diagonal entries which are non-zero in V' are the entries $(2, r+4)$, $(1, r+3)$ which lie at the top right corner

$$V' = \begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & \ddots & \\ & & & 1 \\ & & & & 1 \end{pmatrix}.$$

By Chevalley's commutator formula, we have $[V', V] \subset U_{r+2,r}$. This proves that

$$e_V(1, r+4) e_{V e_{r+4,r}} = (1, r+4) e_{V e_{r+4,r}}.$$

Finally, by moving e_V to the left, we obtain

$$T_{r+4,r} X_{r+2,r} T_{r+4,r} = (-q)^{r+4} e_{r+4,r} (1, r+4) e_{r+4,r} = q^2 X_{r+4,r+2},$$

from which (c) follows. \square

Corollary 4.2. — *The tuple (E, F, X, T) defines a representation datum on $\mathbf{RG}\text{-mod} = \bigoplus_{n \in \mathbf{N}} \mathbf{RG}_n\text{-mod}$.* \square

The next step will be to show that this categorical datum yields a \mathfrak{g} -representation on the full subcategory of unipotent modules of $\mathbf{RG}\text{-mod}$ (see Theorems 4.14 and 4.21).

Remark 4.3. — The reader might argue that we did not check that X was invertible. Nevertheless, we shall only be working with unipotent modules over a field, in which case the eigenvalues of X are powers of $-q$, hence non-zero (see Theorem 4.12). This will ensure that the restriction of X to this category is indeed invertible.

4.3. *The categories of unipotent modules \mathcal{U}_K and \mathcal{U}_k .* — From now on, we fix a prime number ℓ such that $\ell \nmid q(q^2 - 1)$, and an ℓ -modular system (K, \mathcal{O}, k) with $\mathbf{Q}_\ell \subset K$ and $\overline{\mathbf{F}}_\ell \subset k$.

Throughout the following sections, we will denote by d, e and f the order of $q^2, -q$ and q in k . In particular $e \neq 1, 2$. If e is odd, then $d = e$ and $f = 2e$; if e is even, then $d = e/2$ and either $f = e/2$ if $e \equiv 2 \pmod{4}$ or $f = e$ if $e \equiv 0 \pmod{4}$.

4.3.1. *The category \mathcal{U}_K .* — Fix a positive integer n . By [38], the irreducible unipotent \mathbf{KG}_n -modules are labelled by partitions of n . Their character can be directly constructed from the Deligne-Lusztig characters. Namely, for each $w \in \mathfrak{S}_n$, fix an F -stable maximal torus $\mathbf{T}_w \subset \mathbf{GL}_n$ in the G_n -conjugacy class parametrized by w , with the convention that $\mathbf{T}_1 = \mathbf{T}$. Then the class function

$$(4.2) \quad \chi_\lambda = |\mathfrak{S}_n|^{-1} \sum_{w \in \mathfrak{S}_n} \phi_\lambda(w w_0) R_{\mathbf{T}_w}^{\mathbf{GL}_n}(1) \in \mathbf{Z}\text{Irr}(\mathbf{KG}_n)$$

is, up to a sign, an irreducible unipotent character. Since the cohomology groups are defined over $\mathbf{Q}_\ell \subset K$ and since any character of \mathfrak{S}_n has rational values, either χ_λ or $-\chi_\lambda$ is the character of some irreducible \mathbf{KG}_n -module which we denote by E_λ . By abuse of notation we will still denote by E_λ its isomorphism class.

Recall that $G_0 = \{1\}$. We define the category of unipotent \mathbf{KG} -modules by

$$\mathcal{U}_K = \bigoplus_{n \in \mathbf{N}} \mathbf{KG}_n\text{-umod}.$$

This category is abelian semisimple. >From the previous discussion we have $\text{Irr}(\mathcal{U}_K) = \{E_\lambda\}_{\lambda \in \mathcal{P}}$, where by convention $\text{Irr}(KG_0) = \{E_\emptyset\}$.

4.3.2. The category \mathcal{U}_k . — Using the ℓ -modular system we have decomposition maps $d_{\mathcal{O}G_n}$ which by Proposition 3.3 restrict to linear isomorphisms

$$d_{\mathcal{O}G_n} : [KG_n\text{-umod}] \xrightarrow{\sim} [kG_n\text{-umod}].$$

Note that we do not need to assume that KG_n is a split algebra since we are only interested in unipotent KG_n -modules, which are all absolutely irreducible.

For unitary groups, this map is actually unitriangular with respect to the basis of irreducible modules and the dominance order.

Theorem 4.4 ([16]). — *There is a unique labelling $\{D_\lambda\}_{\lambda \in \mathcal{P}_n}$ of the unipotent simple kG_n -modules such that*

$$d_{\mathcal{O}G_n}([E_\lambda]) \in [D_\lambda] + \sum_{v > \lambda} \mathbf{Z}[D_v]$$

where $>$ is the dominance order on partitions of n . □

We define the category of unipotent kG -modules by

$$\mathcal{U}_k = \bigoplus_{n \in \mathbf{N}} kG_n\text{-umod}.$$

This is an abelian category which is not semisimple. The isomorphism classes of simple objects are $\text{Irr}(\mathcal{U}_k) = \{D_\lambda\}_{\lambda \in \mathcal{P}}$ and the decomposition map yields a \mathbf{Z} -linear isomorphism $d_{\mathcal{U}} : [\mathcal{U}_K] \xrightarrow{\sim} [\mathcal{U}_k]$. The following result is a consequence of the unitriangularity of this map.

Proposition 4.5. — *Given a partition $\lambda \vdash n$ there is a unique $\mathcal{O}G_n$ -lattice \tilde{E}_λ such that*

- (a) $K\tilde{E}_\lambda = E_\lambda$ as a KG_n -module,
- (b) $V_\lambda := k\tilde{E}_\lambda$ is an indecomposable kG_n -module with head isomorphic to D_λ ,
- (c) $d_{\mathcal{U}}([E_\lambda]) = [V_\lambda]$.

Proof. — By [8, Example 6.16], any system of orthogonal idempotents of kG_n lifts to a system of orthogonal idempotents of $\mathcal{O}G_n$. Let P_λ be the projective cover of D_λ , and let \tilde{P}_λ be a lift of P_λ to a projective indecomposable $\mathcal{O}G_n$ -lattice. By [16], we have

$$K\tilde{P}_\lambda = E_\lambda \oplus \bigoplus_{\mu < \lambda} E_\mu^{\oplus d_{\lambda,\mu}} \quad \text{with } d_{\lambda,\mu} \in \mathbf{N}.$$

Let $e_\chi \in KG$ be the idempotent associated with an irreducible KG -character χ . Set $e_{\neq \lambda} = \sum_{\chi \neq [E_\lambda]} e_\chi$. The $\mathcal{O}G_n$ -submodule $N := \tilde{P}_\lambda \cap e_{\neq \lambda} K\tilde{P}_\lambda$ of \tilde{P}_λ is pure. We define $\tilde{E}_\lambda = \tilde{P}_\lambda / N$.

It is an $\mathcal{O}G_n$ -lattice. The module $k\tilde{E}_\lambda$ has a simple head equal to D_λ , because $\text{hd}(k\tilde{P}_\lambda) = D_\lambda$. Furthermore, we have $K\tilde{E}_\lambda = K\tilde{P}_\lambda / e_{\neq \lambda} K\tilde{P}_\lambda = E_\lambda$.

Now, let us concentrate on the unicity of \tilde{E}_λ . Let \tilde{E} be an $\mathcal{O}G_n$ -lattice satisfying the properties (a), (b) above. Let $\phi : k\tilde{E} \twoheadrightarrow D_\lambda$ be the obvious surjective map. Since \tilde{P}_λ is projective, there is a morphism $\psi : \tilde{P}_\lambda \longrightarrow \tilde{E}$ such that the following triangle commutes

$$\begin{array}{ccc} k\tilde{P}_\lambda & \longrightarrow & D_\lambda \\ & \searrow k\psi & \uparrow \phi \\ & & k\tilde{E}. \end{array}$$

Since $k\tilde{E}$ has a simple head and the k -algebra kG_n is finite dimensional, the map ϕ is an essential epimorphism. Thus $k\psi$ is onto, and so is ψ by Nakayama's Lemma. We are left with proving that $\text{Ker}(\psi) = N$. Since the module \tilde{E} is free over \mathcal{O} , it is enough to check that $\text{Ker}(K\psi) = e_{\neq \lambda} K\tilde{P}_\lambda$.

To prove this, observe first that since \tilde{P}_λ is projective and \tilde{E} is free over \mathcal{O} , the \mathcal{O} -module $\text{Hom}_{\mathcal{O}G_n}(\tilde{P}_\lambda, \tilde{E})$ is free and we have

$$k \text{Hom}_{\mathcal{O}G_n}(\tilde{P}_\lambda, \tilde{E}) = \text{Hom}_{kG_n}(k\tilde{P}_\lambda, k\tilde{E}) = \text{Hom}_{kG_n}(k\tilde{P}_\lambda, D_\lambda) = k.$$

We deduce that the \mathcal{O} -module $\text{Hom}_{\mathcal{O}G_n}(\tilde{P}_\lambda, \tilde{E})$ is free of rank 1 and generated by the map ψ . Consequently

$$K\psi = K \text{Hom}_{\mathcal{O}G_n}(\tilde{P}_\lambda, \tilde{E}) = \text{Hom}_{KG_n}(K\tilde{P}_\lambda, K\tilde{E}) = \text{Hom}_{KG_n}(K\tilde{P}_\lambda, E_\lambda)$$

and the claim is proved. \square

Remark 4.6. — In the case where V_λ is simple, property (b) is superfluous and V_λ is automatically isomorphic to D_λ . This is for example true when E_λ is cuspidal (see [20, Theorem 6.10]).

4.3.3. Blocks of \mathcal{U}_k . — A block of \mathcal{U}_k is an indecomposable summand of \mathcal{U}_k . Therefore blocks of \mathcal{U}_k correspond to the unipotent blocks of kG_n where n runs over \mathbf{N} . These were first obtained in [13], before the general classification was given in [5] (see Proposition 3.4).

Recall that e is the order of $-q$ modulo ℓ . Given a partition λ , we defined in Section 2.1.3 its e -core $\lambda_{[e]}$, its e -quotient $\lambda^{[e]}$ and its e -weight $w_e(\lambda) = |\lambda^{[e]}|$.

Proposition 4.7. — *The unipotent ℓ -blocks coincide with the fibers of the surjective map $E_\lambda \longmapsto (\lambda_{[e]}, w_e(\lambda)) \in \mathbf{Z}^e(0) \times \mathbf{N}$.* \square

Recall that $\nu \mapsto \nu_{[\ell]}$ induces a bijection between ℓ -cores and $\mathbf{Z}^\ell(0)$. Given ν an ℓ -core we will denote by $B_{\nu,w}$ or $B_{\nu_{[\ell]},w}$ the unipotent ℓ -block containing all the unipotent characters E_λ such that ν is the ℓ -core of λ and $w_\ell(\lambda) = w$. It is a block of kG_n with $n = |\nu| + \ell w$.

It also follows from the classification of blocks that when $w < \ell$ and $\ell \neq 1$, the defect group of $B_{\nu,w}$ is an elementary abelian ℓ -group of rank w (see [5]). In particular, when $w = 0$ the module D_λ is simple and projective, isomorphic to V_λ , and when $w = 1$ the defect group of $B_{\nu,w}$ is a cyclic group. The structure of such blocks was determined in [14].

4.3.4. The weak Harish-Chandra series. — For this section we assume that R is one of the fields K or k .

Let $r, m \geq 0$ and $n = r + 2m$. Recall that $L_{r,1^m} \simeq G_r \times (GL_1)^m$. The finite group $(GL_1)^m$ is a central subgroup of $L_{r,1^m}$ with order $(q^2 - 1)^m$. Since the unipotent characters have the centre in their kernel and $\ell \nmid q^2 + 1$, we deduce that the inflation from G_r to $L_{r,1^m}$ yields an equivalence between RG_r -mod and $RL_{r,1^m}$ -mod. This equivalence intertwines the functors $E_{n,r}, F_{n,r}$ with the parabolic restriction and induction ${}^*R_{L_{r,1^m}}^{G_n}, R_{L_{r,1^m}}^{G_n}$. Therefore working with \mathcal{U}_R and the functors E and F is the same as working in the usual framework of unipotent representations and Harish-Chandra induction/restriction from Levi subgroups. Note however that we do not consider all the standard Levi subgroups, but only the ones that are conjugate to $L_{r,1^m}$. Therefore one needs to consider a slight variation of the usual Harish-Chandra theory.

Definition 4.8 ([24]). — Fix a non-negative integer n .

- (a) An RG_n -module D is weakly cuspidal if ${}^*R_L^{G_n}(D) = 0$ for any Levi subgroup $L \subsetneq G_n$ which is G_n -conjugate to a subgroup of the form $L_{r,1^m}$.
- (b) A weakly cuspidal pair of RG_n is a pair which is G_n -conjugate to $(L_{r,1^m}, D)$ for some $n = r + 2m$ and some weakly cuspidal irreducible RG_r -module D which is viewed as a $kL_{r,1^m}$ -module by inflation.
- (c) The weak Harish-Chandra series $W\text{Irr}(RG_n, D)$ of RG_n determined by the weakly cuspidal pair (L, D) is the set of the constituents of the head of $R_L^{G_n}(D)$ and it coincides with the set of the constituents of its socle.

If D is a unipotent RG_r -module, then D is weakly cuspidal if and only if $E(D) = 0$. Moreover, if D is irreducible, the weak Harish-Chandra series coincides with the set of irreducible constituents in the head of $F^m D$. Therefore it makes sense to define the weak Harish-Chandra of \mathcal{U}_R by

$$W\text{Irr}(RG, D) = \bigsqcup_{n \geq r} W\text{Irr}(RG_n, D).$$

It is the set of irreducible constituents in the head of $F^k D$ for some $k \geq 0$ (or equivalently in the socle). As in the case of the usual theory, the weak Harish-Chandra series of \mathcal{U}_R form a partition of $\text{Irr}(\mathcal{U}_R)$ (see [24, Proposition 2.3] for a proof).

Proposition 4.9. — *Assume that $R = K$ or k . Then*

$$\text{Irr}(\mathcal{U}_R) = \bigsqcup \text{WIrr}(\text{RG}, D)$$

where the union runs over the set of isomorphism classes of weakly cuspidal unipotent irreducible modules D . \square

When $R = K$, weakly cuspidal unipotent modules coincide with cuspidal modules and were determined in [36] (see also Corollary 4.11). One of the main results in this paper is the classification of the weakly cuspidal modules in the case where $R = k$ and $\ell \nmid q(q^2 - 1)$.

4.4. The \mathfrak{g}_∞ -representation on \mathcal{U}_K . — In this section we show how the categorical datum defined in Section 4.2 yields a categorical representation on \mathcal{U}_R in the case where $R = K$. This is achieved by translating in our framework the theory of Lusztig and Howlett-Lehrer on endomorphism algebras of induced representations.

4.4.1. Action of E and F . — Since every unipotent character is a linear combination of Deligne-Lusztig characters—we call such a class function a *uniform function*—it is well-known how to compute the action of E and F on the category \mathcal{U}_K .

Lemma 4.10. — *Let λ be a partition. Then*

$$[E](E_\lambda) = \sum_{\mu} E_{\mu} \quad \text{and} \quad [F](E_\lambda) = \sum_{\mu'} E_{\mu'}$$

where μ (resp. μ') runs over the partitions that are obtained from λ by removing (resp. adding) a 2-hook.

Proof. — By adjunction, it is enough to prove the formula for the restriction functor E . Let $n = |\lambda|$ and let \mathbf{L} be the standard Levi subgroup with $\mathbf{L}^F = \mathbf{L} = \mathbf{L}_{n-2,1}$. Recall from Section 4.3.1 that \mathbf{T}_w denotes an F -stable maximal torus of \mathbf{GL}_n in the G_n -conjugacy class parametrised by the F -conjugacy class of w in \mathbf{W} .

For $x \in G_n$, if ${}^x\mathbf{T}_w \subset \mathbf{L}$ then w is F -conjugate to an element v of the Weyl group $\mathbf{W}_{\mathbf{L}}$ of \mathbf{L} (which is isomorphic to \mathfrak{S}_{n-2}). In addition, given $v \in \mathbf{W}_{\mathbf{L}}$, the set $\mathbf{L} \backslash \{x \in G_n \mid {}^x\mathbf{T}_v \subset \mathbf{L}\} / \mathbf{T}_v$ is in bijection with $\mathbf{W}^{vF} / \mathbf{W}_{\mathbf{L}}^{vF}$. Thus the Mackey formula [10, Theorem 11.13] yields

$$(4.3) \quad {}^*\mathbf{R}_{\mathbf{L}}^{G_n} \mathbf{R}_{\mathbf{T}_w}^{\mathbf{GL}_n}(1) = \frac{|\mathbf{W}^{vF}|}{|\mathbf{W}_{\mathbf{L}}^{vF}|} \mathbf{R}_{\mathbf{T}_v}^{\mathbf{L}}(1).$$

Now if w is not F-conjugate to an element of \mathbf{W}_L , then ${}^*\mathbf{R}_L^{G_n}\mathbf{R}_{\mathbf{T}_w}^{\mathbf{GL}_n}(1) = 0$. Note also that w is F-conjugate to v if and only if wv_0 is conjugate to vw_0 , which we will denote by $vw_0 \sim wv_0$. In this case the F-conjugacy class of v in \mathbf{W} (resp. in \mathbf{W}_L) has size $|\mathbf{W}|/|\mathbf{W}^{vF}|$ (resp. $|\mathbf{W}_L|/|\mathbf{W}_L^{vF}|$). We deduce from (4.2), (4.3) the following formula for the restriction of the character χ_λ :

$$\begin{aligned} {}^*\mathbf{R}_L^{G_n}(\chi_\lambda) &= \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \phi_\lambda(wv_0) {}^*\mathbf{R}_L^{G_n}\mathbf{R}_{\mathbf{T}_w}^{\mathbf{GL}_n}(1) \\ &= \frac{1}{|\mathbf{W}|} \sum_{\substack{w \in \mathbf{W} \\ v \in \mathbf{W}_L \\ vw_0 \sim wv_0}} \frac{|\mathbf{W}_L^{vF}|}{|\mathbf{W}_L|} \phi_\lambda(wv_0) {}^*\mathbf{R}_L^{G_n}\mathbf{R}_{\mathbf{T}_w}^{\mathbf{GL}_n}(1) \\ &= \frac{1}{|\mathbf{W}|} \sum_{\substack{w \in \mathbf{W} \\ v \in \mathbf{W}_L \\ vw_0 \sim wv_0}} \frac{|\mathbf{W}^{vF}|}{|\mathbf{W}_L|} \phi_\lambda(vw_0) \mathbf{R}_{\mathbf{T}_v}^L(1) \\ &= \frac{1}{|\mathbf{W}_L|} \sum_{v \in \mathbf{W}_L} \phi_\lambda(vw_0) \mathbf{R}_{\mathbf{T}_v}^L(1). \end{aligned}$$

To conclude, we write $\phi_\lambda(vw_0) = \phi_\lambda(vw_L(1, n))$, where w_L is the longest element of \mathbf{W}_L , and we apply the Murnaghan-Nakayama rule for the 2-cycle $(1, n)$. This gives a decomposition of ${}^*\mathbf{R}_L^{G_n}(\chi_\lambda)$ in terms of the χ_μ 's, where μ is obtained from λ by removing a 2-hook. By the Murnaghan-Nakayama rule every such χ_μ appears with multiplicity 1 or -1 . Note however that there is no need to worry about the signs in this formula: ${}^*\mathbf{R}_L^{G_n}(\chi_\lambda)$ is a virtual character but $[E](E_\lambda)$ is a true character. \square

Since the only partitions from which one cannot remove any 2-hook are the triangular partitions, we have the following immediate corollary.

Corollary 4.11 ([36]). — A unipotent \mathbf{KG}_n module E_λ is weakly cuspidal if and only if there exists $t \geq 0$ such that $n = t(t+1)/2$ and $\lambda = \Delta_t := (t, t-1, t-2, \dots, 1)$.

We will set $E_t := E_{\Delta_t}$. Consequently, the weakly cuspidal pairs are, up to conjugation, the pairs $(L_{r,1^m}, E_t)$ with $r = t(t+1)/2$. The partition into series is thus

$$\mathrm{Irr}(\mathcal{U}_K) = \bigsqcup_{t \in \mathbf{N}} \mathrm{WIrr}(\mathbf{KG}, E_t).$$

Note that it is proven in [36] that this is also the partition into usual Harish-Chandra series. However we shall not use this fact.

4.4.2. The ramified Hecke algebra. — Let r, m be non-negative integers and $n = r + 2m$. As mentioned in Section 4.3.4, the inflation from G_r to $L_{r,1^m}$ yields an equivalence between KG_r -umod and $\mathrm{KL}_{r,1^m}$ -umod which intertwines the functor $F_{n,r}$ with the parabolic induction $R_{L_{r,1^m}}^{G_n}$. In particular, we have a canonical isomorphism

$$\mathcal{H}(\mathrm{KG}_n, E_t) := \mathrm{End}_{\mathrm{KG}_n}(F^m(E_t))^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{\mathrm{KG}_n}(R_{L_{r,1^m}}^{G_n}(E_t))^{\mathrm{op}}.$$

Now recall from Section 1.3.1 that to the categorical datum (E, F, X, T) is attached a map $\phi_{F^m} : \mathbf{H}_{K,m}^{q^2} \rightarrow \mathrm{End}(F^m)$. The evaluation of this map at the module E_t yields a K -algebra homomorphism

$$\phi_{K,m} : \mathbf{H}_{K,m}^{q^2} \rightarrow \mathcal{H}(\mathrm{KG}_n, E_t), \quad X_k \mapsto X_k(E_t), \quad T_l \mapsto T_l(E_t).$$

By [36] and [29], the algebra $\mathcal{H}(\mathrm{KG}_n, E_t)$ is isomorphic to a Hecke algebra $\mathbf{H}_{K,m}^{Q_t, q^2}$ of type B_m with

$$(4.4) \quad Q_t = \begin{cases} ((-q)^{-1-t}, (-q)^t) & \text{if } t \text{ is even,} \\ ((-q)^t, (-q)^{-1-t}) & \text{if } t \text{ is odd.} \end{cases}$$

We show that the previous map provides such an isomorphism.

Theorem 4.12. — *Let $t, m \geq 0$ and $n = t(t+1)/2 + 2m$. Then the map $\phi_{K,m}$ factors through a K -algebra isomorphism*

$$\mathbf{H}_{K,m}^{Q_t, q^2} \xrightarrow{\sim} \mathcal{H}(\mathrm{KG}_n, E_t).$$

Proof. — Write $Q_t = (Q_1, Q_2)$ and $X = X(E_t)$. We must check that the operator X on $F(E_t)$ satisfies the relation

$$(X - (-q)^{-1-t})(X - (-q)^t) = 0.$$

Then the invertibility of the morphism $\mathbf{H}_{K,m}^{Q_t, q^2} \rightarrow \mathcal{H}(\mathrm{KG}_n, E_t)$ follows from [29, Proposition 3.9]. In fact, it is shown in the proof of [29, Proposition 3.18] that X satisfies the relation

$$(4.5) \quad (X - \epsilon_t (-q)^{-1-t})(X - \epsilon_t (-q)^t) = 0$$

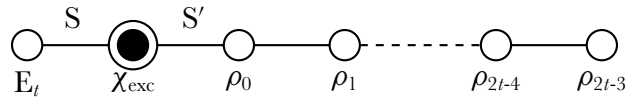
for some $\epsilon_t = \pm 1$. Therefore we must show that $\epsilon_t = 1$, which we will do by induction on t . First observe that the eigenvalues of $X_{2,0}$ on $R_{G_0}^{G_2}(K)$ and of $X_{3,1}$ on $R_{G_1}^{G_3}(K)$ are 1, $(-q)^{-1}$ and $-q$, $(-q)^{-2}$ respectively. So the eigenvalues of $X(E_0)$, $X(E_1)$ on $F(E_0)$, $F(E_1)$ are powers of $-q$. Now fix $t > 1$ and assume that for all $t > s \geq 0$ the eigenvalues of $X(E_s)$ on $F(E_s)$ are powers of $-q$. We will show that the eigenvalues of $X(E_t)$ are also powers of $-q$ using the modular representation theory of unitary groups.

Recall that K is chosen with respect to an ℓ -modular system (K, \mathcal{O}, k) . Since the parametrization of unipotent characters does not depend on ℓ and (E, F, X, T) are defined over $\mathbf{Z}[1/q(q^2 - 1)]$, we can first choose a specific prime number ℓ and prove that the eigenvalues of $X(E_t)$ are powers of $-q$ modulo ℓ . We choose ℓ to be odd and such that the order of $-q$ in k^\times is $e := 2t - 1$.

Fong-Srinivasan determined the repartition of the unipotent characters into ℓ -blocks, and the structure of the blocks with cyclic defect (see Section 4.3.3 and [14]). It follows from their result that the module E_t lies in a block with cyclic defect, since only one e -hook can be removed from Δ_t , the corresponding partition.

Let $r = t(t + 1)/2$. We refer to Section 2.1 for the combinatorics of partitions and β -sets. Let $\beta = \beta_0(\Delta_t)$ be the set of β -numbers of the charged partition $(\Delta_t, 0)$. It equals $\beta = \{t, t - 2, \dots, -t\} \cup \{-t - 1, -t - 2, \dots\}$. As mentioned above, only one e -hook can be removed from Δ_t , therefore E_t lies in a block with cyclic defect. Removing the e -hook corresponds to replacing t by $t - e = -t + 1$ in β to obtain the set $\bar{\beta} = \{t - 2, t - 4, \dots, -t + 2\} \cup \{-t + 1, -t, \dots\}$. This is exactly the set of β -numbers of the charged partition $(\Delta_{t-2}, 0)$. From now on we will assume that all the partitions are charged with the charge 0. Now, the unipotent characters in the block of E_t correspond to the partitions obtained by adding an e -hook to Δ_{t-2} . In terms of β -sets, they are obtained by replacing x by $x + e$ in $\bar{\beta}$, with $x \in \{-3t + 4, -3t + 6, \dots, t - 4, t - 2\} \cup \{-t + 1\}$. We denote by β_x the corresponding β -set. For $l = 0, 1, \dots, 2t - 3$, let ρ_l be the unipotent character of G_r whose partition corresponds to the β -set $\beta_{2l-3t+4}$. The unipotent character E_t correspond to $\beta = \beta_{-t+1}$. Let χ_{exc} be the sum of the non-unipotent characters in the block of E_t , called *exceptional characters*.

The Brauer tree of the block is described in [14, Section 6]. Recall that the vertices in the Brauer tree are labeled by χ_{exc} and the unipotent characters of the block, while edges are labeled by simple kG_r -modules. The simple kG_r -module S labels the edge $\chi \rightarrow \chi'$ if and only if S is a composition factor of the ℓ -reduction of both χ and χ' . The set of the composition factors of the ℓ -reduction of χ is exactly the set of labels of the edges adjacent to χ . The parity of x determines on which branch of the Brauer tree the character associated to the β -set β_x lies. The Brauer tree is



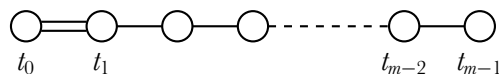
Since E_t is weakly cuspidal, *i.e.* $E(E_t) = 0$, so is S . The character $E(\rho_l)$ can be explicitly computed using Lemma 4.10, and the combinatorics on the $\beta_{2l-3t+4}$ (see Section 2.1 for the effect of removing 2-hooks from a β -set). Two cases arise: if $l = 0$ (resp. $l = 2t - 3$), then $E(\rho_l)$ is irreducible, and the corresponding β -set is obtained by changing $-3t + 6$ to $-3t + 4$ (resp. $3t - 3$ to $3t - 5$); otherwise $E(\rho_l)$ has two constituents whose β -sets are obtained by changing $2l - 3t + 6$ to $2l - 3t + 4$ or $2l - t + 3$ to $2l - t + 1$. We deduce that $\sum (-1)^l E(\rho_l) = 0$ in $[KG_{r-2}\text{-mod}]$. Since $[S'] = \sum (-1)^l d_{\mathcal{L}}([\rho_l])$ in $[kG_r\text{-mod}]$, this implies that the kG_r -module S' is also weakly cuspidal. Therefore the

two composition factors of the ℓ -reduction of the exceptional characters are weakly cuspidal, which forces the exceptional characters to be weakly cuspidal as well.

Let χ be one of the exceptional characters. Since χ is weakly cuspidal, we can use [29] to show that the operator $X(\chi)$ on $F(\chi)$ satisfies a quadratic relation and the product of its eigenvalues equals $-q^{-1}$. In particular, if one of the eigenvalues of $X(\chi)$ is a power of $-q$ modulo ℓ then so is the other. Now ρ_0 is not cuspidal, because E_t is the unique cuspidal unipotent character of KG_r . Therefore ρ_0 lies in the Harish-Chandra series $\mathrm{Irr}(\mathrm{KG}_r, E_s)$ for some $s < t$. The induction hypothesis implies that the parameters of $\mathcal{H}(\mathrm{KG}_r, E_s)$ are Q_s and $(-q)^2$. Hence the eigenvalues of $X(\rho_0)$ are powers of $-q$ modulo ℓ by (4.4), see Section 1.4.1. Since the ℓ -reductions of $F(\rho_0)$ and $F(\chi)$ share a common composition factor, we deduce that the eigenvalues of $X(\chi)$ are powers of $-q$ modulo ℓ . On the other hand, the ℓ -reduction of E_t is isomorphic to S , which is a composition factor of the ℓ -reduction of χ . Therefore, the eigenvalues of $X(E_t)$ are also powers of $-q$ modulo ℓ .

Finally, since $e = 2t - 1$ is odd and $-q$ is of order e modulo ℓ , -1 is not a power of $-q \equiv -q^{-t-1}$ modulo ℓ . We deduce that the eigenvalues of $X(E_t)$ are powers of $-q$. \square

4.4.3. Parametrization of the weak Harish-Chandra series of \mathcal{U}_K . — Let $W(B_m)$ be the Weyl group of type B_m , and t_0, t_1, \dots, t_{m-1} be the generators corresponding to the following Dynkin diagram



Let us first recall the construction of the irreducible characters of $W(B_m)$ (see for example [23, Definition 5.4.4] or [39, Section I.7]). We denote by σ_m the linear character of $W(B_m)$ such that $\sigma_m(t_0) = -1$ and $\sigma_m(t_i) = 1$ for all $i > 0$. Given $\lambda \vdash m$ a partition of m , we write $\tilde{\phi}_\lambda$ for the inflation to $W(B_m)$ of the irreducible character of \mathfrak{S}_m corresponding to λ . Given a, b such that $a + b = m$, one can consider the subgroups $\mathfrak{S}_a \times \mathfrak{S}_b \subset W(B_a) \times W(B_b)$ of $W(B_m)$ where $W(B_a) \times W(B_b)$ is a parabolic subgroup generated by $\{t_0, t_1, \dots, t_{a-1}\} \cup \{t_{a+1}, \dots, t_{m-1}\}$, and $W(B_a) \times W(B_b)$ is obtained by adding the reflection $t_a \cdots t_0 \cdots t_a$. The irreducible character of $W(B_m)$ associated to a bipartition (λ, μ) of m is

$$\mathcal{X}_{\lambda, \mu} = \mathrm{Ind}_{W(B_{|\lambda|}) \times W(B_{|\mu|})}^{W(B_m)} (\tilde{\phi}_\lambda \boxtimes \sigma_{|\mu|} \tilde{\phi}_\mu).$$

For example, $\mathcal{X}_{(m), \emptyset}$ is the trivial character, whereas $\mathcal{X}_{\emptyset, (1^m)}$ is the sign character. We also have $\sigma_m = \mathcal{X}_{\emptyset, (m)}$ and more generally $\mathcal{X}_{\lambda, \mu} = \sigma_m \mathcal{X}_{\mu, \lambda}$.

By Tits deformation theorem, the specialization $q \mapsto 1$ yields a bijection

$$\mathrm{Irr}(\mathbf{H}_{K, m}^{Q, q^2}) \xleftrightarrow{1:1} \mathrm{Irr}(W(B_m))$$

from which we obtain a canonical labelling of the irreducible representations of $\mathbf{H}_{\mathbf{K},m}^{Q_t, q^2}$. We write $\text{Irr}(\mathbf{H}_{\mathbf{K},m}^{Q_t, q^2}) = \{S(\lambda, \mu)_{\mathbf{K}}^{Q_t, q^2}\}_{(\lambda, \mu) \in \mathcal{P}_m^2}$, compare Section 2.2. In [29], Howlett-Lehrer use a renormalization of (4.5). Setting $T_0 = -\epsilon_t(-q)^{-1-t}\mathbf{X} = (-1)^t q^{-1-t}\mathbf{X}$, we have now the quadratic relation

$$(T_0 + 1)(T_0 - q^{2t+1}) = 0.$$

Using this generator instead of \mathbf{X} , we obtain the usual presentation for a Hecke algebra of type B_m with parameters (q^{2t+1}, q^2) . The endomorphism of $\text{KW}(B_m)$ which is obtained from the renormalization $\mathbf{H}_{\mathbf{K},m}^{Q_t, q^2} \xrightarrow{\sim} \mathbf{H}_{\mathbf{K},m}^{(q^{2t+1}, 1), q^2}$ at $q = 1$ is the identity on \mathfrak{S}_m but sends t_0 to $(-1)^t t_0$. Therefore this renormalization sends $S(\lambda, \mu)$ to $S(\lambda, \mu)$ if t is even, and to $S(\mu, \lambda)$ if t is odd. Combining this observation with [14, Appendix], we obtain the following parametrization of the unipotent characters in the series of E_t .

Corollary 4.13. — *Let $t, m \geq 0$ and $n = t(t+1)/2 + 2m$. Then the map $\phi_{\mathbf{K},m}$ and the functor $\mathfrak{E}_{\Gamma^m(E_t)}$ induce a bijection*

$$\text{WIrr}(\text{KG}_n, E_t) \xleftrightarrow{1:1} \text{Irr}(\mathbf{H}_{\mathbf{K},m}^{Q_t, q^2})$$

sending E_λ to $S(\lambda^{[2]})_{\mathbf{K}}^{Q_t, q^2}$ for all partitions $\lambda \vdash n$ with 2-core $\Delta_t = (t, t-1, \dots, 1)$. \square

4.4.4. *The \mathfrak{g}_∞ -representation on $\mathcal{U}_{\mathbf{K}}$.* — The functors E, F preserve the subcategory $\mathcal{U}_{\mathbf{K}}$ by Proposition 3.5, hence (E, F, \mathbf{X}, T) yields a representation datum on $\mathcal{U}_{\mathbf{K}}$. In order to extend it to a categorical representation on $\mathcal{U}_{\mathbf{K}}$, one should consider the quiver $I(q^2)$ with vertices given by the various eigenvalues of \mathbf{X} (which we showed to be all powers of $-q$ in the proof of Theorem 4.12) and arrows $i \longrightarrow q^2 i$.

In this section we will view the integer q as an element of \mathbf{K}^\times in the obvious way. For the construction of Kac-Moody algebras associated to quivers we refer to Section 1.2. Let I_∞ denote the subset $(-q)^{\mathbf{Z}}$ of \mathbf{K}^\times . We define \mathfrak{g}_∞ to be the (derived) Kac-Moody algebra associated to the quiver $I_\infty(q^2)$. Recall that $(\mathfrak{sl}_{\mathbf{Z}})^{\oplus 2} \simeq \mathfrak{g}_\infty$, see Section 1.2.2 for details.

For any $t, m, n \in \mathbf{N}$, let $(\text{KG}_n, E_t)\text{-mod}$ be the Serre subcategory of $\mathcal{U}_{\mathbf{K}}$ generated by the modules $F^m(E_t)$ with $n = r + 2m$ and $r = t(t+1)/2$. We define

$$\mathcal{U}_{\mathbf{K},t} = \bigoplus_{n \geq 0} (\text{KG}_n, E_t)\text{-mod}.$$

Then $\text{Irr}((\text{KG}_n, E_t)\text{-mod}) = \text{WIrr}(\text{KG}_n, E_t)$, which implies $\mathcal{U}_{\mathbf{K}} = \bigoplus_{t \geq 0} \mathcal{U}_{\mathbf{K},t}$ by Proposition 4.9. We can now prove our first categorification result, which says that $\mathcal{U}_{\mathbf{K},t}$ is a \mathfrak{g}_∞ -representation which categorifies the Fock space $\mathbf{F}(Q_t)_\infty$.

Theorem 4.14. — *Let $t \geq 0$ and Q_t be as in (4.4).*

- (a) *The Harish-Chandra induction and restriction functors yield a representation of \mathfrak{g}_∞ on $\mathcal{U}_{K,t}$ which is isomorphic to $\mathcal{L}(\Lambda_{Q_t})_\infty$.*
 (b) *The map $|\mu, Q_t)_\infty \mapsto [E_{\varpi_t(\mu)}]$ induces an isomorphism of \mathfrak{g}_∞ -modules*

$$\mathbf{F}(Q_t)_\infty \xrightarrow{\sim} [\mathcal{U}_{K,t}].$$

Proof. — Composing the functor $\mathfrak{E}_{F^m(E_t)}$ with the algebra isomorphism in Theorem 4.12 and taking the sum over all $m \in \mathbf{N}$, we get an equivalence of semisimple abelian K -categories $\mathfrak{E}_t : \mathcal{L}(\Lambda_{Q_t})_\infty \xrightarrow{\sim} \mathcal{U}_{K,t}$. We claim that it is actually an isomorphism of representation data. To see this, we first observe that the representation datum (E, F, X, T) restricts to a representation datum on $\mathcal{U}_{K,t}$: indeed, by definition $\mathcal{U}_{K,t}$ is stable by F and by the Mackey formula and [24, Proposition 2.2] it is also stable by the adjoint functor E . Therefore to prove our claim we must show that

- (i) there are isomorphisms $\mathfrak{E}_t E \simeq E \mathfrak{E}_t$ and $\mathfrak{E}_t F \simeq F \mathfrak{E}_t$ of functors $\mathcal{L}(\Lambda_{Q_t})_\infty \rightarrow \mathcal{U}_{K,t}$,
 (ii) the isomorphisms $\mathfrak{E}_t F \simeq F \mathfrak{E}_t$ and $\mathfrak{E}_t F^2 \simeq F^2 \mathfrak{E}_t$ intertwine the endomorphisms $\mathfrak{E}_t X$, $X \mathfrak{E}_t$ and $\mathfrak{E}_t T$, $T \mathfrak{E}_t$.

Note that to prove (i), it is enough to show $\mathfrak{E}_t F \simeq F \mathfrak{E}_t$, since $\mathfrak{E}_t E \simeq E \mathfrak{E}_t$ will follow by adjunction.

For the rest of the proof we will write for short $\mathbf{H}_m := \mathbf{H}_{K,m}^{Q_t, q^2}$. Set $n = r + 2m$ and $r = t(t+1)/2$. Then the functors

$$F \mathfrak{E}_{F^m(E_t)}, \mathfrak{E}_{F^{m+1}(E_t)} F : \mathbf{H}_m\text{-mod} \longrightarrow (KG_{n+2}, E_t)\text{-mod}$$

are both obtained by tensoring with the (KG_{n+2}, \mathbf{H}_m) -bimodule $F^{m+1}(E_t)$. More precisely, the left action of KG_{n+2} is the same in both cases, while the right action of \mathbf{H}_m comes from the right action of \mathbf{H}_m on $F^m(E_t)$ and the functoriality of F in the first case, and from the right action of \mathbf{H}_{m+1} on $F^{m+1}(E_t)$ and the obvious inclusion $\mathbf{H}_m \subset \mathbf{H}_{m+1}$ in the second case. Assertion (a) follows.

Claim (ii) is obvious. Indeed, given an integer $d > 0$, let $x \in \mathbf{H}_{K,d}^{q^2}$ be an element in the affine Hecke algebra and let $M \in \mathbf{H}_m\text{-mod}$. Then, we have $F^d(M) \in \mathbf{H}_{d+m}\text{-mod}$. The action of x on $F^d \mathfrak{E}_{F^m(E_t)}(X)$ and on $\mathfrak{E}_{F^{d+m}(E_t)} F^d(M)$ are represented respectively by the action of $\phi_{F^d}(x) \otimes 1$ on

$$\begin{aligned} F^d(F^m(E_t)) \otimes_{\mathbf{H}_m} M &= F^{d+m}(E_t) \otimes_{\mathbf{H}_{d+m}} \mathbf{H}_{d+m} \otimes_{\mathbf{H}_m} M \\ &= F^{d+m}(E_t) \otimes_{\mathbf{H}_{d+m}} F^d(M) \end{aligned}$$

and by the action of $1 \otimes x$ on $F^d(M)$ in $F^{d+m}(E_t) \otimes_{\mathbf{H}_{d+m}} F^d(M)$. They obviously coincide. The claim follows, taking $d = 1$, $x = X$ or $d = 2$, $x = T$.

Now we can finish the proof of the theorem. We equip $\mathcal{U}_{K,t}$ with the \mathfrak{g}_∞ -representation which is transferred from the \mathfrak{g}_∞ -representation on $\mathcal{L}(\Lambda_{Q_t})_\infty$ via the equivalence \mathfrak{E}_t . This proves (a). We deduce that \mathfrak{E}_t induces on the Grothendieck groups a

\mathfrak{g}_∞ -module isomorphism $\mathbf{L}(\Lambda_{Q_t})_\infty = [\mathcal{L}(\Lambda_{Q_t})_\infty] \xrightarrow{\sim} [\mathcal{U}_k]$. Then (b) follows from Corollary 4.13 and the \mathfrak{g}_∞ -module isomorphism $\mathbf{F}(Q_t)_\infty = \mathbf{L}(\Lambda_{Q_t})_\infty$ given in Propositions 2.1, 2.2. \square

Remark 4.15. — (a) The functor F on \mathcal{U}_k is represented by the sum of bimodules $\bigoplus_{n \in \mathbb{N}} \mathbf{KG}_{n+2} e_{n+2,n}$ on which the endomorphism $X \in \text{End}(F)$ acts by right multiplication by the element

$$(4.6) \quad \sum_n (-1)^n q^n e_{n+2,n} (1, n+2) e_{n+2,n}.$$

(b) For $i \in I$, the functor $F_i : (\mathbf{KG}_n, E_t)\text{-mod} \longrightarrow (\mathbf{KG}_{n+2}, E_t)\text{-mod}$ is the generalized eigenspace of $X \in \text{End}(F)$ associated with the eigenvalue i . For each bipartition μ we have

$$F_i(E_{\varpi_t(\mu)}) = \begin{cases} E_{\varpi_t(v)} & \text{if } \text{res}(v - \mu, Q_t)_\infty = i, \\ 0 & \text{otherwise.} \end{cases}$$

4.5. *The \mathfrak{g}_e -representation on \mathcal{U}_k .* — By Proposition 3.5, the representation datum (E, F, X, T) on $\mathbf{kG}\text{-mod}$ induces a representation datum on \mathcal{U}_k . Since the abelian category \mathcal{U}_k is not semisimple, to extend the representation datum to a categorical \mathfrak{g} -representation one needs to prove that weight spaces of \mathcal{U}_k are sums of blocks. This will be done combinatorially by studying a representation of a bigger Lie algebra \mathfrak{g}_\circ , which is a $(-q)$ -analogue of the action of Harish-Chandra induction and restriction on unipotent representations of $\text{GL}_n(q)$. By definition, this action is compatible with the decomposition into ℓ -blocks of $\text{GL}_n(q)$ and one can transfer this property to unitary groups by analogy with $\text{GU}_n(q)$.

4.5.1. *The Lie algebras \mathfrak{g}_e and $\mathfrak{g}_{e,\circ}$.* — Recall from Theorem 4.12 that the eigenvalues of X on E and F are all powers of $-q$. If we denote again by q the image of q under the canonical map $\mathcal{O} \twoheadrightarrow k$, then the eigenvalues of X on kE and kF belong to the finite set $(-q)^{\mathbb{Z}} \subset k^\times$. This set has exactly e elements, where e is the order of $-q$ in k^\times . We define I_e to be the subset $(-q)^{\mathbb{Z}}$ of k^\times . We denote by I_e and $I_{e,\circ}$ the finite quivers $I_e(q^2)$ and $I_e(-q)$ as in Section 1.2.2.

Recall that we have defined a Lie algebra \mathfrak{g}_e such that: if e is odd, then $\mathfrak{g}_e \simeq \widehat{\mathfrak{sl}}_e$, if e is even, then $\mathfrak{g}_e \subsetneq (\widehat{\mathfrak{sl}}_{e/2})^{\oplus 2}$. We have also defined a Lie algebra $\mathfrak{g}_{e,\circ}$ associated with $I_{e,\circ}$ which is isomorphic to $\widehat{\mathfrak{sl}}_e$.

4.5.2. *Action of \mathfrak{g}'_e on $[\mathcal{U}_k]$.* — The quotient map $\mathcal{O} \twoheadrightarrow k$ yields a morphism of quivers $\text{sp} : I_\infty \longrightarrow I_e$ and a surjective morphism of abelian groups $\text{sp} : P_\infty \twoheadrightarrow P_e$ sending Λ_i to $\Lambda_{\text{sp}(i)}$. In addition any integrable representation V of \mathfrak{g}_∞ can be “restricted” to an integrable representation of the derived algebra \mathfrak{g}'_e , where $e_i \in \mathfrak{g}'_e$ (resp. $f_i \in \mathfrak{g}'_e$) act

as $\sum_{\text{sp}(j)=i} e_j$ (resp. $\sum_{\text{sp}(j)=i} f_j$). From the definition of the action of \mathfrak{g}_∞ and \mathfrak{g}_e on Fock spaces, see (2.2), we deduce that the map $|\mu, Q_i\rangle_\infty \mapsto |\mu, Q_i\rangle_e$ induces the following isomorphism of \mathfrak{g}'_e -modules

$$(4.7) \quad \text{sp} : \text{Res}_{\mathfrak{g}'_e}^{\mathfrak{g}_\infty} \mathbf{F}(Q_i)_\infty \xrightarrow{\sim} \mathbf{F}(Q_i)_e.$$

We show that under the decomposition map, this isomorphism endows $[\mathcal{U}_k]$ with a structure of \mathfrak{g}'_e -module which is compatible with the one coming from the representation datum.

Proposition 4.16. — *For each $i \in I_e$, let kE_i (resp. kF_i) be the generalized i -eigenspace of X on kE (resp. kF).*

- (a) *The operators $[kE_i]$, $[kF_i]$ endow $[\mathcal{U}_k]$ with a structure of \mathfrak{g}'_e -module.*
- (b) *The decomposition map $d_{\mathcal{U}} : [\mathcal{U}_K] \longrightarrow [\mathcal{U}_k]$ is a \mathfrak{g}'_e -module isomorphism*

$$\text{Res}_{\mathfrak{g}'_e}^{\mathfrak{g}_\infty} [\mathcal{U}_K] \xrightarrow{\sim} [\mathcal{U}_k].$$

Proof. — Recall from Section 4.2 that the endofunctor F of $\mathcal{O}G$ -mod is represented by the bimodule $\bigoplus_{n \in \mathbf{N}} \mathcal{O}G_{n+2} e_{n+2,n}$. Under base change from \mathcal{O} to K and k , it yields a functor KF on KG -mod and a functor kF on kG -mod. They are represented by the bimodules $\bigoplus_{n \in \mathbf{N}} KG_{n+2} e_{n+2,n}$ and $\bigoplus_{n \in \mathbf{N}} kG_{n+2} e_{n+2,n}$ respectively. For any $\mathcal{O}G$ -module M we write $KM = K \otimes_{\mathcal{O}} M$ and $kM = k \otimes_{\mathcal{O}} M$. The associativity of the tensor product implies that $KF(KM) = K(FM)$ and $kF(kM) = k(FM)$.

Similarly, the endomorphism X of F which acts by right multiplication by the element (4.6) yields an endomorphism of KF and of kF . For each $i \in I_\infty$ (resp. $i \in I_e$), let KF_i (resp. kF_i) be the generalized i -eigenspace of X acting on KF (resp. on kF). We define KE_i and kE_i in a similar way. Finally, for $i \in I_e$, we write $E_i = (\bigoplus_{\text{sp}(j)=i} KE_j) \cap E$ and $F_i = (\bigoplus_{\text{sp}(j)=i} KF_j) \cap F$. Notice that, although the sums above are a priori infinite sums, they are well-defined as subfunctors of E and F . Then, we have

- (i) $E = \bigoplus_{i \in I_e} E_i$ and $F = \bigoplus_{i \in I_e} F_i$,
- (ii) for each $i \in I_e$, the functors kE_i , kF_i are isomorphic to the specialization of E_i , F_i to k .

Alternatively, the decompositions $kE = \bigoplus_{i \in I_e} kE_i$ and $kF = \bigoplus_{i \in I_e} kF_i$ give rise to some idempotents in $\text{End}_k(kE)$, $\text{End}_k(kF)$ which lift to $\text{End}_{\mathcal{O}}(E)$, $\text{End}_{\mathcal{O}}(F)$ because \mathcal{O} is a complete local ring, yielding decompositions $E = \bigoplus_{i \in I_e} E_i$ and $F = \bigoplus_{i \in I_e} F_i$ such that $k \otimes_{\mathcal{O}} E_i = kE_i$, $k \otimes_{\mathcal{O}} F_i = kF_i$ and $K \otimes_{\mathcal{O}} E_i = \bigoplus_{\text{sp}(j)=i} KE_j$, $K \otimes_{\mathcal{O}} F_i = \bigoplus_{\text{sp}(j)=i} KF_j$.

We deduce that the decomposition map $d_{\mathcal{U}} : [\mathcal{U}_K] \longrightarrow [\mathcal{U}_k]$ is a \mathbf{C} -linear isomorphism which intertwines the operators $\bigoplus_{\text{sp}(j)=i} KE_j$, $\bigoplus_{\text{sp}(j)=i} KF_j$ on $[\mathcal{U}_K]$ with the operators kE_i , kF_i on $[\mathcal{U}_k]$ for each $i \in I_e$. \square

We can fit the three isomorphisms written in Proposition 4.16(b), (4.7) and Theorem 4.14(b) into the following diagram:

$$\begin{array}{ccc}
 \mathrm{Res}_{\mathfrak{g}'_e}^{\mathfrak{g}_\infty} [\mathcal{U}_K] & \xrightarrow[\sim]{d_{\mathcal{U}}} & [\mathcal{U}_K] \\
 \downarrow \wr & & \downarrow \wr \\
 \mathrm{Res}_{\mathfrak{g}'_e}^{\mathfrak{g}_\infty} \left(\bigoplus_{t \in \mathbf{N}} \mathbf{F}(Q_t)_\infty \right) & \xrightarrow[\sim]{\mathrm{sp}} & \bigoplus_{t \in \mathbf{N}} \mathbf{F}(Q_t)_e.
 \end{array}$$

By composition, we get an explicit description of the \mathfrak{g}'_e -module structure on $[\mathcal{U}_K]$.

Recall from Section 2.1.3 that $\varpi_t(\mu)$ is the unique partition with 2-quotient μ and 2-core Δ_t .

Corollary 4.17. — *The map $|\mu, Q_t\rangle_e \mapsto [V_{\varpi_t(\mu)}]$ induces a \mathfrak{g}'_e -module isomorphism*

$$\bigoplus_{t \in \mathbf{N}} \mathbf{F}(Q_t)_e \xrightarrow{\sim} [\mathcal{U}_K]. \quad \square$$

4.5.3. Action of \mathfrak{g}_e on $[\mathcal{U}_K]$. — We now define an action of \mathfrak{g}_e on $[\mathcal{U}_K]$ by extending the action from \mathfrak{g}'_e to \mathfrak{g}_e on $\bigoplus_{t \in \mathbf{N}} \mathbf{F}(Q_t)_e$. This amounts to defining the action of ∂ , or equivalently to extending the grading from P_e to $X_e = P_e \oplus \mathbf{Z}\delta$. To this end we shall use the (level 1) action of $\mathfrak{g}_{e,\circ}$ on the Fock space $\mathbf{F}(1)_{e,\circ}$.

Recall from Lemma 1.1 that $\kappa : \mathfrak{g}_e \rightarrow \mathfrak{g}_{e,\circ}$ is a Lie algebra homomorphism such that $e_i \mapsto [e_{-q_i,\circ}, e_{i,\circ}]$, $f_i \mapsto [f_{-q_i,\circ}, f_{i,\circ}]$ and $\partial \mapsto \partial_\circ$. Any integrable $\mathfrak{g}_{e,\circ}$ -representation (resp. $\mathfrak{g}'_{e,\circ}$ -representation) can be “restricted” to an integrable \mathfrak{g}_e -representation (resp. \mathfrak{g}'_e -representation) through κ . We denote by $\mathrm{Res}_{\mathfrak{g}_e}^{\mathfrak{g}_{e,\circ}}$ and $\mathrm{Res}_{\mathfrak{g}'_e}^{\mathfrak{g}'_{e,\circ}}$ the corresponding operations.

Lemma 4.18. — *The map $|\mu, Q_t\rangle_e \mapsto (-1)^{a(\varpi_t(\mu))} |\varpi_t(\mu), 1\rangle_{e,\circ}$ induces an isomorphism of \mathfrak{g}'_e -modules*

$$\bigoplus_{t \in \mathbf{N}} \mathbf{F}(Q_t)_e \xrightarrow{\sim} \mathrm{Res}_{\mathfrak{g}'_e}^{\mathfrak{g}'_{e,\circ}} \mathbf{F}(1)_{e,\circ}$$

where a is Lusztig’s a -function (see [37, 4.4.2]).

Proof. — The map is clearly an isomorphism of vector spaces. We first show the compatibility of the action for the Lie algebras coming from the quiver in characteristic zero. Recall that $I_\infty = (-q)^{\mathbf{Z}}$ with the action of q^2 , where q is seen as an element of \mathbf{K} . Let $I_{\infty,\circ}$ be the quiver with the same set of vertices, but with arrows given by multiplication by $-q$. Let $\mathfrak{g}_{\infty,\circ}$ be the corresponding derived Lie algebra. It is isomorphic to $\mathfrak{sl}_{\mathbf{Z}}$ whereas \mathfrak{g}_∞ is isomorphic to $(\mathfrak{sl}_{\mathbf{Z}})^{\oplus 2}$. As before, we can embed \mathfrak{g}_∞ into $\mathfrak{g}_{\infty,\circ}$ by sending e_i to

$[e_{-qi,\circ}, e_{i,\circ}]$ and f_i to $[f_{-qi,\circ}, f_{i,\circ}]$. Now, the isomorphism in the lemma can be deduced from the isomorphism of \mathfrak{g}_∞ -modules

$$(4.8) \quad \bigoplus_{t \in \mathbf{N}} \mathbf{F}(\mathbf{Q}_t)_\infty \xrightarrow{\sim} \text{Res}_{\mathfrak{g}_\infty}^{\mathfrak{g}_{\infty,\circ}} \mathbf{F}(1)_{\infty,\circ}$$

using the transitivity of the restriction and (4.7).

We prove the isomorphism (4.8) using the explicit description of the action of the Chevalley generators on Fock spaces in terms of β -sets. Fix $t \geq 0$ and μ a bipartition. Let $\lambda = \varpi_t(\mu)$, i.e., we have $\tau_2(\lambda, 0) = (\mu, \sigma_t)$. Given $i = (-q)^j$, we want to compare the action of f_i on $|\mu, \mathbf{Q}_t\rangle_\infty$ and $[f_{-qi,\circ}, f_{i,\circ}]$ on $|\lambda, 1\rangle_{\infty,\circ}$. Let β be the set of β -numbers of the charged partition $(\lambda, 0)$.

First, we consider the right hand side of (4.8). We distinguish four cases:

Case 1. If $j \notin \beta$, then one can add an i -node neither on $|\lambda, 1\rangle$ nor on $f_{-qi,\circ}|\lambda, 1\rangle$ (if the later is $\neq 0$), therefore $[f_{-qi,\circ}, f_{i,\circ}]|\lambda, 1\rangle = 0$.

Case 2. If $j, j+2 \in \beta$, then one can add a $(-qi)$ -node neither on $f_{i,\circ}|\lambda, 1\rangle$ nor on $|\lambda, 1\rangle$, therefore $[f_{-qi,\circ}, f_{i,\circ}]|\lambda, 1\rangle = 0$.

Case 3. If $j, j+1 \in \beta$ and $j+2 \notin \beta$, then $[f_{-qi,\circ}, f_{i,\circ}]|\lambda, 1\rangle = -f_{i,\circ}f_{-iq,\circ}|\lambda, 1\rangle = -|\lambda', 1\rangle$ where λ' is obtain by adding first a $(-qi)$ -node, then an i -node. The charged β -set of $(\lambda', 0)$ is obtained from β by changing j to $j+2$.

Case 4. If $j \in \beta$ and $j+1, j+2 \notin \beta$, then $[f_{-qi,\circ}, f_{i,\circ}]|\lambda, 1\rangle = f_{-qi,\circ}f_{i,\circ}|\lambda, 1\rangle = |\lambda'', 1\rangle$ where λ'' is obtain by adding first an i -node, then a $(-qi)$ -node. The charged β -set of $(\lambda'', 0)$ is again obtained from β by changing j to $j+2$.

Now, we consider the left hand side of (4.8). Let $\sigma_t = (\sigma_1, \sigma_2)$ be the 2-core of λ . By definition, the bipartition μ is the unique bipartition whose charged β -set satisfies

$$\beta = \beta_0(\lambda) = (-1 + 2\beta_{\sigma_1}(\mu^1)) \sqcup 2\beta_{\sigma_2}(\mu^2).$$

By Proposition 2.3, the Fock space $\mathbf{F}(\mathbf{Q}_t)_\infty$ is identified with the tensor product of the level 1 Fock spaces $\mathbf{F}(-q^{-1}(q^2)^{\sigma_1})_\infty \otimes \mathbf{F}((q^2)^{\sigma_2})_\infty$. If j is odd, then $i = (-q)^j = -q^{-1}(q^2)^{(j+1)/2}$ and the action of f_i on $|\mu_1, -q^{-1}(q^2)^{\sigma_1}\rangle_\infty$ corresponds to changing $(j+1)/2$ to $(j+1)/2 + 1$ in $\beta_{\sigma_1}(\mu_1)$. If j is even, then $i = (q^2)^{j/2}$ and f_i acts on $|\mu_2, (q^2)^{\sigma_2}\rangle_\infty$ by changing $j/2$ to $j/2 + 1$. Using the previous equality of β -sets, this amounts to changing j to $j+2$ in β . This proves that the action of f_i on $|\mu, \mathbf{Q}_t\rangle_\infty$ and of $[f_{-qi,\circ}, f_{i,\circ}]$ on $|\lambda, 1\rangle_{\infty,\circ}$ coincide up to a sign.

Finally, it remains to see that in the case (3) the difference $a(\lambda') - a(\lambda)$ is odd, whereas in the case (4) the difference $a(\lambda'') - a(\lambda)$ is even. This is a straightforward computation using the formula for the a -function given in [37, 4.4.2]. \square

We can now define the action of ∂ on $\bigoplus_{t \in \mathbf{N}} \mathbf{F}(\mathbf{Q}_t)_e$ and $[\mathcal{U}_k]$. To do that, consider the Fock space $\mathbf{F}(1)_{e,\circ}$ as a charged Fock space for the charge $s = 0$. This endows $\mathbf{F}(1)_{e,\circ}$ with an integrable representation of $\mathfrak{g}_{e,\circ}$ as in Section 2.3.1. Consequently, by Corollary

4.17 and Lemma 4.18 we can endow $\bigoplus_{i \in \mathbf{N}} \mathbf{F}(Q_i)_e$, and therefore $[\mathcal{U}_k]$, with an integrable representation of \mathfrak{g}_e such that the maps

$$|\mu, Q_i\rangle_e \mapsto [V_\lambda] \mapsto (-1)^{a(\lambda)} |\lambda, 1\rangle_{e,\circ}$$

with $\lambda = \varpi_i(\mu)$ induce \mathfrak{g}_e -module isomorphisms

$$(4.9) \quad \bigoplus_{i \in \mathbf{N}} \mathbf{F}(Q_i)_e \xrightarrow{\sim} [\mathcal{U}_k] \xrightarrow{\sim} \text{Res}_{\mathfrak{g}_e}^{\mathfrak{g}_{e,\circ}} \mathbf{F}(1)_{e,\circ}.$$

Remark 4.19. — The map $[V_\lambda] \mapsto (-1)^{a(\lambda)} |\lambda, 1\rangle_{e,\circ}$ also endows $[\mathcal{U}_k]$ with a structure of $\mathfrak{g}_{e,\circ}$ -module, but which does not come from Harish-Chandra induction and restriction. On the other hand, this would be the case if we were working with finite linear groups instead of unitary groups. In that case, the Fock space one should consider would be $\mathbf{F}(1)$ with the level 1 action of the Kac-Moody algebra associated with the quiver $q^{\mathbf{Z}}(q)$. Using the analogy between $\text{GL}_n(-q)$ and $\text{GU}_n(q)$ which intertwines (at the level of characters) the Harish-Chandra induction with the 2-Harish-Chandra induction, see [5, Section 3], one could expect that the action of $\mathfrak{g}_{e,\circ}$ on $[\mathcal{U}_k]$ comes from some truncation of induction and restriction functors associated with Deligne-Lusztig varieties. These functors are no longer exact but only triangulated. This could explain the appearance of signs in the formulae for the action of $e_{i,\circ}$ and $f_{i,\circ}$ on the standard basis elements $[V_\lambda]$.

4.5.4. The \mathfrak{g}_e -action on \mathcal{U}_k . — From Corollary 4.17 and (4.9), we know that the representation datum on \mathcal{U}_k yields an integrable representation of \mathfrak{g}_e on $[\mathcal{U}_k]$. In order to show that it endows \mathcal{U}_k with a structure of categorical \mathfrak{g}_e -representation, it only remains to prove that there is a decomposition of the category into weight spaces. To this end, we show that unipotent characters lying in the same ℓ -block have the same weight.

Lemma 4.20. — *Let λ and μ be partitions of n . If V_μ, V_λ belong to the same block of \mathbf{kG}_n then $[V_\lambda], [V_\mu]$ have the same weight for the action of \mathfrak{g}_e .*

Proof. — Recall that $[\mathcal{U}_k]$ is isomorphic to $\text{Res}_{\mathfrak{g}_e}^{\mathfrak{g}_{e,\circ}} \mathbf{F}(1)_{e,\circ}$ as a \mathfrak{g}_e -module by (4.9). Here, the restriction is obtained through the map $\kappa : \mathfrak{g}_e \longrightarrow \mathfrak{g}_{e,\circ}$ defined in Lemma 1.1. In particular, we have the following equality of weights in X_e

$$(4.10) \quad \text{wt}([V_\lambda]) = \kappa^*(\text{wt}(|\lambda, 1\rangle_{e,\circ})).$$

The indecomposable modules V_λ, V_μ lie in the same block of \mathbf{kG}_n if and only if the unipotent character E_λ, E_μ lie in the same ℓ -block. Now, by Propositions 2.4, 4.7 this is equivalent to the weights of $|\lambda, 1\rangle_{e,\circ}$ and $|\mu, 1\rangle_{e,\circ}$ being equal. The lemma follows from (4.10). \square

From the lemma we deduce that the classes of the simple modules $[D_\lambda]$ are also weight vectors. Indeed, they are linear combination of $[V_\mu]$'s in the same block, therefore

with the same weight. Given $\omega \in X_e$, we define $\mathcal{U}_{k,\omega}$ to be the Serre subcategory of \mathcal{U}_k generated by the simple modules D_μ such that $[D_\mu]$ has weight ω . From Lemma 4.20 we deduce that

$$\mathcal{U}_k = \bigoplus_{\omega \in X_e} \mathcal{U}_{k,\omega}$$

and we obtain our second and main categorification theorem for the unipotent representations of finite unitary groups.

Theorem 4.21. — *The representation datum (E, F, T, X) associated with Harish-Chandra induction and restriction and the decomposition $\mathcal{U}_k = \bigoplus_{\omega \in X_e} \mathcal{U}_{k,\omega}$ yields a categorical representation of \mathfrak{g}_e on \mathcal{U}_k . Furthermore, the map $|\mu, Q_i\rangle_e \mapsto [V_{\varpi_i(\mu)}]$ induces a \mathfrak{g}_e -module isomorphism $\bigoplus_{i \in \mathbf{N}} \mathbf{F}(Q_i)_e \xrightarrow{\sim} [\mathcal{U}_k]$. \square*

4.6. Derived equivalences of blocks of \mathcal{U}_k . — In this section we apply the categorical techniques of [7] to produce some derived equivalences between blocks of \mathcal{U}_k . In the linear prime case (when e is even), we use the existence of *good blocks* [34] to deduce Broué’s abelian defect group conjecture for finite unitary groups.

Recall that d , e and f denote respectively the order of q^2 , $-q$ and q modulo ℓ .

4.6.1. Characterization of the blocks of \mathcal{U}_k . — Recall that we proved in Lemma 4.20 that each weight category of \mathcal{U}_k is a union of blocks of \mathcal{U}_k (or equivalently unipotent blocks of $kG\text{-mod}$). Here we investigate which block can occur in a given weight space. More precisely, we show that each weight space (resp. each weight space on a Harish-Chandra series) is indecomposable when e is odd (resp. when e is even).

Throughout this section, we will denote by $\omega_{\lambda,\circ}$ the weight of $|\lambda, 1\rangle_{e,\circ}$ for the action of $\mathfrak{g}_{e,\circ}$ on $\mathbf{F}(1)_{e,\circ}$ and by ω_λ the weight of $[V_\lambda]$ for the action of \mathfrak{g}_e on $[\mathcal{U}_k]$. Recall from (2.6), (2.7) that

$$\begin{aligned} \omega_{\lambda,\circ} &= t_{\pi_{\lambda_{[e]}}}(\Lambda_{1,\circ}) - w_e(\lambda)\delta_\circ \\ (4.11) \quad &= \Lambda_{1,\circ} + \pi_{\lambda_{[e]}} - \Delta(\lambda_{[e]}, 1)\delta_\circ - w_e(\lambda)\delta_\circ \end{aligned}$$

where $\lambda_{[e]}$ is the e -core of λ , and $w_e(\lambda) = |\lambda^{[e]}|$ is its e -weight. Using Proposition 4.7, this shows that $\omega_{\lambda,\circ}$ characterizes the ℓ -block in which V_λ lies. In other words, ℓ -blocks correspond to weight spaces for the action of $\mathfrak{g}_{e,\circ}$ on $\mathbf{F}(1)_{e,\circ}$. Since the equality (4.10) gives

$$\omega_\lambda = \kappa^*(\omega_{\lambda,\circ}),$$

we are left with computing the different weights that can appear in $(\kappa^*)^{-1}(\omega_\lambda)$. We shall start with the case where e is odd.

Lemma 4.22. — *Assume e is odd. Let λ, ν be partitions. Then $\omega_{\lambda, \circ} = \omega_{\nu, \circ}$ if and only if $\omega_\lambda = \omega_\nu$.*

Proof. — Since $\omega_\lambda = \kappa^*(\omega_{\lambda, \circ})$, it is enough to show that $\kappa^* : X_{e, \circ} \rightarrow X_e$ is injective. With e being odd, we have $2\alpha_{i, \circ}^\vee = \sum_j (-1)^j \kappa(\alpha_{(-q)^j i}^\vee)$ for each i , hence $2Q_{e, \circ}^\vee \subseteq \kappa(Q_e^\vee)$. If $\kappa^*(\alpha_\circ) = 0$ then $0 = \langle \kappa(X_e^\vee), \alpha_\circ \rangle_{e, \circ} = \langle \kappa(Q_e^\vee) + \mathbf{Z}\partial_\circ, \alpha_\circ \rangle_{e, \circ}$. We deduce that $\langle X_{e, \circ}^\vee, \alpha_\circ \rangle_{e, \circ} = 0$, hence $\alpha_\circ = 0$. \square

We now assume that e is even. Then κ^* is no longer injective, therefore weight spaces of \mathcal{U}_k might contain several blocks in general. However, one can show that κ^* is injective on weights coming from partitions with the same 2-core.

Lemma 4.23. — *Assume e is even. Let λ, ν be partitions. Then $\omega_{\lambda, \circ} = \omega_{\nu, \circ}$ if and only if $\omega_\lambda = \omega_\nu$ and $\lambda_{[2]} = \nu_{[2]}$.*

Proof. — If $\omega_{\lambda, \circ} = \omega_{\nu, \circ}$ then the partitions λ and ν have the same e -core. With e being even, they also have the same 2-core.

We now prove the converse. An easy computation shows that the kernel of κ^* is spanned by the weight $\sum_{j=0}^{e-1} (-1)^j \Lambda_{(-q)^j \circ}$. In particular, if $\omega_\lambda = \omega_\nu$ then $\omega_{\lambda, \circ} - \omega_{\nu, \circ}$ lies in $P_{e, \circ}$. Therefore by (4.11), this difference equals $\pi_s = \sum_{i \in I_e} (s_i - s_{-qi}) \Lambda_{i, \circ}$ where $s = \lambda_{[e]} - \nu_{[e]}$. Since $\kappa^*(\pi_s) = 0$, we deduce from the formulas in Lemma 1.1 that $s_i = s_{q^2 i}$ for all $i \in I_e$. On the other hand, since $\lambda_{[2]} = \nu_{[2]}$ then by Lemma 4.28 below we must have $\sum_{j \text{ odd}} s_{-q^j} = 0$ and $\sum_{j \text{ even}} s_{q^j} = 0$. This proves that $s = 0$, thus $\pi_s = 0$ and therefore $\omega_{\lambda, \circ} = \omega_{\nu, \circ}$. \square

Assume e is even. Given $t \in \mathbf{N}$ and $\omega \in X_e$, we define $\mathcal{U}_{k, t}$ (resp. $\mathcal{U}_{k, t, \omega}$) to be the Serre subcategory of \mathcal{U}_k generated by simple modules D_λ where λ has 2-core $\Delta_t = (t, t-1, \dots, 1)$ (resp. with in addition $\omega_\lambda = \omega$). With e being even, any pair of partitions with the same e -core have the same 2-core, therefore $\mathcal{U}_{k, t}$ is a direct summand of \mathcal{U}_k . From the previous lemmas and the characterization of blocks of \mathcal{U}_k (see Proposition 4.7) we get the following more precise result.

Proposition 4.24. — *Let $\omega \in X_e$.*

- (a) *If e is odd, then the category $\mathcal{U}_{k, \omega}$ is an indecomposable summand of \mathcal{U}_k .*
- (b) *If e is even, then the category $\mathcal{U}_{k, t, \omega}$ is an indecomposable summand of $\mathcal{U}_{k, t}$ and \mathcal{U}_k for all $t \in \mathbf{N}$.* \square

Remark 4.25. — The indecomposable categories in (a) and (b) are both equal to the unique block $B_{\nu, w}$ of \mathcal{U}_k such that $\omega = \omega_{\nu, \circ} - w \delta$ for some $w \in \mathbf{N}$. We will call ω the *degree* of the block $B_{\nu, w}$.

When e is even $[\mathcal{U}_{k, t}]$ is spanned by the classes $[V_\lambda]$ where the 2-core of λ is the triangular partition Δ_t . In particular, $[\mathcal{U}_{k, t}]$ is stable by the action of $[E_i]$ and $[F_i]$, and

since $\mathcal{U}_{k,t}$ is a direct summand of \mathcal{U}_k , the category $\mathcal{U}_{k,t}$ itself is stable by E_i and F_i . This allows to refine Theorem 4.21 into the following result when e is even.

Corollary 4.26. — *Assume e is even, and let $t \in \mathbf{N}$. The Harish-Chandra induction and restriction functors yield a representation of \mathfrak{g}_e on $\mathcal{U}_{k,t}$ which categorifies $\mathbf{F}(\mathbf{Q}_e)_e$.* \square

4.6.2. Action of the affine Weyl group. — In this section we study the action of the affine Weyl group of \mathfrak{g}_e on the weight spaces of \mathcal{U}_k and $\mathcal{U}_{k,t}$ with a view to understanding the action on unipotent blocks. As above, the results will depend on the parity of e .

We start with the case where e is even. To avoid cumbersome notation we fix t and we write $\mathbf{Q} = \mathbf{Q}_t = (\mathbf{Q}_1, \mathbf{Q}_2)$. Recall from (2.1) that $\sigma = (\Delta_t)_{[2]} = (\sigma_1, \sigma_2)$ is given by

$$\sigma = \begin{cases} (-t/2, t/2) & \text{if } t \text{ is even,} \\ ((1+t)/2, -(1+t)/2) & \text{if } t \text{ is odd,} \end{cases}$$

so that we have $\mathbf{Q} = (-q^{-1}q^{2\sigma_1}, q^{2\sigma_2})$ by (4.4).

In Section 4.5.1 we defined \mathfrak{g}_e as a subalgebra of $\tilde{\mathfrak{g}}_e := \mathfrak{g}_{e,1} \oplus \mathfrak{g}_{e,2}$ where each $\mathfrak{g}_{e,p}$ is isomorphic to $\widehat{\mathfrak{sl}}_{e/2}$. The derived Lie algebras of $\mathfrak{g}_e, \tilde{\mathfrak{g}}_e$ coincide, but the derivation ∂ of \mathfrak{g}_e is equal to $(\partial_1, \partial_2) = (\Lambda_{-q^{-1}}^\vee, \Lambda_1^\vee)$. Recall also that the Lie algebra $\mathfrak{g}_{e,1}$ is generated by $\partial_1 = \Lambda_{-q^{-1}}^\vee$ and e_i, f_i for $i \in (-q)^{-1+2\mathbf{Z}}$, whereas $\mathfrak{g}_{e,2}$ is generated by $\partial_2 = \Lambda_1^\vee$ and e_i, f_i for $i \in q^{2\mathbf{Z}}$. For each $p = 1, 2$, the Fock space $\mathbf{F}(\mathbf{Q}_p)_{e,p}$ can be endowed with an action of $\mathfrak{g}_{e,p}$. To do so, in order to define the action of ∂_p , we must fix a charge, see Section 2.3.1. First, observe that the bijection $\phi : (-q)^{p-2+2\mathbf{Z}} \rightarrow q^{2\mathbf{Z}}$ such that $\phi(i) = i(-q)^{2-p}$ identifies the quiver associated with $\mathfrak{g}_{e,p}$ with the quiver $q^{2\mathbf{Z}}(q^2)$, and that $\phi(\mathbf{Q}_p) = q^{2\sigma_p}$. Thus, we can fix the charge to be σ_p . Consequently, there is a well-defined action of $\tilde{\mathfrak{g}}_e$ on $\mathbf{F}(\mathbf{Q}_1)_{e,1} \otimes \mathbf{F}(\mathbf{Q}_2)_{e,2}$.

Proposition 4.27. — *Assume e is even. Then the linear map such that $|\mu, \mathbf{Q}\rangle_e \mapsto |\mu^1, \mathbf{Q}_1\rangle_{e,1} \otimes |\mu^2, \mathbf{Q}_2\rangle_{e,2}$ is an isomorphism of \mathfrak{g}_e -modules*

$$\mathbf{F}(\mathbf{Q})_e \xrightarrow{\sim} \text{Res}_{\tilde{\mathfrak{g}}_e}^{\tilde{\mathfrak{g}}_e}(\mathbf{F}(\mathbf{Q}_1)_{e,1} \otimes \mathbf{F}(\mathbf{Q}_2)_{e,2}).$$

Proof. — The map is an isomorphism of \mathfrak{g}'_e -modules by Proposition 2.3 (recall that the derived algebras of \mathfrak{g}_e and $\tilde{\mathfrak{g}}_e$ coincide). Therefore we are left with comparing the action of the derivation ∂ on the elements $|\mu, \mathbf{Q}\rangle_e$ and $|\mu^1, \mathbf{Q}_1\rangle_{e,1} \otimes |\mu^2, \mathbf{Q}_2\rangle_{e,2}$.

First, we compute the action of ∂ on $|\mu, \mathbf{Q}\rangle_e$. Set $\lambda = \varpi_i(\mu)$. By (4.9), the weight of $|\mu, \mathbf{Q}\rangle_e$ is ω_λ . By Lemma 1.1 and (2.4), we get

$$\langle \partial, \omega_\lambda \rangle_e = \langle \partial, \kappa^*(\omega_{\lambda, \circ}) \rangle_e = \langle \kappa(\partial), \omega_{\lambda, \circ} \rangle_{e, \circ} = \langle \partial_\circ, \omega_{\lambda, \circ} \rangle_{e, \circ} = -n_1(\lambda, 1).$$

Now, we compute the action of ∂ on $|\mu^1, \mathbf{Q}_1\rangle_{e,1} \otimes |\mu^2, \mathbf{Q}_2\rangle_{e,2}$. The embedding $\mathfrak{g}_e \hookrightarrow \tilde{\mathfrak{g}}_e$ sends ∂ to (∂_1, ∂_2) , therefore the δ -part of the weight of $|\mu^1, \mathbf{Q}_1\rangle_{e,1} \otimes |\mu^2, \mathbf{Q}_2\rangle_{e,2}$ is

$$-n_{-q^{-1}}(\mu^1, \mathbf{Q}_1) - \Delta(\sigma_1, e/2) - n_1(\mu^2, \mathbf{Q}_2) - \Delta(\sigma_2, e/2).$$

Since e is even, removing an e -hook from λ amounts to removing an $e/2$ -hook from μ^1 or from μ^2 . Since $(-q)$ has order e and q^2 has order $e/2$, this has the effect of subtracting 1 from $n_1(\lambda, 1)$ and from $n_{-q^{-1}}(\mu^1, Q_1) + n_1(\mu^2, Q_2)$. Therefore to prove the equality between δ -weights we can assume that λ is an e -core. In that case both μ^1 and μ^2 are $e/2$ -cores and we use the following technical lemma together with Remark 2.5 to conclude. \square

Lemma 4.28. — *Let $\mu = (\mu^1, \mu^2)$ be a bipartition and $\lambda = \varpi_i(\mu)$. Let $s = (s_i)_{i \in I_e} = \lambda_{[e]}$, $s^1 = (s_{-q^i})_{j \text{ odd}}$ and $s^2 = (s_{q^i})_{j \text{ even}}$. Then, we have*

- (a) $\tau_{e/2}(\mu^p, \sigma_p) \in \mathcal{P}^{e/2} \times \{s^p\}$ for $p = 1, 2$,
- (b) $w_e(\lambda) = w_{e/2}(\mu^1) + w_{e/2}(\mu^2)$. \square

From Proposition 4.27 we deduce that the action of the affine Weyl group W_e of \mathfrak{g}_e on $\mathbf{F}(Q)_e$ can be read off from the action of $W_{e,1} \times W_{e,2}$ on $\mathbf{F}(Q_1)_{e,1} \otimes \mathbf{F}(Q_2)_{e,2}$. We have the following partition of weights into orbits under W_e .

Proposition 4.29. — *Assume e is even. Let λ, ν be partitions. The weights ω_λ and ω_ν are conjugate under W_e if and only if $\lambda_{[2]} = \nu_{[2]}$ and $w_e(\lambda) = w_e(\nu)$.*

Proof. — Since each $[\mathcal{U}_{k,t}]$ is stable by \mathfrak{g}_e , the partitions λ and ν must have the same 2-core in order to be conjugate under W_e . We will therefore assume that $\lambda_{[2]} = \nu_{[2]} = \sigma_t$ and we denote by μ and γ the bipartitions such that $\lambda = \varpi_t(\mu)$ and $\nu = \varpi_t(\gamma)$.

We write $X_{e,1} = P_{e,1} \oplus \mathbf{Z}\delta_1$ and $X_{e,2} = P_{e,2} \oplus \mathbf{Z}\delta_2$. Recall that X_e is the quotient of the lattice $\tilde{X}_e = X_{e,1} \oplus X_{e,2}$ by the line spanned by $\delta_1 - \delta_2$. To avoid cumbersome notation, we will write $\tilde{\omega}_\mu, \tilde{\omega}_\gamma$ for the weights of $|\mu^1, Q_1\rangle_{e,1} \otimes |\mu^2, Q_2\rangle_{e,2}$ and $|\gamma^1, Q_1\rangle_{e,1} \otimes |\gamma^2, Q_2\rangle_{e,2}$ in \tilde{X}_e and $\omega_\mu, \omega_\gamma$ for their image in X_e . Notice that, by Proposition 4.27, the weights $\omega_\lambda, \omega_\nu$ are conjugate under W_e if and only if $\omega_\mu, \omega_\gamma$ are conjugate under $W_{e,1} \times W_{e,2}$.

Now, by Proposition 2.4 we have

$$\begin{aligned} \tilde{\omega}_\mu &\in W_{e,1}(\Lambda_{Q_1}) + W_{e,2}(\Lambda_{Q_2}) + w_{e/2}(\mu^1)\delta_1 + w_{e/2}(\mu^2)\delta_2, \\ \tilde{\omega}_\gamma &\in W_{e,1}(\Lambda_{Q_1}) + W_{e,2}(\Lambda_{Q_2}) + w_{e/2}(\gamma^1)\delta_1 + w_{e/2}(\gamma^2)\delta_2. \end{aligned}$$

In particular, the weights $\tilde{\omega}_\mu, \tilde{\omega}_\gamma$ are conjugate under $W_{e,1} \times W_{e,2}$ if and only if

$$w_{e/2}(\mu^1)\delta_1 + w_{e/2}(\mu^2)\delta_2 = w_{e/2}(\gamma^1)\delta_1 + w_{e/2}(\gamma^2)\delta_2$$

in \tilde{X}_e . Now, if $w_e(\lambda) = w_e(\nu)$ then $w_{e/2}(\mu^1) + w_{e/2}(\mu^2) = w_{e/2}(\gamma^1) + w_{e/2}(\gamma^2)$ by Lemma 4.28, hence there is an integer m such that the weights $\tilde{\omega}_\mu, \tilde{\omega}_\gamma + m(\delta_1 - \delta_2)$ are conjugate under $W_{e,1} \times W_{e,2}$. We deduce that the weights $\omega_\mu, \omega_\gamma$ are conjugate under $W_{e,1} \times W_{e,2}$, hence $\omega_\lambda, \omega_\nu$ are conjugate under W_e .

Conversely, assume that $\omega_\lambda, \omega_\nu$ are conjugate under W_e or, equivalently, that $\omega_\mu, \omega_\gamma$ are conjugate under $W_{e,1} \times W_{e,2}$. This means that there exist $w \in W_{e,1} \times W_{e,2}$ and

$m \in \mathbf{Z}$ such that $w(\tilde{\omega}_\mu) = \tilde{\omega}_\gamma + m(\delta_1 - \delta_2)$. If m is negative, we can add $-m$ $e/2$ -hooks to μ^1 and to γ^2 to obtain bipartitions μ' and γ' such that $\tilde{\omega}_{\mu'} = \tilde{\omega}_\mu - m\delta_1$ and $\tilde{\omega}_{\gamma'} = \tilde{\omega}_\gamma - m\delta_2$. Since w acts trivially on δ_1 and δ_2 , we now have $w(\tilde{\omega}_{\mu'}) = \tilde{\omega}_{\gamma'}$. By Proposition 2.4, we deduce that $w_{e/2}((\mu')^1) = w_{e/2}((\gamma')^1)$ and $w_{e/2}((\mu')^2) = w_{e/2}((\gamma')^2)$. In particular, we have $w_{e/2}((\mu')^1) + w_{e/2}((\mu')^2) = w_{e/2}((\gamma')^1) + w_{e/2}((\gamma')^2)$, and the same equality holds for μ and γ . This proves that $w_e(\lambda) = w_e(\nu)$ by Lemma 4.28. The same argument applies when m is nonnegative by adding hooks to μ^2 and γ^1 . \square

We now turn to the case where e is odd. Let $\rho_o = \sum \Lambda_{i,o}^{\text{cl}} \in Q_{e,o}^{\text{cl}}$ be the sum of the classical fundamental weights. For each $\alpha_o \in X_{e,o}$ and $w \in W_{e,o}$, we abbreviate $w \bullet \alpha_o = w(\alpha_o + \rho_o) - \rho_o$.

Proposition 4.30. — *Assume e is odd. Let λ, ν be partitions. The weights ω_λ and ω_ν are conjugate under W_e if and only if $\lambda_{[e]} \in \mathfrak{S}_{I_e} \bullet \nu_{[e]} + 2\mathbf{Z}^e$ and $w_e(\lambda) = w_e(\nu)$.*

Proof. — Let $W_{e,o}^{(2)}$ be the subgroup of $W_{e,o}$ which is equal to $\mathfrak{S}_{I_e} \times 2Q_{e,o}^{\text{cl}}$ under the isomorphism $W_{e,o} = \mathfrak{S}_{I_e} \times Q_{e,o}^{\text{cl}}$. Similarly, let $\mathfrak{g}_{e,o}^{(2)}$ be the Lie subalgebra of $\mathfrak{g}_{e,o}$ which is equal to $\mathfrak{sl}_e(\mathbf{C}) \otimes \mathbf{C}[t^2, t^{-2}] \oplus \mathbf{C}c \oplus \mathbf{C}\partial$ under the isomorphism $\mathfrak{g}_{e,o} = \widehat{\mathfrak{sl}}_e$ and let π be the Lie algebra automorphism of $\mathfrak{g}_{e,o}$ which takes $x \otimes t^n$ to $x \otimes t^{n-\text{ht}(\alpha)}$ for each α -root vector $x \in \mathfrak{sl}_e(\mathbf{C})$. The embedding $\kappa : \mathfrak{g}_e \rightarrow \mathfrak{g}_{e,o}$ defined in Lemma 1.1 yields an embedding $\kappa : W_e \rightarrow W_{e,o}$.

Lemma 4.31. — *We have*

- (a) $\kappa(\mathfrak{g}_e) = \pi(\mathfrak{g}_{e,o}^{(2)})$,
- (b) $\kappa(W_e) = (1, -\rho_o) \cdot W_{e,o}^{(2)} \cdot (1, \rho_o)$.

By Lemma 4.22, we must characterize the pairs $\{\lambda, \nu\}$ such that $\omega_{\lambda,o}$ and $\omega_{\nu,o}$ are W_e -conjugate in $X_{e,o}$. By (4.11) and Proposition 2.4, we deduce that $\omega_{\lambda,o}$ and $\omega_{\nu,o}$ are W_e -conjugate if and only if $t_{\pi_s + \rho_o}(\Lambda_{1,o}) - w_e(\lambda)\delta_o$ and $t_{\pi_u + \rho_o}(\Lambda_{1,o}) - w_e(\nu)\delta_o$ are $W_{e,o}^{(2)}$ -conjugate, where we abbreviate $s = \lambda_{[e]}$ and $u = \nu_{[e]}$. We have $w_e(\lambda) = w_e(\nu)$ if and only if $t_{\pi_s + \rho_o}(\Lambda_{1,o}) - w_e(\lambda)\delta_o$ and $t_{\pi_u + \rho_o}(\Lambda_{1,o}) - w_e(\nu)\delta_o$ are $W_{e,o}$ -conjugate. Hence $\omega_{\lambda,o}$ and $\omega_{\nu,o}$ are W_e -conjugate if and only if $w_e(\lambda) = w_e(\nu)$ and $\pi_s \in \mathfrak{S}_{I_e} \bullet \pi_u + 2Q_{e,o}^{\text{cl}}$. The proposition is proved. \square

Proof of Lemma 4.31. — It is enough to prove claim (a). Let $\tilde{\kappa} = \pi^{-1} \circ \kappa$. We must prove that $\tilde{\kappa}(\mathfrak{g}_e) = \mathfrak{g}_{e,o}^{(2)}$. The inclusion $\tilde{\kappa}(\mathfrak{g}_e) \subseteq \mathfrak{g}_{e,o}^{(2)}$ is obvious. Let us concentrate on the reverse inclusion.

Let $\mathfrak{g}_{e,o}^{(0)}$ be the Lie subalgebra of $\mathfrak{g}_{e,o}$ which is equal to $\mathfrak{sl}_e(\mathbf{C})$ under the isomorphism $\mathfrak{g}_{e,o} = \widehat{\mathfrak{sl}}_e$. First, we consider the root vectors $\tilde{\kappa}(e_i)$ of $\mathfrak{g}_{e,o}$ as i runs over the set I_e . If $i = 1$ or $-q^{-1}$ then $\tilde{\kappa}(e_i)$ is of the form $x \otimes t^{3-e}$ where x is a root vector of $\mathfrak{g}_{e,o}^{(0)}$ whose weight has height $2 - e$, else it is of the form $y \otimes t^2$ where y is a root vector of $\mathfrak{g}_{e,o}^{(0)}$ whose

weight has height 2. Now, it is easy to see that all root vectors of $\mathfrak{g}_{e,\circ}^{(0)}$ associated with the negative simple roots can be decomposed as the Lie bracket of one vector of the first type and $(e-3)/2$ vectors of the second type. A similar result holds for the root vectors of $\mathfrak{g}_{e,\circ}^{(0)}$ associated with the (positive) simple roots, by considering the f_i 's instead of the e_i 's. This implies that $\mathfrak{g}_{e,\circ}^{(0)}$ is contained in $\tilde{\kappa}(\mathfrak{g}_e)$.

To conclude, observe that the Lie algebra $\mathfrak{g}_{e,\circ}^{(2)}$ is generated by $\mathfrak{g}_{e,\circ}^{(0)}$ and any vector $\tilde{\kappa}(e_i)$ of the second type mentioned above. \square

4.6.3. Derived equivalences of blocks of \mathcal{U}_k . — Now that we know the orbits of the affine Weyl group on the weight spaces of $[\mathcal{U}_k]$ (hence on the blocks of \mathcal{U}_k), we can apply Proposition 4.24 and the work of Chuang and Rouquier [7] to produce derived equivalences between blocks of \mathcal{U}_k in the same W_e -orbit.

The following definition is taken from [34] for e even and from [35] for e odd.

Definition 4.32. — Let $\nu = (s_{(-q)^i}) \in \mathbf{Z}^L$ be an e -core and $w \geq 0$. The unipotent block $B_{\nu,w}$ is a good block if for every $i = 0, \dots, e-2$ we have $s_{(-q)^i} \leq s_{(-q)^{i+2}} + w - 1$.

We will say that two unipotent blocks are W_e -conjugate (or conjugate under W_e) if the corresponding weight spaces in \mathcal{U}_k (or $\mathcal{U}_{k,t}$ when e is even) are W_e -conjugate.

Lemma 4.33. — Any unipotent block is W_e -conjugate to a good block.

Proof. — Let λ be a partition and $w = w_e(\lambda)$. Assume first that e is even. Let ν be any other partition such that $w_e(\lambda) = w_e(\nu)$. Write $\lambda_{[e]} = (s_{(-q)^i})$ and $\nu_{[e]} = (t_{(-q)^i})$. By Lemma 4.28, the partitions λ, ν have the same 2-core if and only if $\sum s_{q^{2i}} - \sum s_{-q^{2i+1}} = \sum t_{q^{2i}} - \sum t_{-q^{2i+1}}$. In such case, Proposition 4.29 ensures that the degrees of the blocks $B_{\lambda_{[e]},w}$ and $B_{\nu_{[e]},w}$ are W_e -conjugate (see Remark 4.25 for the definition of the degree of a block). If one chooses for ν the partition with e -core $t_{(-q)^i} = s_{(-q)^i} + \lfloor i/2 \rfloor (w-1)$, then $B_{\nu_{[e]},w}$ is a good block.

Assume now that e is odd. Let $\varepsilon_i = 0$ if $s_{q^{2i+2}} - s_{q^{2i}}$ is even, and 1 otherwise. We choose for ν a partition with $w_e(\nu) = w_e(\lambda)$ and with e -core $\nu_{[e]} = (t_{q^i})$ such that $t_{q^{2i}} = s_1 + 2i(w-1) + \varepsilon_0 + \dots + \varepsilon_{i-1}$. Then $B_{\nu_{[e]},w}$ is a good block, and $t_{q^{2i}} - s_{q^{2i}} = 2i(w-1) + s_1 - s_{q^2} + s_{q^2} - \dots - s_{q^{2i}} + \varepsilon_0 + \dots + \varepsilon_{i-1}$ is even. We deduce from Proposition 4.30 that the blocks $B_{\lambda_{[e]},w}$ and $B_{\nu_{[e]},w}$ are W_e -conjugate. \square

Together with the results of Livesey [34] on the structure of good blocks in the linear prime case, we deduce that Broué's abelian defect group conjecture holds for unipotent blocks when e is even.

Theorem 4.34. — Assume e is even. Let B be a unipotent block of $G_n = \mathrm{GU}_n(q)$ over k or \mathcal{O} , and D a defect group of B . If D is abelian, then B is derived equivalent to the Brauer correspondent of B in $N_{G_n}(D)$.

Proof. — Let B' be a good block which is W_e -conjugate to B , which exists by Lemma 4.33. By [7, Theorem 6.4], the blocks B and B' are derived equivalent. Now it follows from [34, Theorem 7.1] that any good block is derived equivalent to its Brauer correspondent. \square

Remark 4.35. — Let B be any block of a finite general unitary group. Then, by the Jordan decomposition [4, Théorème B], the block B is Morita equivalent to a direct product of unipotent blocks of unitary groups $GU_m(q')$ with r odd and linear groups $GL_n(q')$ with s even. However, it is not yet proven that this Morita equivalence preserves the local structure of the blocks. If it were the case, then Theorem 4.34 would hold for any block of G_n .

4.7. The crystals of \mathcal{U}_K and \mathcal{U}_k . — In this section we show how to compare the crystals of the categorical representations \mathcal{U}_K and \mathcal{U}_k (which are related to Harish-Chandra induction and restriction) with the crystals of the Fock spaces related to $[\mathcal{U}_K]$ and $[\mathcal{U}_k]$. This solves the main conjecture of Gerber-Hiss-Jacon [24] and gives a combinatorial way to compute the (weak) Harish-Chandra branching graph and the Hecke algebras associated to the weakly cuspidal unipotent modules.

4.7.1. Crystals and Harish-Chandra series. — Recall that to any categorical representation one can associate a perfect basis, and hence an abstract crystal (see Proposition 1.12). In the previous section we constructed a categorical action on the categories of unipotent representations of finite unitary groups over K (denoted by \mathcal{U}_K) and over k (denoted by \mathcal{U}_k). From these two categorical representations we get:

- (a) an abstract crystal $B(\mathcal{U}_K) = (\text{Irr}(\mathcal{U}_K), \tilde{E}_i, \tilde{F}_i)$ with the canonical labeling $\text{Irr}(\mathcal{U}_K) = \{[E_\lambda] \mid \lambda \in \mathcal{P}\}$. Since the representation of \mathfrak{g}_∞ on \mathcal{U}_K is the direct sum of its representations on the subcategories $\mathcal{U}_{K,t}$ for each $t \in \mathbf{N}$, we deduce that $B(\mathcal{U}_K)$ is the direct sum of the abstract crystals $B(KG, E_t) = (\text{Irr}(KG, E_t), \tilde{E}_i, \tilde{F}_i)$,
- (b) an abstract crystal $B(\mathcal{U}_k) = (\text{Irr}(\mathcal{U}_k), \tilde{E}_i, \tilde{F}_i)$ with the canonical labeling $\text{Irr}(\mathcal{U}_k) = \{[D_\lambda] \mid \lambda \in \mathcal{P}\}$.

By construction of E and F , these crystals are related to the (weak) Harish-Chandra series, as stated in the following proposition.

Proposition 4.36. — Let $R = K$ or k , and $I = (-q)^{\mathbf{Z}} \subset R^\times$.

- (a) $D \in \text{Irr}(\mathcal{U}_R)$ is weakly cuspidal if and only if $\tilde{F}_i D = 0$ for all $i \in I$.
- (b) If $M, N \in \text{Irr}(\mathcal{U}_R)$, then N appears in the head of $F(M)$ if and only if there exists $i \in I$ such that $N \simeq \tilde{F}_i M$.
- (c) If $D \in \text{Irr}(\mathcal{U}_R)$ is weakly cuspidal, then

$$\text{WIrr}(RG, D) = \{\tilde{F}_{i_1} \cdots \tilde{F}_{i_m}(D) \mid m \in \mathbf{N}, i_1, \dots, i_m \in I\}.$$

In other words, the (uncolored) crystal graph associated with $B(\mathcal{U}_R)$ coincides with the weak Harish-Chandra branching graph, and its connected components with the weak Harish-Chandra series.

Proof. — Assertions (a) and (b) follow from the definition of \tilde{E}_i and \tilde{F}_i and the fact that $E = \bigoplus E_i$ and $F = \bigoplus F_i$.

Let D be a cuspidal unipotent RG_r -module. Since the F_i 's are exact functors, we have $F_{i_1} \cdots F_{i_m}(D) \twoheadrightarrow \tilde{F}_{i_1} \cdots \tilde{F}_{i_m}(D)$ which shows that $\tilde{F}_{i_1} \cdots \tilde{F}_{i_m}(D) \in \text{WIrr}(RG_n, D)$. Therefore to prove (c) it is enough to show that given $n = r + 2m$ we have

$$\text{WIrr}(kG_n, D) \subset \{\tilde{F}_{i_1} \cdots \tilde{F}_{i_m}(D) \mid i_1, \dots, i_m \in I\}.$$

We argue by induction on m . Let $M \in \text{WIrr}(kG_{n+2}, D)$. Then $F^{m+1}(D)$ maps onto M . Thus there is a vertex $i \in I$ such that $F_i F^m(D)$ maps onto M . Since F_i is exact we deduce that there is a composition factor N of $F^m(D)$ such that $F_i N$ maps onto M , *i.e.*, such that $\tilde{F}_i N = M$. The module N lies in a weak Harish-Chandra series $\text{WIrr}(RG_n, E)$ for some weakly cuspidal module E . Since F and therefore F_i preserves the series, we must have $E \simeq D$ (using Proposition 4.9). By induction on m , one can write $N = \tilde{F}_{i_1} \cdots \tilde{F}_{i_m}(D)$, hence $M = \tilde{F}_i N = \tilde{F}_i \tilde{F}_{i_1} \cdots \tilde{F}_{i_m}(D)$. \square

Note that by [36], the ordinary Harish-Chandra series and weak Harish-Chandra series on unipotent modules coincide when $R = K$.

4.7.2. Comparison of the crystals. — Throughout this section we will assume that e , the order of $(-q)$ modulo ℓ is odd. Recall from Section 4.5.2 that $[\mathcal{U}_k]$ is isomorphic to a direct sum of Fock spaces $\mathbf{F}(\mathbf{Q}_t)_e$ where t runs over \mathbf{N} and $\mathbf{Q}_t = (\mathbf{Q}_1, \mathbf{Q}_2)$ is defined by

$$\mathbf{Q}_t = \begin{cases} ((-q)^{-1-t}, (-q)^t) & \text{if } t \text{ is even,} \\ ((-q)^t, (-q)^{-1-t}) & \text{if } t \text{ is odd.} \end{cases}$$

To any charged Fock space one can associate an abstract crystal, see Section 2.3.2. We now show how to choose the charge for each $\mathbf{F}(\mathbf{Q}_t)_e$ so that the crystal will coincide with the Harish-Chandra branching graph. Recall from (2.1) that

$$\sigma_t = \begin{cases} (-t/2, t/2) & \text{if } t \text{ is even,} \\ ((1+t)/2, -(1+t)/2) & \text{if } t \text{ is odd.} \end{cases}$$

Now, we define

$$(4.12) \quad s_t = (s_1, s_2) = -\frac{1}{2}(e+1, 0) + \sigma_t,$$

so that with the assumption on e we have $\mathbf{Q}_p = q^{2s_p}$ for each $p = 1, 2$. In other words s_t is a charge for $\mathbf{F}(\mathbf{Q}_t)_e$ with respect to q^2 . We denote by $\mathbf{B}(s_t)_e$ the corresponding abstract crystal of $\mathbf{F}(\mathbf{Q}_t)_e$, with the canonical labeling $\mathbf{B}(s_t)_e = \{b(\mu, s_t) \mid \mu \in \mathcal{P}^2, t \in \mathbf{N}\}$. Finally, we set $\mathbf{B}_e = \bigsqcup_{t \in \mathbf{N}} \mathbf{B}(s_t)_e$.

We can now prove our main theorem, which compares the crystal \mathbf{B}_e with the abstract crystal coming from the categorical action of $\widehat{\mathfrak{sl}}_e$ on \mathcal{U}_k .

Theorem 4.37. — Assume that e is odd. The map $b(\mu, s_t) \mapsto [D_{\varpi_t(\mu)}]$ is a crystal isomorphism $B_e \xrightarrow{\sim} B(\mathcal{U}_k)$.

Proof. — Recall from Proposition 1.12 that the set $B(\mathcal{U}_k) = \{[D_\lambda] \mid \lambda \in \mathcal{P}\}$ is a perfect basis of $[\mathcal{U}_k]$. Further, the discussion in Section 2.3.2 implies that the family $\mathbf{B}_e^\vee := \bigoplus_{t \in \mathbf{N}} \mathbf{B}^\vee(s_t)$ is a perfect basis of $\bigoplus_{t \in \mathbf{N}} \mathbf{F}(Q_t)_e$ which comes with a canonical labeling $\mathbf{B}_e^\vee = \{\mathbf{b}^\vee(\mu, s_t) \mid \mu \in \mathcal{P}^2, t \in \mathbf{N}\}$. Finally, by Proposition 2.6 the map $\mathbf{b}^\vee(\mu, s_t) \mapsto b(\mu, s_t)$ is a crystal isomorphism $\mathbf{B}_e^\vee \xrightarrow{\sim} B_e$.

Hence, we must prove that the bijection $\varphi : B(\mathcal{U}_k) \rightarrow \mathbf{B}_e^\vee$ such that $[D_{\varpi_t(\mu)}] \mapsto \mathbf{b}^\vee(\mu, s_t)$ is a crystal isomorphism. Since both bases are perfect, by Proposition 1.14, it is enough to prove that there is a partial order $>$ on the set of partitions such that, under the \mathbf{C} -linear isomorphism $\bigoplus_{t \in \mathbf{N}} \mathbf{F}(Q_t)_e \xrightarrow{\sim} [\mathcal{U}_k]$ in (4.9) which maps $|\mu, Q_t\rangle_e$ to $[V_{\varpi_t(\mu)}]$ for each μ, t , we have

$$\mathbf{b}^\vee(\mu, s_t) = \varphi([D_{\varpi_t(\mu)}]) \in [D_{\varpi_t(\mu)}] + \sum_{\varpi_u(\gamma) > \varpi_t(\mu)} \mathbf{c}[D_{\varpi_u(\gamma)}].$$

To do so, we use the main result of [42] to show that one can take $>$ to be the dominance order. Given a positive integer d , and elements $\kappa, s_1, \dots, s_l \in \mathbf{C}$, let $\mathcal{O}^{s, \kappa}\{n\}$ be the category \mathcal{O} of the rational double affine Hecke algebra of type $G(l, 1, n)$ with parameters κ and $s = (s_1, \dots, s_l)$. The parameters are normalized as in [42, Section 6.2.1]. The category $\mathcal{O}^{s, \kappa}\{n\}$ is a highest weight category. We will abbreviate $\mathcal{O}^{s, \kappa} = \bigoplus_{n \in \mathbf{N}} \mathcal{O}^{s, \kappa}\{n\}$, where $\mathcal{O}^{s, \kappa}\{0\}$ is the category of finite dimensional vector spaces. Let $\Delta(\mu)^{s, \kappa}$ and $T(\mu)^{s, \kappa}$ be the standard and the tilting module with lowest weight $\mu \in \mathcal{P}^l$. To characterize the highest weight structure on $\mathcal{O}^{s, \kappa}$ we must fix a partial order $\geq_{s, \kappa}$ on the set of l -partitions such that

$$(T(\mu)^{s, \kappa} : \Delta(\gamma)^{s, \kappa}) \neq 0 \implies \mu \geq_{s, \kappa} \gamma.$$

There are several choices for this order. >From now on we will assume that $s \in \mathbf{Z}^l$ and $\kappa = -e$ with e a positive integer. Then, we choose the same order as in [42, Section 6.2.2], following an idea of [11]. More precisely, recall the combinatorics of l -partitions in Section 2.1. If A, B are boxes of Young diagrams of l -partitions, then we write $A >_s B$ if we have $\text{ct}^s(A) < \text{ct}^s(B)$ or if $\text{ct}^s(A) = \text{ct}^s(B)$ and $p(A) > p(B)$. We'll abbreviate $\tilde{\text{ct}}(A) = \text{ct}^s(A) - ep(A)/l$. Then, we define the partial order on \mathcal{P}^l by setting $\mu \geq_{s, -e} \gamma$ if and only if there are orderings $Y(\mu) = \{A_n\}$ and $Y(\gamma) = \{B_n\}$ such that $A_n \geq_s B_n$ for all n . Note that, since $-e < 0$, if $\mu \geq_{s, -e} \gamma$ and $Y(\mu) = \{A_n\}$ and $Y(\gamma) = \{B_n\}$ are as above, then we have $\tilde{\text{ct}}(A_n) \leq \tilde{\text{ct}}(B_n)$ for all n .

For each $p = 1, 2, \dots, l$ we set $Q_p = u^s$ where $u = \exp(2\sqrt{-1}\pi/e)$. We consider the charged Fock space $\mathbf{F}(s) = (\mathbf{F}(Q), s)$ of \mathfrak{sl}_e . It is equipped with the Uglov canonical basis $\mathbf{B}^+(s) = \{\mathbf{b}^+(\mu, s) \mid \mu \in \mathcal{P}^l\}$. Now, set $s^* = (-s_l, \dots, -s_2, -s_1)$ and

$\mu^\star = ({}^t\mu^1, \dots, {}^t\mu^2, {}^t\mu^1)$ where ${}^t\lambda$ denotes the transposed partition of λ . By [42, Theorem 7.3] there is a \mathbf{C} -linear isomorphism $[\mathcal{O}^{s^\star, -e}] \xrightarrow{\sim} \mathbf{F}(\mathbf{Q})$ such that

$$[\Delta(\gamma^\star)^{s^\star, -e}] \mapsto |\gamma, \mathbf{Q}\rangle, \quad [\mathrm{T}(\mu^\star)^{s^\star, -e}] \mapsto \mathbf{b}^+(\mu, s)$$

from which we deduce that

$$\mathbf{b}^+(\mu, s) \in |\mu, \mathbf{Q}\rangle + \sum_{\mu^\star >_{s^\star, -e} \gamma^\star} \mathbf{C} |\gamma, \mathbf{Q}\rangle.$$

Using the transpose and the fact that $\gamma^\star >_{s^\star, -e} \mu^\star$ if and only if $\mu >_{s, -e} \gamma$ we obtain

$$(4.13) \quad \mathbf{b}^\vee(\mu, s) \in |\mu, \mathbf{Q}\rangle + \sum_{\mu >_{s, -e} \gamma} \mathbf{C} |\gamma, \mathbf{Q}\rangle.$$

Now, we consider the particular case where e is odd and $s = s_t$ for some $t \in \mathbf{N}$ as in (4.12). The following holds.

Lemma 4.38. — *Assume that the positive integer e is odd. For all bipartitions μ, γ we have $\mu >_{s_t, -e} \gamma \Rightarrow \varpi_t(\gamma) > \varpi_t(\mu)$, where $>$ is the dominance order.*

Proof. — Let $\lambda = \varpi_t(\mu)$ and $\nu = \varpi_t(\gamma)$. By construction, the charged β -sets of λ and μ are related by

$$\beta_0(\lambda) = \{2u - 1 \mid u \in \beta_{\sigma_1}(\mu^1)\} \sqcup \{2u \mid u \in \beta_{\sigma_2}(\mu^2)\}.$$

In order to compare λ and ν we will compare the numbers of elements in the sets $\{A \in Y(\lambda) \mid \mathrm{ct}(A) \geq j\}$ and $\{B \in Y(\nu) \mid \mathrm{ct}(B) \geq j\}$ and invoke [11, Lemma 4.2]. To this end we proceed as in [11, p. 814] by computing, for all integers j

$$\begin{aligned} & \sharp\{A \in Y(\lambda) \mid \mathrm{ct}(A) \geq j\} \\ &= \sum_{\substack{k \in \mathbf{Z} \\ k \geq j}} \sharp\{u \in \beta_0(\lambda) \mid u \geq k + 1\} + a_j \\ &= \sum_{\substack{k \in \mathbf{Z} \\ 2k \geq j}} (\sharp\{u \in \beta_{\sigma_1}(\mu^1) \mid u \geq k + 1\} + \sharp\{u \in \beta_{\sigma_2}(\mu^2) \mid u \geq k + 1\}) \\ &\quad + \sum_{\substack{k \in \mathbf{Z} \\ 2k+1 \geq j}} (\sharp\{u \in \beta_{\sigma_1}(\mu^1) \mid u \geq k + 2\} + \sharp\{u \in \beta_{\sigma_2}(\mu^2) \mid u \geq k + 1\}) \\ &\quad + a_j \\ &= \sum_{\substack{k \in \mathbf{Z} \\ 2k \geq j}} \sharp\{A \in Y(\mu^1) \mid \mathrm{ct}(A) = k - \sigma_1\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{k \in \mathbf{Z} \\ 2k \geq j}} \sharp \{A \in Y(\mu^2) \mid \text{ct}(A) = k - \sigma_2\} \\
& + \sum_{\substack{k \in \mathbf{Z} \\ 2k+1 \geq j}} \sharp \{A \in Y(\mu^1) \mid \text{ct}(A) = k - \sigma_1 + 1\} \\
& + \sum_{\substack{k \in \mathbf{Z} \\ 2k+1 \geq j}} \sharp \{A \in Y(\mu^2) \mid \text{ct}(A) = k - \sigma_2\} + a_{j,t},
\end{aligned}$$

where the integers a_j and $a_{j,t}$ are constants which depend only on j and t . Next, assume that e is odd. Recall from (4.12) that $s_1 = \sigma_1 + d$ and $s_2 = \sigma_2$ with $d = -(e+1)/2$. Now, we identify $Y(\mu)$ with $Y(\mu^1) \sqcup Y(\mu^2)$. We deduce that

$$\begin{aligned}
\sharp \{A \in Y(\lambda) \mid \text{ct}(A) \geq j\} &= \sharp \{A \in Y(\mu^1) \mid \text{ct}^{s_1}(A) \geq \lceil j/2 \rceil + d\} \\
&+ \sharp \{A \in Y(\mu^2) \mid \text{ct}^{s_2}(A) \geq \lceil j/2 \rceil\} \\
&+ \sharp \{A \in Y(\mu^1) \mid \text{ct}^{s_1}(A) \geq \lceil (j-1)/2 \rceil + d + 1\} \\
&+ \sharp \{A \in Y(\mu^2) \mid \text{ct}^{s_2}(A) \geq \lceil (j-1)/2 \rceil\} + a_{j,t}.
\end{aligned}$$

In particular, we get

$$\begin{aligned}
\sharp \{A \in Y(\lambda) \mid \text{ct}(A) \geq j\} &= \sharp \{A \in Y(\mu^1) \mid \widetilde{\text{ct}}(A) \geq \lceil j/2 \rceil - e - 1/2\} \\
&+ \sharp \{A \in Y(\mu^2) \mid \widetilde{\text{ct}}(A) \geq \lceil j/2 \rceil - e\} \\
&+ \sharp \{A \in Y(\mu^1) \mid \widetilde{\text{ct}}(A) \geq \lceil (j-1)/2 \rceil - e + 1/2\} \\
&+ \sharp \{A \in Y(\mu^2) \mid \widetilde{\text{ct}}(A) \geq \lceil (j-1)/2 \rceil - e\} + a_{j,t}.
\end{aligned}$$

Finally, since e is odd and $\text{ct}^{s_i}(A)$ is an integer we can remove the symbols $\lceil \bullet \rceil$ and we obtain

$$\begin{aligned}
\sharp \{A \in Y(\lambda) \mid \text{ct}(A) \geq j\} &= \sharp \{A \in Y(\mu) \mid \widetilde{\text{ct}}(A) \geq j/2 - e - 1/2\} \\
&+ \sharp \{A \in Y(\mu) \mid \widetilde{\text{ct}}(A) \geq j/2 - e\} + a_{j,t}.
\end{aligned}$$

Now, assume that $\mu \geq_{s_t, -e} \gamma$. Then we can choose orderings $Y(\mu) = \{A_n\}$ and $Y(\gamma) = \{B_n\}$ such that $\widetilde{\text{ct}}(A_n) \leq \widetilde{\text{ct}}(B_n)$ for all n . We deduce that for all integer j we have

$$\sharp \{A \in Y(\nu) \mid \text{ct}(A) \geq j\} \geq \sharp \{A \in Y(\lambda) \mid \text{ct}(A) \geq j\},$$

from which we deduce that $\nu \geq \lambda$ (for the dominance order) by [11, Lemma 4.2]. \square

Proof of Theorem 4.37. Continued. From the lemma above together with (4.13) we deduce that

$$\mathbf{b}^\vee(\mu, s_i) \in |\mu, Q_i\rangle_e + \sum_{\varpi_i(\gamma) > \varpi_i(\mu)} \mathbf{c} |\gamma, Q_i\rangle_e.$$

Hence, under the identification of $[\mathcal{U}_k]$ with $\bigoplus_{i \in \mathbf{N}} \mathbf{F}(Q_i)_e$ we obtain

$$\mathbf{b}^\vee(\mu, s_i) \in [V_{\varpi_i(\mu)}] + \sum_{\varpi_i(\gamma) > \varpi_i(\mu)} \mathbf{c} [V_{\varpi_i(\gamma)}].$$

On the other hand, by Proposition 4.4 we have

$$[D_\lambda] \in [V_\lambda] + \sum_{v > \lambda} \mathbf{c} [V_v].$$

Therefore one can use Proposition 1.14, to deduce that the map $[D_{\varpi_i(\mu)}] \mapsto \mathbf{b}^\vee(\mu, s_i)$ is a crystal isomorphism $B(\mathcal{U}_k) \xrightarrow{\sim} \mathbf{B}_e^\vee$. \square

As a direct consequence of this theorem and Proposition 4.36, the unipotent module $D_{\varpi_i(\mu)}$ in \mathcal{U}_k is weakly cuspidal if and only if we have $\tilde{E}_i(b(\mu, Q_i)) = 0$ for all $i \in I_e$.

Remark 4.39. — Our choice for the charge s_i differs from the choice used by Gerber-Hiss-Jacon in [24], where the charge considered is $\mathbf{c}_i = (t + (1 - e)/2, 0)$. N. Jacon observed the following relation between s_i and \mathbf{c}_i .

- If t is odd, then $\mathbf{c}_i = s_i + ((1 + t)/2, (1 + t)/2)$, so that $b(\mu, \mathbf{c}_i) \mapsto b(\mu, s_i)$ is a crystal isomorphism.
- If t is even, let us consider $\tilde{s}_i = \mathbf{c}_i - ((1 + t - e)/2, (1 + t - e)/2) = (t/2, (-t + e - 1)/2)$. Then s_i is obtained from \tilde{s}_i by the operation $(a, b) \mapsto (b - e, a)$ which induces a crystal isomorphism $b(\mu, s_i) \mapsto b(\mu^\dagger, \tilde{s}_i)$ where $(\mu^1, \mu^2)^\dagger = (\mu^2, \mu^1)$. Consequently, $b(\mu, s_i) \mapsto b(\mu^\dagger, \mathbf{c}_i)$ induces a crystal isomorphism.

The discrepancy with respect to the parity of t comes from the parametrization of the simple modules of the Hecke algebra $\mathbf{H}_{K,m}^{Q_e, q^2}$ (see Section 4.4.3).

Remark 4.40. — When e is even and $t \geq 0$, the category $\mathcal{U}_{k,t}$ categorifies the level 2 Fock space $\mathbf{F}(Q_i)_e$ by Corollary 4.26. It has a tensor product decomposition into level 1 Fock spaces $\mathbf{F}(-q^{-1}q^{2\sigma_1})_e \otimes \mathbf{F}(q^{2\sigma_2})_e$. Here, we view q as an element of k . One can therefore construct an abstract crystal of $\mathbf{F}(Q_i)_e$ out of the crystals of the charged level 1 Fock spaces $(\mathbf{F}(q^{2\sigma_1}), \sigma_1)$, $(\mathbf{F}(q^{2\sigma_2}), \sigma_2)$ under the identification $\mathbf{F}(Q_i)_e = \mathbf{F}(q^{2\sigma_1}) \otimes \mathbf{F}(q^{2\sigma_2})$ in Section 4.6.2. This crystal coincides with the crystal $B(\mathcal{U}_{k,t})$ coming from the categorification.

Remark 4.41. — By Theorem 4.14, the category $\mathcal{U}_{K,t}$ categorifies the level 2 Fock space $\mathbf{F}(Q_t)_\infty$. It has a tensor product decomposition into level 1 Fock spaces $\mathbf{F}(-q^{-1}q^{2\sigma_1})_\infty \otimes \mathbf{F}(q^{2\sigma_2})_\infty$. Here, we view q as an element of K . One can therefore construct an abstract crystal of $\mathbf{F}(Q_t)_\infty$ out of the crystals of the charged level 1 Fock spaces $(\mathbf{F}(q^{2\sigma_1}), \sigma_1)$, $(\mathbf{F}(q^{2\sigma_2}), \sigma_2)$ under the identification $\mathbf{F}(Q_t)_\infty = \mathbf{F}(q^{2\sigma_1}) \otimes \mathbf{F}(q^{2\sigma_2})$ associated with the map $(-q)^{b-2+2\mathbf{Z}} \rightarrow q^{2\mathbf{Z}}$ such that $i \mapsto i(-q)^{2-p}$ for each $p = 1, 2$. This crystal coincides with the crystal $B(\mathcal{U}_{K,t})$ coming from the categorification.

4.7.3. Corollaries. — As a byproduct of our main theorem, we obtain a proof of the conjectures of Gerber-Hiss-Jacon stated in [24].

Corollary 4.42. — *Assume that e is odd. The modules D_λ and D_ν lie in the same weak Harish-Chandra series if and only if the corresponding vertices of the abstract crystal B_e belong to the same connected component. In particular, if this holds then λ and ν have the same 2-core.*

Proof. — Let μ, γ be bipartitions and t, u be non-negative integers such that $\lambda = \varpi_t(\mu)$ and $\nu = \varpi_u(\gamma)$. Then by Proposition 4.36 and Theorem 4.37, the vertices $b(\mu, s_t)$ and $b(\gamma, s_u)$ of the abstract crystal B_e lie in the same connected component. This implies $t = u$. \square

Corollary 4.43. — *Assume that e is odd and that λ is a partition of n . If D_λ is weakly cuspidal, then λ is a 2-core.*

Proof. — It follows from the combination of [24, Theorem 7.6] and Theorem 4.37. \square

Let $r > 0$, $m \geq 0$ and $n = r + 2m$. Recall from Section 1.3.1 that there is k -algebra homomorphism

$$\phi_{k,m} : \mathbf{H}_{k,m}^{q^2} \longrightarrow \text{End}_{kG_n}(F^m)^{\text{op}}.$$

As in Theorem 4.12, one can show that the evaluation at a weakly cuspidal module is a Hecke algebra of type B_m .

Corollary 4.44. — *Let λ be a partition of $r > 0$ such that D_λ is weakly cuspidal. Let $(t, t - 1, \dots, 1)$ be the e -core of λ . Then for $m \geq 0$ and $n = r + 2m$, the map $\phi_{k,m}$ factors through an algebra isomorphism*

$$\mathbf{H}_{k,m}^{Q, q^2} \xrightarrow{\sim} \mathcal{H}(kG_n, D_\lambda).$$

Proof. — We first compute the weight of $[D_\lambda]$. By Corollary 4.43, the e -core of λ is a 2-core $\Delta_t = (t, t - 1, \dots, 1)$ for some $t \geq 0$. Then from Proposition 2.4 and (4.10) we deduce that the projection of the weights ω_λ and ω_{Δ_t} to the weight lattice P_e of \mathfrak{g}_e are

equal. By Lemma 4.18 and (4.9) we deduce that this projection equals $\Lambda_{\mathcal{Q}_\ell}$. We can now invoke Proposition 1.8 for the module D_λ . We have $ED_\lambda = 0$ since D_λ is weakly cuspidal, and $\text{End}_{\mathcal{U}_k}(D_\lambda) = k$ since D_λ is simple and k is a splitting field for G_r . This shows that $\phi_{k,m}$ induces the expected isomorphism. \square

Remark 4.45. — Corollary 4.42 proves Conjecture 5.4 in [24, Section 5.4]. Corollary 4.43 proves Conjecture 5.5 in [24, Section 5.4]. Corollary 4.44 generalizes the argument in [18, Proposition 5.21], in the particular case of the unitary group, and it proves the conjecture in [25, Section 5].

5. The representation of the Heisenberg algebra on \mathcal{U}_k

Every cuspidal kG_n -module is weakly cuspidal. Therefore, every Harish-Chandra series in $\text{Irr}(\mathcal{U}_k)$ is partitioned into weak Harish-Chandra series. Proposition 4.36 and Theorem 4.37 yield a complete (combinatorial) description of the partition of $\text{Irr}(\mathcal{U}_k)$ in weak Harish-Chandra series using the decomposition of $[\mathcal{U}_k]$ for the action of \mathfrak{g}_e . In this section we generalize this result to Harish-Chandra series by considering an action of some Heisenberg algebra on $[\mathcal{U}_k]$. The results in this section are partly conjectural. We'll be sketchy, the details will appear elsewhere. Throughout this section we will assume that e , the order of $-q$ modulo ℓ is odd, and hence equal to d , the order of q^2 modulo ℓ .

5.1. The Heisenberg action on $\mathbf{F}(\mathcal{Q})$. — Set $\Lambda = \bigoplus_{m \in \mathbf{N}} \mathbf{CIrr}(K\mathfrak{S}_m)$ where \mathfrak{S}_0 is the trivial group. We equip the vector space Λ with the bilinear form $\langle \bullet, \bullet \rangle_\Lambda = \bigoplus_{m \in \mathbf{N}} \langle \bullet, \bullet \rangle_{\mathfrak{S}_m}$ and the vector space $\Lambda \otimes \Lambda$ with its tensor square. The induction and restriction of representations of symmetric groups yield a pair of linear maps $\text{Ind}_{n,m}^{n+m}$ and $\text{Res}_{n,m}^{n+m}$ between the \mathbf{C} -vector spaces Λ and $\Lambda \otimes \Lambda$ which are adjoint relatively to these scalar products.

Fix a positive integer e . A level e representation of the Heisenberg algebra \mathfrak{H}_e on a \mathbf{C} -vector space V is a family of endomorphisms b_n, b_n^* of V with $n \in e\mathbf{Z}_{>0}$ satisfying the relations $[b_n, b_m] = [b_n^*, b_m^*] = 0$ and $[b_n, b_m^*] = -n\delta_{n,m}$ for $n, m > 0$. For each partition λ we write $e\lambda = (e\lambda_1, e\lambda_2, \dots)$ and $b_{e\lambda} = \prod_i b_{e\lambda_i}$, $b_{e\lambda}^* = \prod_i b_{e\lambda_i}^*$.

If λ is a partition of m and $w \in \mathfrak{S}_m$ is of *cycle-type* $\rho(w) = \lambda$, let $c_\lambda = c_w$ denote the conjugacy class of w . The characteristic function of this class is a class function on \mathfrak{S}_m , and as such it can be viewed as an element in Λ . We abbreviate $c_m = c_{(m)}$. Let z_λ be the number of elements of the centralizer of w in \mathfrak{S}_m .

There is a unique representation of \mathfrak{H}_e on Λ such that for each $\phi \in \mathbf{CIrr}(\mathfrak{S}_r)$ and $\psi \in \mathbf{CIrr}(\mathfrak{S}_n)$ with $n = r + em$ we have

$$(5.1) \quad b_{e\lambda}(\phi) = \text{Ind}_{r,em}^n(\phi \otimes z_{e\lambda} c_{e\lambda}), \quad b_{e\lambda}^*(\psi) = \langle \text{Res}_{r,em}^n(\psi), z_{e\lambda} c_{e\lambda} \rangle_\Lambda.$$

We can now define the representation of \mathfrak{H}_e on $\mathbf{F}(\mathcal{Q})$. >From now on we will assume that e is odd and that \mathfrak{g} is isomorphic to \mathfrak{g}_e (which is isomorphic to $\widehat{\mathfrak{sl}}_e$).

- Assume that $l = 1$. Then, the representation of \mathfrak{g}' on $\mathbf{F}(\mathbf{Q})$ admits a commuting action of \mathfrak{H}_e which is identified with (5.1) under the \mathbf{C} -linear isomorphism $\mathbf{F}(\mathbf{Q}) \xrightarrow{\sim} \mathbf{\Lambda}$ such that $|\nu, \mathbf{Q}\rangle \mapsto \phi_\nu$.
- Assume that $l > 1$. Using (2.2) we get a \mathfrak{g}' -module isomorphism $\mathbf{F}(\mathbf{Q}) \xrightarrow{\sim} \mathbf{\Lambda}^{\otimes l}$. The representation of \mathfrak{g}' on $\mathbf{F}(\mathbf{Q})$ admits a commuting action of \mathfrak{H}_{le} which is the l -th tensor power of the representations of \mathfrak{H}_e on $\mathbf{\Lambda}$.

Composing the assignment $\varepsilon_\lambda \chi_\lambda = [E_\lambda] \mapsto (-1)^{a(\lambda)} \phi_\lambda$ with the decomposition map $d_{\mathcal{U}}$, we get a \mathbf{C} -linear isomorphism

$$(5.2) \quad [\mathcal{U}_k] \xrightarrow{\sim} \mathbf{\Lambda}, \quad [V_\lambda] \mapsto (-1)^{a(\lambda)} \phi_\lambda.$$

It fits into the commutative square of isomorphisms

$$\begin{array}{ccc} [\mathcal{U}_k] & \xrightarrow{5.2} & \mathbf{\Lambda} \\ \uparrow 4.21 & & \uparrow \phi \\ \bigoplus_t \mathbf{F}(\mathbf{Q})_e & \xrightarrow{4.18} & \mathbf{F}(1)_{e,\circ}. \end{array}$$

The maps (4.21), (4.18) are \mathfrak{g}_e -linear and ϕ is \mathfrak{H}_e -linear. We equip $[\mathcal{U}_k]$ with the unique representation of \mathfrak{H}_{2e} such that (4.21) is \mathfrak{H}_{2e} -linear. Under the obvious inclusion $\mathfrak{H}_{2e} \subseteq \mathfrak{H}_e$ the vector spaces $\mathbf{F}(1)_{e,\circ}$ and $\mathbf{\Lambda}$ can be viewed as \mathfrak{H}_{2e} -modules.

Lemma 5.1. — *The map (5.2) is an isomorphism of \mathfrak{H}_{2e} -modules $[\mathcal{U}_k] \xrightarrow{\sim} \text{Res}_{\mathfrak{H}_{2e}}^{\mathfrak{H}_e} \mathbf{\Lambda}$.* \square

5.2. *The modular Harish-Chandra series of GL_m .* — Given non-negative integers m_j with $j \geq -1$, we will say that a partition λ of m is *cuspidal of type (m_j)* if

$$\lambda = (1^{(m_{-1})}, e^{(m_0)}, (e\ell)^{(m_1)}, (e\ell^2)^{(m_2)}, \dots)$$

in exponential notation. For such a partition we consider the standard Levi subgroup of \mathbf{GL}_m given by

$$\mathbf{GL}_\lambda = (\mathbf{GL}_1)^{m_{-1}} \times \prod_{j \geq 0} (\mathbf{GL}_{e\ell^j})^{m_j}.$$

The unipotent module $S_{(1^m)}$ is the head of $W_{(1^m)}$, which is a modular reduction of the Steinberg character $L_{(1^m)}$ of KGL_m . We call it the *modular Steinberg representation* of kGL_m and we write $St_m = S_{(1^m)}$. Let St_λ be the kGL_λ -module given by

$$St_\lambda = (St_1)^{\otimes m_{-1}} \otimes \bigotimes_{j \geq 0} (St_{e\ell^j})^{\otimes m_j},$$

and set

$$X_\lambda = R_{\mathbf{GL}_\lambda}^{\mathbf{GL}_m}(St_\lambda) = (St_1)^{\odot m-1} \odot \bigodot_{j \geq 0} (St_{e^j})^{\odot m_j}.$$

The following is due to Dipper-James and Dipper-Du.

Proposition 5.2.

- (a) Any unipotent cuspidal pair of \mathbf{kGL}_m is \mathbf{GL}_m -conjugate to a pair $(\mathbf{GL}_\lambda, St_\lambda)$ where λ runs over the set of all cuspidal partitions of m .
- (c) If λ is cuspidal of type (m_j) , then we have a \mathbf{k} -algebra isomorphism

$$\mathcal{H}(\mathbf{kGL}_m, St_\lambda) \simeq \mathbf{H}_{\mathbf{k}, m-1}^{1; q^2} \otimes \bigotimes_{j \geq 0} \mathbf{H}_{\mathbf{k}, m_j}^{1; 1}. \quad \square$$

Recall that e is odd. For each partition λ of m we fix an F_{q^2} -stable maximal torus \mathbf{T}_λ of \mathbf{GL}_m of cycle-type λ . For a future use, let us quote the following result.

Lemma 5.3. — If λ is a cuspidal partition of m then \mathbf{T}_λ is a maximal torus of \mathbf{GL}_λ and we have $[X_\lambda] = R_{\mathbf{T}_\lambda}^{\mathbf{GL}_m}(1)$ in $[\mathcal{U}_\mathbf{k}]$. \square

5.3. The functors \mathbf{B} and \mathbf{B}^* . — The set of unipotent characters of \mathbf{KGL}_n is

$$\mathrm{Unip}(\mathbf{KGL}_n) = \{L_\lambda \mid \lambda \in \mathcal{P}_n\},$$

where L_λ is given by

$$L_\lambda = \frac{1}{|\mathfrak{S}_n|} \sum_{w \in \mathfrak{S}_n} \phi_\lambda(w) R_{\mathbf{T}_w}^{\mathbf{GL}_n}(1).$$

Here \mathbf{T}_w is an F_{q^2} -stable maximal torus of \mathbf{GL}_n of cycle-type $\rho(w)$, with the convention that $\mathbf{T}_1 = \mathbf{T}$ is the split torus of diagonal matrices. There is a unique labeling

$$\mathrm{Unip}(\mathbf{kGL}_n) = \{S_\lambda \mid \lambda \in \mathcal{P}_n\}$$

such that

$$d_{\mathcal{O}\mathbf{GL}_n}([L_\lambda]) = [S_\lambda] \bmod \bigoplus_{\mu > \lambda} \mathbf{Z}[S_\mu].$$

Let W_λ be the unique indecomposable unipotent \mathbf{kG}_n -module such that

$$[W_\lambda] = d_{\mathcal{O}\mathbf{GL}_n}([L_\lambda]), \quad \mathrm{hd}(W_\lambda) \simeq S_\lambda.$$

Recall that the vector space $\mathbf{\Lambda}$ is equipped with the bilinear form $\langle \bullet, \bullet \rangle_{\mathbf{\Lambda}}$. Let $\langle \bullet, \bullet \rangle_{\text{GL}}$ be the bilinear form on $[\text{kGL}_m\text{-umod}]$ such that the \mathbf{C} -linear isomorphism

$$(5.3) \quad \bigoplus_m [\text{kGL}_m\text{-umod}] \xrightarrow{\sim} \mathbf{\Lambda}, \quad [W_\lambda] \mapsto \phi_\lambda$$

is an isometry. Similarly, let $\langle \bullet, \bullet \rangle_{\mathcal{U}}$ be the bilinear form on $[\mathcal{U}_k]$ such that the isomorphism (5.2) is an isometry.

Recall that $R = K$ or k . Given $n, r, m > 0$ such that $n = r + 2m$, we write $L_{r,m} \simeq G_r \times \text{GL}_m$. For any unipotent RGL_m -module X we have the following functors

$$B_X : \text{RG}_r\text{-umod} \rightarrow \text{RG}_n\text{-umod}, \quad M \mapsto R_{L_{r,m}}^{G_n}(M \otimes X),$$

$$B_X^* : \text{RG}_n\text{-umod} \rightarrow \text{RG}_r\text{-umod}, \quad M \mapsto \text{Hom}_{\text{RGL}_m}(X, {}^*R_{L_{r,m}}^{G_n}(M)).$$

The functor B_X^* is right adjoint to B_X . In the particular case where $m = 1$ and $X = R$ is the trivial module, we recover the functors F, E defined in Section 4.2. The functor B_X is exact, hence it yields a \mathbf{C} -linear endomorphism $[B_X]$ of $[\mathcal{U}_k]$. The functor B_X^* is not exact and the module X may not have a finite projective dimension in $\text{kGL}_m\text{-umod}$. Hence, the right derived functor of B_X^* may not yield any endomorphism of $[\mathcal{U}_k]$. Nevertheless, we can define the \mathbf{C} -linear map

$$[B_X^*] = \langle [X], [{}^*R_{L_{r,m}}^{G_n}](\bullet) \rangle_{\text{GL}}.$$

Proposition 5.4. — *Let X, X' be modules in $\bigoplus_{m>0} \text{kGL}_m\text{-umod}$.*

- (a) *There is an isomorphism of functors $B_X B_{X'} \simeq B_{X \otimes X'} \simeq B_{X'} B_X$.*
- (b) *There is an equality of linear maps $[B_X^*] \circ [B_{X'}^*] = [B_{X \otimes X'}^*] = [B_{X'}^*] \circ [B_X^*]$.*
- (c) *The linear maps $[B_X], [B_X^*]$ are adjoint relatively to the bilinear form $\langle \bullet, \bullet \rangle_{\mathcal{U}}$.*
- (d) *If $M \in \mathcal{U}_k$ is cuspidal, then $[B_X^*](M) = 0$.* □

Now, we define the functors given by $B_\lambda = B_{X_\lambda}$ if λ is a cuspidal partition, and $A_\lambda = B_{W_\lambda}$ for any partition λ . Note that the endofunctors F, E of \mathcal{U}_k are given by $F = B_1$ and $E = B_1^*$. The functors B_λ^* and A_λ^* are not well-defined, but we may still consider the maps $[B_\lambda^*]$ and $[A_\lambda^*]$.

We will be mostly interested by the functors A_λ associated with partitions λ whose parts are all divisible by e , and by the functors B_λ associated with cuspidal partitions λ of type (m_j) with $m_{-1} = 0$. We will call these partitions **-cuspidal* partitions.

The representation of \mathfrak{H}_{2e} on $[\mathcal{U}_k]$ is determined by the linear operators b_n, b_n^* with $n \in 2e\mathbf{Z}_{>0}$ which satisfy the relations $[b_n, b_m] = [b_n^*, b_m^*] = 0$ and $[b_n, b_m^*] = -n \delta_{n,m} \text{Id}$. Our goal in this section is to re-interpret these operators in representation theoretic terms using the category \mathcal{U}_k .

For any partition λ whose parts are divisible by e we define the operators

$$a_{2\lambda} = \sum_v \chi_v^\lambda z_v^{-1} b_{2v}, \quad a_{2\lambda}^* = \sum_v \chi_v^\lambda z_v^{-1} b_{2v}^*,$$

where χ_v^λ is the value of the irreducible character ϕ_λ of the symmetric group on a permutation of cycle-type v .

Proposition 5.5. — Assume that $\ell \neq 2$. We have the following equalities in $\text{End}([\mathcal{U}_k])$

- (a) $a_{2\lambda} = [A_\lambda]$ and $a_{2\lambda}^* = [A_\lambda^*]$ if the parts of λ are divisible by e ,
- (b) $b_{2\lambda} = [B_\lambda]$ and $b_{2\lambda}^* = [B_\lambda^*]$ if the partition λ is $*$ -cuspidal.

□

5.4. The conjecture. — By Proposition 4.36, the set of isomorphism classes of weakly cuspidal modules in \mathcal{U}_k is a basis of the space $[\mathcal{U}_k]^{\leq 0}$ consisting of all elements in $[\mathcal{U}_k]$ which are killed by the map $[E_i]$ for each $i \in I_e$. We define

$$[\mathcal{U}_k]^{\text{hw}} = \{x \in [\mathcal{U}_k]^{\leq 0} \mid a_v^*(x) = 0, \forall v \in \mathcal{P}\}.$$

From Propositions 5.4, 5.5 we get the following inclusion.

Lemma 5.6. — $\{[D] \mid D \in \text{Irr}(\mathcal{U}_k) \text{ is cuspidal}\} \subseteq [\mathcal{U}_k]^{\text{hw}}.$

□

However, it is not obvious that the vector space $[\mathcal{U}_k]^{\text{hw}}$ is spanned by classes of irreducible unipotent modules.

Conjecture 5.7. — The set $\{[D] \mid D \in \text{Irr}(\mathcal{U}_k) \text{ is cuspidal}\}$ is a basis of $[\mathcal{U}_k]^{\text{hw}}$.

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O. D.

Université Paris Diderot, UFR de Mathématiques,
Bâtiment Sophie Germain, 8 place Aurélie Nemours,
75013 Paris, France
olivier.dudas@imj-prg.fr

M. V.

Université de Cergy-Pontoise, laboratoire AGM,
2 avenue A. Chauvin,
95302 Cergy-Pontoise Cedex, France
michela.varagnolo@u-cergy.fr

E. V.

Université Paris Diderot, UFR de Mathématiques,
Bâtiment Sophie Germain, 8 place Aurélie Nemours,
75013 Paris, France
eric.vasserot@imj-prg.fr

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