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MODULI OF REPRESENTATIONS OF THE FUNDAMENTAL GROUP OF A SMOOTH PROJECTIVE VARIETY. II

by CARLOS T. SIMPSON

Introduction

This second part is devoted to the subject of the title, moduli spaces of representations of the fundamental group of a smooth complex projective variety X. We study three moduli spaces for related objects. The *Betti moduli space* $\mathbf{M}_B(X, n)$ is a coarse moduli space for rank n representations of the fundamental group of the usual topological space X^{top}. A *vector bundle with integrable connection* is a pair (E, ∇) where E is a vector bundle and $\nabla : E \to E \otimes \Omega^1_X$ is an operator satisfying the Leibniz rule and $\nabla^2 = 0$. The *de Rham moduli space* $\mathbf{M}_{DR}(X, n)$ is a coarse moduli space for rank n vector bundles with integrable connection on X. A *Higgs bundle* [Hi1] [Si5] is a pair (E, φ) where E is a vector bundle and $\varphi : E \to E \otimes \Omega^1_X$ is a morphism of $\mathcal{O}_X$-modules such that $\varphi \wedge \varphi = 0$. There is a condition of semistability analogous to that for vector bundles, but only concerning subsheaves preserved by φ. The *Dolbeault moduli space* $\mathbf{M}_{Dol}(X, n)$ is a coarse moduli space for rank n semistable Higgs bundles with Chern classes vanishing in rational cohomology. In all three cases, the objects in question form an abelian category in which we can apply the Jordan-Hölder theorem. Let gr(E) denote the direct sum of the subquotients in a Jordan-Hölder series for E, and say that E_1 is *Jordan equivalent* to E_2 if $\mathrm{gr}(E_1) \cong \mathrm{gr}(E_2)$. The points of the coarse moduli spaces parametrize Jordan equivalence classes of objects.

The constructions of these moduli spaces are reviewed in § 5. The construction of $\mathbf{M}_B$ is a classical one from the theory of representations of discrete groups. The construction of $\mathbf{M}_{DR}$ follows from the construction of Part I, § 4, for the case where $\Lambda^{DR} = \mathscr{D}_X$ is the full sheaf of rings of differential operators on X. We give two constructions of $\mathbf{M}_{Dol}$. One is based on an interpretation of Higgs sheaves as coherent sheaves on $T^* X$, and uses the construction of the moduli space of coherent sheaves constructed in Part I, § 1. The other consists of applying the general construction of Part I, § 4, to the case $\Lambda^{Dol} = \mathrm{Sym}^{\bullet}(TX)$.

The three types of objects are related to each other. The *Riemann-Hilbert correspondence* between systems of ordinary differential equations and their monodromy representations provides an equivalence of categories between vector bundles with integrable connection and representations of the fundamental group. To (E, ∇) corresponds the monodromy of the system of equations $\nabla(e) = 0$. The correspondence between Higgs bundles and local systems of [Hi1], [Do3], [Co], [Si2], and [Si5] gives an equivalence of categories between semistable Higgs bundles with vanishing rational Chern classes, and representations of the fundamental group. Together, these correspondences give isomorphisms of sets of points

$$\mathbf{M}_{B}(X, n) \cong \mathbf{M}_{DR}(X, n) \cong \mathbf{M}_{Dol}(X, n).$$

In § 7 we use the analytic results of Part I, § 5, to show that the first map is an isomorphism of the associated complex analytic spaces, and that the second is a homeomorphism of usual topological spaces.

There is a natural algebraic action of the groupe $\mathbf{C}^*$ on the moduli space $\mathbf{M}_{Dol}(X, n)$, given by $t(E, \varphi) = (E, t\varphi)$, and our identifications thus give a natural action—no longer algebraic—on the space of representations. The fixed points of this action are exactly those representations which come from complex variations of Hodge structure [Si5]. Although $\mathbf{M}_{Dol}$ is not compact, the properness of Hitchin's map (Theorem 6.11) implies that $\mathbf{M}_{Dol}(X, n)$ contains the limits of points tE as $t \to 0$. This yields the conclusion that any representation of the fundamental group may be deformed to a complex variation of Hodge structure (Corollary 7.19 below). This theorem was in some sense the principal motivation for constructing the moduli spaces. See [Si5] for more details on some consequences.

The reason for the terminologies Betti, de Rham and Dolbeault is that these moduli spaces may be considered as the analogues for the first nonabelian cohomology, of the Betti cohomology, the algebraic de Rham cohomology, and the Dolbeault cohomology $\bigoplus_{p+q=i} H^q(X, \Omega^p_X)$ of X. The first nonabelian cohomology set $H^1(X, Gl(n, \mathbf{C}))$ is the set of isomorphism classes of rank n representations of $\pi_1(X)$. This has a structure of topological space, but it is not Hausdorff. The universal Hausdorff space to which it maps is the Betti moduli space $\mathbf{M}_B(X, n)$. To explain the analogies for the de Rham and Dolbeault spaces, we have to digress to discuss the Čech realizations of the cohomology groups with complex coefficients.

The algebraic de Rham cohomology is the hypercohomology of X with coefficients in the algebraic de Rham complex $\Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots$ If $X = \bigcup U_\alpha$ is an affine open covering of X, and if we denote multiple intersections by multiple indices, then the cocycles defining $H^1_{DR}(X, \mathbf{C})$ consist of the pairs of collections $(\{g_{\alpha\beta}\}, \{a_\alpha\})$ where $g_{\alpha\beta}$ are regular functions $U_{\alpha\beta} \to \mathbf{C}$, and a_α are one-forms on U_α, such that

$$g_{\alpha\beta} = g_{\beta\gamma} + g_{\alpha\gamma},$$

$$d(g_{\alpha\beta}) = a_\alpha - a_\beta,$$

and

$$d(a_\alpha) = 0.$$

Addition of the coboundary of a collection $\{ s_\alpha \}$, where s_α are regular functions $U_\alpha \to \mathbf{C}$, changes the pair $(\{ g_{\alpha\beta} \}, \{ a_\alpha \})$ to $(\{ g_{\alpha\beta} + s_\alpha - s_\beta \}, \{ a_\alpha + d(s_\alpha) \})$. The group of cocycles modulo coboundaries is $H^1_{DR}(X, \mathbf{C})$. The nonabelian case has formulas which are more complicated, but which reduce to the above if the coefficient group is abelien. A vector bundle with integrable connection is defined by a pair $(\{ g_{\alpha\beta} \}, \{ A_\alpha \})$, where $g_{\alpha\beta} : U_{\alpha\beta} \to Gl(n, \mathbf{C})$ are the gluing functions for the vector bundle, and A_α are $n \times n$ matrix-valued one forms defining the connection $\nabla = d + A_\alpha$. These are subject to the conditions

$$g_{\beta\gamma}\, g_{\alpha\beta} = g_{\alpha\gamma},$$

$$A_\alpha = g_{\alpha\beta}^{-1}\, d(g_{\alpha\beta}) + g_{\alpha\beta}^{-1}\, A_\beta\, g_{\alpha\beta},$$

and

$$d(A_\alpha) + A_\alpha \wedge A_\alpha = 0.$$

A change of local frames by a collection of regular functions $s_\alpha : U_\alpha \to Gl(n, \mathbf{C})$ changes the pair $(\{ g_{\alpha\beta} \}, \{ A_\alpha \})$ to $(\{ s_\beta^{-1}\, g_{\alpha\beta}\, s_\alpha \}, \{ s_\alpha^{-1}\, A_\alpha\, s_\alpha + s_\alpha^{-1}\, d(s_\alpha) \})$. The set of pairs up to equivalence given by such changes of frames, is the first nonabelian de Rham cohomology set $H^1_{DR}(X, Gl(n, \mathbf{C}))$, the set of isomorphism classes of vector bundles with integrable connection on X.

A similar if somewhat looser interpretation gives an analogy between the abelian Dolbeault cohomology group $H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X)$ and the first nonabelian Dolbeault cohomology set $H^1(X, Gl(n, \mathbf{C}))$, the set of isomorphism classes of *Higgs bundles* (E, φ) which are semistable with vanishing rational Chern classes. Here E is a vector bundle and $\varphi \in H^0(X, End(E) \otimes_{\mathcal{O}_X} \Omega^1_X)$. Such a pair may be given a cocycle description similar to the above (just eliminate the terms involving d). The conditions of semistability and vanishing Chern classes are new.

Following this interpretation, we can think of the first nonabelian cohomology as a *nonabelian motive* in a way analogous to [DM], with its Betti, de Rham and Dolbeault realizations. It would be good to have ℓ-adic, and crystalline interpretations in characteristic p.

We treat everything in the relative case of a smooth projective morphism $X \to S$ to a base scheme of finite type over $\mathbf{C}$. This creates some difficulties for the Betti moduli spaces: we have to introduce the notion of *local system of schemes* over a topological space. The relative Betti spaces $\mathbf{M}_B(X/S, n)$ are local systems of schemes over S^{top}. The de Rham and Dolbeault moduli spaces are schemes over S, whose fibers are the de Rham and Dolbeault moduli spaces for the fibers X_s. The interpretation in terms of nonabelian cohomology suggests the existence of a *Gauss-Manin connection* on $\mathbf{M}_{DR}(X/S, n)$, a foliation transverse to the fibers which when integrated gives the transport corresponding to the local system of complex analytic spaces $\mathbf{M}^{an}_B(X/S, n)$. We construct this connection in § 8, using Grothendieck's idea of the crystalline site.

In § 9, we treat the case of other coefficient groups. If G is a reductive algebraic group, we may define the Betti moduli space $\mathbf{M}_B(X, G)$ to be the coarse moduli space for representations of $\pi_1(X)$ in G. We construct the de Rham moduli space $\mathbf{M}_{DR}(X, G)$ for principal G-bundles with integrable connection, and the Dolbeault moduli space $\mathbf{M}_{Dol}(X, G)$ for principal Higgs bundles for the group G, which are semistable with vanishing rational Chern classes, and extend the results of §§ 7 and 8 to these cases.

One corollary is a result valid for representations of any finitely generated discrete group Υ: if $G \to H$ is a morphism of reductive algebraic groups with finite kernel, then the resulting morphism of coarse moduli spaces $\mathbf{M}(\Upsilon, G) \to \mathbf{M}(\Upsilon, H)$ is finite (Corollary 9.16).

Parallel to the discussion of moduli spaces, we discuss the Betti, de Rham and Dolbeault *representation spaces* $\mathbf{R}_B(X, x, n)$, $\mathbf{R}_{DR}(X, x, n)$, and $\mathbf{R}_{Dol}(X, x, n)$. These are fine moduli spaces for objects provided with a frame for the fiber over a base point $x \in X$ (here we assume that X is connected). There are relative versions for X/S where the frames are taken along a section $\xi : S \to X$, and there are versions for principal objects for any linear algebraic group. In § 10, we discuss the local structure of the singularities of the representation spaces, using the deformation theory associated to a differential graded Lie algebra developed by Goldman and Millson [GM]. By Luna's étale slice theorem, this also gives information about the local structure of the moduli spaces. The differential graded Lie algebra controlling the deformation theory of a principal vector bundle with integrable connection or a principal Higgs bundle is *formal* if the object is reductive (in other words, corresponds to a closed orbit under the action of G on $\mathbf{R}(X, x, G)$). By the theory of [GM], this implies that the representation space has a singularity defined by a quadratic form on its Zariski tangent space. Furthermore, the differential graded Lie algebras controling the deformation theories of the flat bundle and the corresponding Higgs bundle are the same. This gives a formal isomorphism between the singularities of the de Rham (or Betti) representation space and the singularities of the Dolbeault representation space, at semisimple points which correspond to each other. This *isosingularity principle* holds also for the singularities of the moduli spaces. The homeomorphism between $\mathbf{M}_{DR}$ and $\mathbf{M}_{Dol}$ is not complex analytic, so these local formal isomorphisms are not directly related to the global homeomorphism.

Finally, in § 11 we discuss the case of representations of the fundamental group of a Riemann surface of genus $g \geqslant 2$. Hitchin calculated the cohomology in the case of rank two projective representations of odd degree, where the moduli space is smooth [Hi1]. We do not attempt to go any further in this direction. We simply treat the most elementary property, irreducibility (which is more or less a calculation of H^0). We treat the case of representations of degree zero, so the moduli space has singularities corresponding to reducible representations; we prove that the singularities are normal. The technique is to use the fact that the Betti moduli space is a complete intersection, and apply Serre's criterion (following a suggestion of M. Larsen, and prompted by a question of E. Witten). We have to verify that there are no singularities in codimension one. To prove irreducibility it suffices to prove that the space is connected, which we do by a simple argument derived from Hitchin's method for calculating the cohomology.

Relationship with part I

It is worth reiterating the nature of the connections between this second part and Part I of the paper. The sections are numbered globally, so we begin with § 6. References to lemmas and such, numbered for sections 1-5, are references to the statements in " Moduli of representations of the fundamental group of a smooth projective variety I ". We rely on the technical work done in part I for many of the constructions of moduli spaces, identifications, and criteria for convergence used here. For the most part, we apply statements from the first part, so Part II can be read without having read Part I in a very detailed way, but just having a copy at hand for reference.

Origins

The correspondence between Higgs bundles and local systems, reflected in the homeomorphism between the Dolbeault and the de Rham or Betti moduli spaces, comes from work of Hitchin [Hi1], Colette [Co] and Donaldson [Do3], as well as [Si2]. The formalism of this correspondence is developed in [Si5].

The original correspondence of this type was the result of Narasimhan and Seshadri [NS] between unitary representations and stable vector bundles. This was subsequently generalized by Donaldson [Do1] [Do2], Mehta and Ramanathan [MR1] [MR2], and Uhlenbeck and Yau [UY].

The idea of obtaining a correspondence between all representations into a noncompact group, and vector bundles provided with the additional structure of a Higgs field, comes from Hitchin's paper [Hi1] with the appendix [Do3] of Donaldson. Hitchin established the correspondence between rank two Higgs bundles and rank two representations on a Riemann surface (and his arguments are easily extended to any rank). Independently, I had arrived at a correspondence between certain representations with noncompact structure group (the complex variations of Hodge structure), and certain holomorphic objects involving an endomorphism valued one-form (systems of Hodge bundles) [Si1]. Deligne and Beilinson had also arrived at a correspondence between systems of Hodge bundles and variations of Hodge structure over a Riemann surface (unpublished work). My definitions and very first results were independent of those of Delinge and Beilinson, then Deligne explained their work and made some important suggestions. I didn't see, until W. Goldman directed me to Hitchin's paper which had just appeared, that one could hope to obtain a correspondence involving all representations. In light of Hitchin's definition, systems of Hodge bundles could be seen as special types of Higgs bundles, and my arguments in higher dimensions could be generalized to the case of Higgs bundles [Si2]. This provided one direction of the correspondence. The other direction (corresponding to Donaldson's appendix to Hitchin's paper) was provided by the results on equivariant harmonic maps and the Bochner formula obtained by Colette in his thesis [Co].

This correspondence provided the motivation for the construction of the moduli space of Higgs bundles. Hitchin gave an analytic construction in his paper, and he obtained all of the interesting properties, such as the properness of the map given by taking the characteristic polynomial of the Higgs field. In my thesis, I had constructed a moduli space for systems of Hodge bundles, using Mumford's construction for vector bundles on curves. The construction presented in part I grew out of this, but uses methods of geometric invariant theory more suited to higher dimensions, as pioneered by Gieseker [Gi] and Maruyama [Ma1] [Ma2]. See the introduction of Part I for further details. When I first discussed this with him, Hitchin advised me that Nitsure had given an algebraic construction for Higgs bundles over a curve [Ni1].

Early on, while I was looking at systems of Hodge bundles, J. Bernstein made the comment that a system of Hodge bundles could be considered as a sheaf on the cotangent bundle of the variety. This remark, generalized to the case of Higgs bundles, forms the basis for one of the constructions of the moduli space of Higgs bundles presented in § 6.

The discussion of the Gauss-Manin connection in § 8 was prompted by a discussion with S. Mochizuki, wherein he pointed out that the analytic connection provided by the Betti trivialisation of $\mathbf{M}^{an}_{DR}(X/S, n)$ over S, was not *a priori* algebraic. The methods used to prove algebraicity are the crystalline methods envisioned by Grothendieck in connection with his construction of the Gauss-Manin connection for abelian cohomology [Gr3]. The existence of the Gauss-Manin connection was announced in [Si3], and a brief sketch of the proof was given.

The material in § 10 about deformation theory is an easy extension of the work of Goldman and Millson [GM]. Their work was, in turn, based on a deformation theory developed by Schlessinger, Stasheff and Deligne.

The proof of irreducibility in § 11 was motivated by an old question posed to me by J. Bloch, and made possible by Hitchin's method of using Morse theory or the $\mathbf{C}^*$ action to calculate the topology of the moduli space (which we use just to show connectedness). My original proof contained a long and technical part showing that the singularities were locally irreducible. E. Witten later posed the question of whether the singularities were normal, and M. Larsen helped by directing me to the place in [Ha] explaining how to use Serre's criterion to prove normality of a complete intersection. The only technical part now needed is an inductive verification that the singularities are in codimension at least two, which makes the argument much shorter.

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6. Moduli spaces for representations

The Betti moduli spaces

We begin with a classical construction from the theory of spaces of representations of discrete groups. Suppose Γ is a finitely generated group. Fix n. Put

$$\mathbf{R}(\Gamma, n) = \mathrm{Hom}(\Gamma, \mathrm{Gl}(n, \mathbf{C})).$$

It is a scheme over $\mathrm{Spec}(\mathbf{C})$ representing the functor which to a $\mathbf{C}$-scheme S associates the set $\mathrm{Hom}(\Gamma, \mathrm{Gl}(n, \mathrm{H}^0(\mathrm{S}, \mathcal{O}_\mathrm{S})))$. The scheme $\mathbf{R}(\Gamma, n)$ can be constructed by choosing generators $\gamma_1, \ldots, \gamma_k$ for Γ. Let Rel denote the set of relations among the γ_i. Then

$$\mathbf{R}(\Gamma, n) \subset \mathrm{Gl}(n, \mathbf{C}) \times \ldots \times \mathrm{Gl}(n, \mathbf{C}) \quad (k \text{ times})$$

is the closed subset defined by the equations $r(m_1, \ldots, m_k) = 1$ for $r \in \mathrm{Rel}$. It is easy to see that this subset represents the required functor—a representation $\rho : \Gamma \to \mathrm{Gl}(n)$ corresponds to the point $(m_1, \ldots, m_k)$ with $m_i = \rho(\gamma_i)$. Note that $\mathbf{R}(\Gamma, n)$ is a closed subset of an affine variety, so it is affine.

The group $\mathrm{Gl}(n, \mathbf{C})$ acts on $\mathbf{R}(\Gamma, n)$ by simultaneous conjugation of the matrices. The orbits under this action are the isomorphism classes of representations.

Two representations ρ and ρ' are said to be *Jordan equivalent* if there exist composition series for each such that the associated graded representations are isomorphic. The theorem of Jordan-Hölder says that the associated graded doesn't depend on the choice of composition series; this semisimple representation is an invariant of the representation, known as its *semisimplification.*

Proposition **6.1.** — *There exists a universal categorical quotient* $\mathbf{R}(\Gamma, n) \to \mathbf{M}(\Gamma, n)$ *by the action of* $\mathrm{Gl}(n, \mathbf{C})$. *The scheme* $\mathbf{M}(\Gamma, n)$ *is an affine scheme of finite type over* $\mathbf{C}$. *The closed points of* $\mathbf{M}(\Gamma, n)$ *represent the Jordan equivalence classes of representations.*

Proof. — This is well known. The quotient is constructed by taking the coordinate ring $\mathrm{A} = \mathrm{H}^0(\mathbf{R}(\Gamma, n), \mathcal{O}_{\mathbf{R}(\Gamma, n)})$, setting $\mathrm{B} = \mathrm{A}^{\mathrm{Gl}(n, \mathbf{C})}$ to be the subring of invariants, and putting $\mathbf{M}(\Gamma, n) = \mathrm{Spec}(\mathrm{B})$. Hilbert proved that B is finitely generated. Mumford shows in [Mu] that Spec(B) is a universal categorical quotient of $\mathrm{Spec}(\mathrm{A}) = \mathbf{R}(\Gamma, n)$. Finally, Seshadri shows that the closed points of the quotient are in one to one correspondence with the closed orbits [Se]. The closed orbits are the orbits corresponding to semisimple representations, and the closed orbit in the closure of a given orbit is the one corresponding to the semisimplification of the given representation. □

Suppose X is a conected smooth projective variety over Spec($\mathbf{C}$). Choose $x \in X$ and let $\Gamma = \pi_1(X^{an}, x)$. We will use the notation

$$\mathbf{R}_B(X, x, n) \stackrel{\text{def}}{=} \mathbf{R}(\Gamma, n)$$

and call this the *Betti representation space*; and the notation

$$M_B(X, n) \stackrel{\text{def}}{=} \mathbf{M}(\Gamma, n)$$

calling this the *Betti moduli space*. This terminology is suggested by the terminology of [DM].

The space $\mathbf{M}_B(X, n)$ does not depend on the choice of x. More precisely, if we include the choice of x in the notation then there are canonical isomorphisms

$$\tau(x, y) : \mathbf{M}_B(X, x, n) \cong \mathbf{M}_B(X, y, n)$$

such that $\tau(y, z)\,\tau(x, y) = \tau(x, z)$, given as follows. We may choose a path from x to y, giving $\pi_1(X^{an}, x) \cong \pi_1(X^{an}, y)$ and hence $\mathbf{R}_B(X, x, n) \cong \mathbf{R}_B(X, y, n)$. This isomorphism is compatible with the action of $Gl(n, \mathbf{C})$ so it descends to the desired $\tau(x, y)$. Choice of a different path gives a different isomorphism of representation spaces which differs by the action of a section $g : \mathbf{R}_B(X, y, n) \to Gl(n, \mathbf{C})$. By the definition of quotient, the two natural maps

$$\mathbf{R}_B(X, y, n) \times Gl(n, \mathbf{C}) \to \mathbf{M}_B(X, y, n)$$

are equal. Hence the two maps from the graph of g to $\mathbf{M}_B(X, y, n)$ are the same. Thus the two maps from $\mathbf{R}_B(X, x, n)$ to $\mathbf{M}_B(X, y, n)$ are the same, so the two isomorphisms $\tau(x, y)$ are the same. Thus $\tau(x, y)$ is canonically defined; and this independence of the choice of path implies the formula $\tau(y, z)\,\tau(x, y) = \tau(x, z)$. We will identify the spaces obtained from different choices of base point, and drop the base point from the notation for $\mathbf{M}_B(X, n)$.

Local systems of schemes

Suppose T is a topological space. A *local system of schemes* Z *over* T is a functor from the category of $\mathbf{C}$-schemes to the category of sheaves of sets over T, denoted (backward)

$$Z : (S \in Sch,\ U \subset T) \mapsto Z(U)(S),$$

such that: there exists a covering by open sets $T = \bigcup_\alpha U_\alpha$ such that for any open set V contained in one of the U_α, the functor $S \mapsto Z(V)(S)$ is represented by a scheme $Z(V)$ over $\mathbf{C}$; and such that if $W \subset V$ are connected open sets contained in one of the U_α

then the restriction map $Z(V) \to Z(W)$ is an isomorphism (note that the restriction morphisms of functors are automatically morphisms of schemes).

Choose a point $t \in T$. The stalk

$$Z_t \stackrel{\text{def}}{=} \varinjlim_{t \in V} Z(V)$$

is a scheme.

Lemma **6.2.** — *If* Z *is a local system of schemes on* T, *the group* $\pi_1(T, t)$ *acts on the stalk* Z_t *by* **C**-*scheme automorphisms. If* T *is connected and locally simply connected, the construction* $Z \mapsto Z_t$ *is an equivalence between the category of local systems of schemes over* T *and the category of schemes with action of* $\pi_1(T, t)$.

Proof. — Suppose Z is a local system of schemes over T. Let $T = \bigcup_\alpha U_\alpha$ be an open covering as in the definition. We may suppose that the U_α are connected. We have schemes $Z(U_\alpha)$. If $v \in U_\alpha \cap U_\beta$ then we have isomorphisms of schemes

$$Z(U_\alpha) \stackrel{\cong}{\leftarrow} Z_v \stackrel{\cong}{\rightarrow} Z(U_\beta).$$

If $v \in U_\alpha \cap U_\beta \cap U_\gamma$ then the resulting hexagon commutes. If $\sigma : [0, 1] \to T$ is a path with $\sigma(0) = \sigma(1) = t$ then we may choose $0 = s_0 < s_1 < \ldots < s_k = 1 \in [0, 1]$ and $\alpha_0, \ldots, \alpha_k$ such that

$$\sigma([s_i, s_{i+1}]) \subset U_{\alpha_i}.$$

We obtain $Z_{\sigma(s_i)} \cong Z(U_{\alpha_i}) \cong Z_{\sigma(s_{i+1})}$. Putting these isomorphisms together we get an isomorphism of schemes $Z_t \cong Z_t$. One can check that a homotopic path $\sigma' \sim \sigma$ gives the same isomorphism, so we get an action of $\pi_1(T, t)$ on Z_t.

Suppose T is connected and locally simply connected, so the universal covering $\widetilde{T}$ exists. Given a scheme Z_t with action of $\pi_1(T, t)$, form the constant local system of schemes $\widetilde{Z}$ over the universal covering $\widetilde{T}$, whose fiber at the base point t is Z_t (the sheaf is given by the rule $\widetilde{Z}(U)(S) = Z_t(S)$ for connected open sets U; the restriction maps are the identity). The group of covering transformations $\mathrm{Aut}(\widetilde{T}/T) = \pi_1(T, t)$ acts on $\widetilde{Z}$ over its action on $\widetilde{T}$, by the given action on Z_t. Now define $Z(U)(S)$ to be the set of invariants in $\widetilde{Z}(\widetilde{U})(S)$, where $\widetilde{U}$ is the inverse image of U in the universal covering. This gives a local system of schemes Z over T. This construction is the inverse of the previous construction. □

We can make a similar definition of local system of complex analytic spaces over a topological space T; and the analogue of the previous lemma still holds. If Z is a local system of schemes then we obtain a corresponding local system of complex analytic spaces Z^{an}. The stalk Z_t^{an} is the complex analytic space associated to Z_t.

If T is a complex analytic space and Z is a local system of schemes over T then we denote by $Z^{(an)}$ the *total analytic space* over T constructed as follows. Choose an open covering $T = \bigcup_i T_i$ such that the restriction of Z to T_i is a constant local system with stalk Z_i. For each connected component of $T_i \cap T_j$ there is an isomorphism $Z_i \cong Z_j$, satisfying a compatibility relation for triples of indices. Let $Z^{(an)}$ be the space obtained by glueing together the complex analytic spaces $Z_i^{an} \times T_i$ using these isomorphisms.

The Betti moduli spaces in the relative case

Suppose that $f: X \to S$ is a smooth projective morphism to a scheme S of finite type over $\mathbf{C}$. Suppose that S and the fibers X_s are connected. The associated map of complex analytic spaces f^{an} is a fibration of the underlying topological spaces. Choose base points $t \in S$ and $x \in X_t$. Let $\Gamma = \pi_1(X_t^{an}, x)$. Let $\operatorname{Aut}(\Gamma)$ denote the group of automorphisms of Γ; $\operatorname{Inn}(\Gamma) \subset \operatorname{Aut}(\Gamma)$ the image of the natural map $\operatorname{Ad} : \Gamma \to \operatorname{Aut}(\Gamma)$ (which sends γ to the inner automorphism $\operatorname{Ad}(\gamma)(g) = \gamma g \gamma^{-1}$); and $\operatorname{Out}(\Gamma) = \operatorname{Aut}(\Gamma)/\operatorname{Inn}(\Gamma)$. The group $\pi_1(S^{an}, t)$ acts on Γ by outer automorphisms, in other words there is a map

$$\pi_1(S^{an}, t) \to \operatorname{Out}(\Gamma).$$

This may be defined as follows: if $\sigma : [0, 1] \to S^{an}$ is a loop representing an element of $\pi_1(S^{an}, t)$, then the pullback $\sigma^*(X^{an})$ is a fibration over $[0, 1]$; it is trivial, so we obtain a homeomorphism $X_t^{an} \cong X_t^{an}$ between the fibers over 0 and 1; this gives a map $\pi_1(X_t^{an}, x) \cong \pi_1(X_t^{an}, y)$ for some other point y; finally, choose a path joining x and y, to get an automorphism of $\Gamma = \pi_1(X_t^{an}, x)$—which is well defined independent of the choice, up to inner automorphism. The resulting outer automorphism is independent of the homotopy class of the path σ.

The group $\operatorname{Aut}(\Gamma)$ acts on the representation space $\mathbf{R}(\Gamma, n)$. This descends to an action of $\operatorname{Out}(\Gamma)$ on the moduli space $\mathbf{M}(\Gamma, n)$, as inner automorphisms act on the representation space through functions $\mathbf{R}(\Gamma, n) \to \operatorname{Gl}(n, \mathbf{C})$ and hence trivially on the moduli space. In our case, we have denoted $\mathbf{M}(\Gamma, n)$ by $\mathbf{M}_B(X_t, n)$. Composing this action with the action of $\pi_1(S^{an}, t)$ on Γ, we obtain an action of $\pi_1(S^{an}, t)$ on $\mathbf{M}_B(X_t, n)$ by $\mathbf{C}$-scheme automorphisms. From Lemma 6.2, we obtain a local system of schemes $\mathbf{M}_B(X/S, n)$ over the topological space underlying S^{an}.

The relative version of the Betti moduli space is this local system of schemes $\mathbf{M}_B(X/S, n)$. It is independent of the choice of base points t and x. The stalk over $s \in S$ is

$$\mathbf{M}_B(X/S, n)_s = \mathbf{M}_B(X_s, n).$$

Suppose $\xi : S \to X$ is a section. Then $\pi_1(S^{an}, t)$ acts on $\pi_1(X_s, \xi(t))$ by automorphisms. We obtain a local system of schemes $\mathbf{R}_B(X/S, \xi, n)$, which is again independent of the choice of t. The stalk over $s \in S$ is

$$\mathbf{R}_B(X/S, \xi, n)_s = \mathbf{R}_B(X_s, \xi(s)\ n).$$

The constructions $\mathbf{M}_B(X/S, n)$ and $\mathbf{R}_B(X/S, \xi, n)$ may be extended to the case of non-connected base S by taking the disjoint union of the spaces over each connected component of S. The construction $\mathbf{M}_B(X/S, n)$ may also be extended to the case where the fibers are not connected. If $s \in S$ and if $X_s = X_1 \cup \ldots \cup X_k$ is the decomposition of the fiber into connected components, then

$$\mathbf{M}_B(X/S, n)_s = \mathbf{M}_B(X_1, n) \times \ldots \times \mathbf{M}_B(X_k, n).$$

The action of $\pi_1(S^{an}, s)$ permutes the factors in the product appropriately.

We close the discussion of the Betti moduli spaces by giving the universal and co-universal properties they satisfy.

Proposition **6.3.** — *Suppose $f : X \to S$ is smooth and projective with connected fibers, and suppose that $\xi : S \to X$ is a section. Then for any scheme Y and any open set $U \subset S^{an}$ the set $\mathbf{R}_B(X/S, \xi, n)(U)(Y)$ is equal to the set of isomorphism classes of pairs (L, β) where L is a locally constant sheaf of $H^0(Y, \mathcal{O}_Y)$-modules on $f^{-1}(U)$ and $\beta : \xi^{-1}(L) \cong H^0(Y, \mathcal{O}_Y)^n$.*

Proof. — It suffices to prove this for small open sets U, for example connected open sets over which the topological fibration $(X^{an} \times_S^{an} U, \xi)$ is trivial. In this case, choose $s \in U$. A locally constant sheaf L of $H^0(Y, \mathcal{O}_Y)$-modules on $f^{-1}(U)$ together with a fram β is the same thing as a representation of $\pi_1(X_s, \xi(s))$ in $Gl(n, H^0(Y, \mathcal{O}_Y))$, hence the same thing as a morphism $Y \to \mathbf{R}_B(X_s, \xi(s), n)$. This is the set of Y-valued points of the local system of schemes over the set U, since the local system of schemes is trivial and U is connected. □

Proposition **6.4.** — *Suppose $f : X \to S$ is a smooth projective morphism. Let $\mathbf{M}_B^\natural(X/S, n)$ denote the functor from **C**-schemes to sheaves of sets over S^{an} which associates to each scheme Y and each open set $U \subset S^{an}$ the set of isomorphism classes of locally constant sheaves of free $H^0(Y, \mathcal{O}_Y)$-modules of rank n on $f^{-1}(U)$. There is a map of functors from $\mathbf{M}_B^\natural(X/S, n)$ to $\mathbf{M}_B(X/S, n)$. If Z is any local system of schemes over S^{an} with a natural transformation of functors $\mathbf{M}_B^\natural(X/S, n) \to Z$, there is a unique factorization through a map $\mathbf{M}_B(X/S, n) \to Z$.*

Proof. — This is a translation to the case of local systems of schemes, of the property that the fiber $\mathbf{M}_B(X_s, n)$ universally co-represents the functor $\mathbf{M}_B^\natural(X_s, n)$. □

Moduli of Higgs bundles

Suppose that $f : X \to S$ is a smooth projective morphism to a scheme of finite type over **C**. A *Higgs sheaf* on X over S is a coherent sheaf E on X together with a holomorphic map $\varphi : E \to E \otimes \Omega^1_{X/S}$ such that $\varphi \wedge \varphi = 0$. Similarly, a *Higgs bundle* is a Higgs sheaf (E, φ) such that E is a locally free sheaf.

Higgs bundles on curves were introduced by Hitchin in [Hi2] and [Hi1]. The condition $\varphi \wedge \varphi = 0$ for higher dimensional varieties was introduced in [Si2] and [Si5]. Hitchin gave an analytic construction of the moduli space [Hi1] (this part of his argument works for any rank). Nitsure gave an algebraic construction of the moduli space of Higgs bundles over a curve [Ni1].

We give two constructions of the moduli space of Higgs bundles, based on two different interpretations. The first is simply to note that a Higgs bundle is a Λ-module for an appropriate sheaf of rings Λ. This does not give too much other information, and is based on all of Part 1. The second construction uses only the moduli space of coherent sheaves of § 1, and it gives some additional information about the moduli space: the properness of Hitchin's map.

Lemma **6.5**. — *Let* $\Lambda^{\mathrm{Higgs}} = \mathrm{Sym}^{\cdot}(\mathrm{T}(\mathrm{X}/\mathrm{S}))$. *Then a Higgs sheaf on* X *over* S *is the same thing as an* $\mathcal{O}_{\mathrm{X}}$*-coherent* Λ^{Higgs}*-module on* X.

Proof. — This follows from the discussion of split almost polynomial rings Λ at the end of § 2, Part I (Lemma 2.13). In this case it is easy to see that an action of the symmetric algebra on a sheaf E is the same thing as a map $\varphi : \mathrm{E} \to \mathrm{E} \otimes \mathrm{T}^*(\mathrm{X}/\mathrm{S})$ such that $\varphi \wedge \varphi = 0$. □

Fix a relatively very ample $\mathcal{O}_{\mathrm{X}}(1)$. Define the notions of pure dimension, p-semistability, p-stability, μ-semistability and μ-stability for Higgs sheaves to be the same as the corresponding notions for Λ^{Higgs}-modules. These coincide with the notions defined in [Si5] for the case when $\mathrm{S} = \mathrm{Spec}(\mathbf{C})$ (pure dimension $d = \dim(\mathrm{X})$ is the same thing as torsion-free). Recall that in the relative case, the conditions of semistability and stability contain the hypothesis that the sheaf is flat over the base S. Let $\mathbf{M}^{\natural}_{\mathrm{Higgs}}(\mathrm{X}/\mathrm{S}, \mathrm{P})$ denote the functor which associates to an S-scheme S′ the set of isomorphism classes of p-semistable Higgs sheaves E on X′ over S′ with Hilbert polynomial P. This is universally co-represented by the *moduli space* $\mathbf{M}_{\mathrm{Higgs}}(\mathrm{X}/\mathrm{S}, \mathrm{P}) \stackrel{\mathrm{def}}{=} \mathbf{M}(\Lambda^{\mathrm{Higgs}}, \mathrm{P})$ constructed in Theorem 4.7, Part I. The points of $\mathbf{M}_{\mathrm{Higgs}}(\mathrm{X}/\mathrm{S}, \mathrm{P})$ parametrize Jordan equivalence classes of p-semistable Higgs sheaves with Hilbert polynomial P on the fibers X_s. If P has degree $\dim(\mathrm{X})$, then these are the same as torsion-free p-semistable Higgs sheaves which were discussed in [Si5].

Let P_0 denote the Hilbert polynomial of $\mathcal{O}_{\mathrm{X}}$. Let $\mathbf{M}^{\natural}_{\mathrm{Dol}}(\mathrm{X}/\mathrm{S}, n)$ denote the functor which to an S-scheme S′ associates the set of isomorphism classes of p-semistable Higgs sheaves E on X′ over S′ of Hilbert polynomial $n\mathrm{P}_0$, such that the Chern classes $c_i(\mathrm{E}_s)$ vanish in $\mathrm{H}^{2i}(\mathrm{X}_s, \mathbf{C})$ for all closed points $s \in \mathrm{S}'$.

In general, if $f : \mathrm{X} \to \mathrm{S}$ is a smooth projective morphism and E is a coherent sheaf on X which is flat over S, then the Chern classes $c_i(\mathrm{E})$ are sections of the relative algebraic de Rham cohomology $\mathrm{R}^{2i}f_*(\Omega^{\cdot}_{\mathrm{X}/\mathrm{S}}, d)$ which are flat with respect to the Gauss-Manin connection. Thus the condition that the Chern classes $c_i(\mathrm{E}_s) \in \mathrm{H}^{2i}(\mathrm{X}_s, \mathbf{C})$ vanish depends only on the connected component of S containing s.

The functor $\mathbf{M}^{\natural}_{\mathrm{Dol}}(\mathrm{X}/\mathrm{S}, n)$ is universally corepresented by a scheme $\mathbf{M}_{\mathrm{Dol}}(\mathrm{X}/\mathrm{S}, n)$ which is a disjoint union of some of the connected components of $\mathbf{M}_{\mathrm{Higgs}}(\mathrm{X}/\mathrm{S}, n\mathrm{P}_0)$ (the fact that $\mathbf{M}_{\mathrm{Dol}}(\mathrm{X}/\mathrm{S}, n)$ may be a proper subset of $\mathbf{M}_{\mathrm{Higgs}}(\mathrm{X}/\mathrm{S}, n\mathrm{P}_0)$ was pointed out to me by J. Le Potier [Le]). The points of $\mathbf{M}_{\mathrm{Dol}}(\mathrm{X}/\mathrm{S}, n)$ correspond to Jordan equivalence classes of p-semistable torsion-free Higgs sheaves of rank n on the fibers X_s, with Chern classes vanishing in the complex valued (or equivalently, rational) cohomology of X_s. We call $\mathbf{M}_{\mathrm{Dol}}(X/S, n)$ the *Dolbeault moduli space*. There is an open set $\mathbf{M}^s_{\mathrm{Dol}}(X/S, n)$ parametrizing p-stable Higgs sheaves, and there a universal family exists étale locally.

Proposition **6.6.** — *Suppose* X *is a smooth projective variety over* $S = \mathrm{Spec}(\mathbf{C})$. *If* E *is a* μ*-semistable torsion free Higgs sheaf with Chern classes equal to zero, then* E *is a bundle, and is in fact an extension of* μ*-stable Higgs bundles whose Chern classes vanish. Any sub-Higgs sheaf of degree zero is a strict subbundle with vanishing Chern classes.*

Proof. — [Si5] Theorem 2. □

Corollary **6.7.** — *If* X *is smooth and projective over a base* S, *if* S′ *is an* S*-scheme, and if* E *is an element of* $\mathbf{M}^\natural_{\mathrm{Dol}}(X/S, n)(S')$, *then* E *is locally free over* X′. *The points of* $\mathbf{M}_{\mathrm{Dol}}(X/S, n)$ *correspond to direct sums of* μ*-stable Higgs bundles with vanishing rational Chern classes on the fibers* X_s.

Proof. — This follows from the previous proposition and Lemma 1.27, Part I. □

Remark. — For Higgs sheaves with vanishing Chern classes, p-semistability (resp. p-stability) is equivalent to μ-semistability (resp. μ-stability). This follows from Proposition 6.6.

Suppose X is smooth and projective over S. A Higgs bundle E on X (flat over S) is *of semiharmonic type* if the restrictions to the fibers E_s are semistable Higgs bundles with vanishing rational Chern classes. Say that E_s is of *harmonic type* if it is a direct sum of stable Higgs bundles with vanishing rational Chern classes. The Higgs bundles of semiharmonic type are those which correspond to representations of the fundamental group in [Si5]. Those of harmonic type correspond to semisimple representations. The closed points of $\mathbf{M}_{\mathrm{Dol}}(X/S, n)$ parametrize the Higgs bundles of harmonic type of rank n.

Suppose $X \to S$ is a smooth projective morphism with connected fibers, and suppose $\xi : S \to X$ is a section. Let $\mathbf{R}(\Lambda^{\mathrm{Higgs}}, \xi, nP_0)$ denote the representation space for framed Λ^{Higgs}-modules constructed in Theorem 4.10, Part I. By Proposition 6.6, all p-semistable Higgs sheaves with vanishing rational Chern classes satisfy condition LF(X) and hence condition LF(ξ). Let $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n)$ denote the disjoint union of those connected components of $\mathbf{R}(\Lambda^{\mathrm{Higgs}}, \xi, nP_0)$ corresponding to Higgs sheaves with vanishing rational Chern classes. Then $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n)$ represents the functor which associates to an S-scheme S′ the set of isomorphism classes of pairs (E, β) where E is a Higgs bundle of semiharmonic type on X′ over S′ and $\beta : \xi^*(E) \cong \mathcal{O}^n_{S'}$ is a frame. We call this scheme the *Dolbeault representation space*.

The $\mathbf{C}^*$ action

Recall that an action of an algebraic group G on a scheme Z is a morphism $G \times Z \to Z$ satisfying the usual axioms for a group action, with the axioms written in terms of diagrams of morphisms: the two maps $G \times G \times Z \to Z$ are the same (associativity); and the map $Z \to Z$ induced by the identity element $e \in G$ is the identity. We can define similarly the notion of an action on a functor $Y^{\natural}$: this is a natural transformation of functors $G \times Y^{\natural} \to Y^{\natural}$ satisfying the same axioms. If G acts on a functor $Y^{\natural}$, and $\varphi : Y^{\natural} \to Y$ is a natural transformation so that the scheme Y universally corepresents $Y^{\natural}$, then there is a unique action of G on the scheme Y which is compatible with φ. The morphism $G \times Y \to Y$ is obtained from the natural transformation of functors $G \times Y^{\natural} \to Y$ by applying the universality hypothesis, that $G \times Y$ corepresents the functor $(G \times Y) \times_Y Y^{\natural} = G \times Y^{\natural}$. The axioms are checked using the uniqueness part of the definition of universally co-representing a functor.

The algebraic group $\mathbf{C}^*$ acts on the functor $\mathbf{M}^{\natural}_{\mathrm{Higgs}}(X/S, P)$ in the following way. If S′ is an S-scheme, $t : S' \to \mathbf{C}^*$ is an S′-valued point, and $(E, \varphi) \in \mathbf{M}^{\natural}_{\mathrm{Higgs}}(X/S, P)(S')$ is a p-semistable Higgs sheaf with Hilbert polynomial P on X′ over S′, then $(E, t\varphi)$ is again an element of $\mathbf{M}^{\natural}_{\mathrm{Higgs}}(X/S, P)(S')$ (the property of p-semistability is preserved because the subsheaves preserved by $t\varphi$ are the same as those preserved by φ). We obtain a morphism of functors giving the group action. By the above discussion, there is a unique compatible action of $\mathbf{C}^*$ on $\mathbf{M}_{\mathrm{Higgs}}(X/S, P)$. This gives an action of $\mathbf{C}^*$ on $\mathbf{M}_{\mathrm{Dol}}(X/S, n)$. Similarly, if the fibers X_s are connected and ξ is a section, the formula $t((E, \varphi), \beta) = ((E, t\varphi), \beta)$ gives an action of $\mathbf{C}^*$ on the Dolbeault representation space $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n)$. This commutes with the action of $\mathrm{Gl}(n, \mathbf{C})$ and the good quotient $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n) \to \mathbf{M}_{\mathrm{Dol}}(X/S, n)$ is compatible with the action of $\mathbf{C}^*$.

The subspace $\mathbf{M}_{\mathrm{Dol}}(X/S, n)^{\mathbf{C}^*}$ of points fixed by $\mathbf{C}^*$ is a closed subvariety. Algebraically the structure of a Higgs bundle on X_s fixed by $\mathbf{C}^*$ is the following (cf. [Si5] Lemma 4.1). If $(E, \varphi) \cong (E, t\varphi)$ for some $t \in \mathbf{C}^*$ which is not a root of unity, let f be the isomorphism. By appropriately combining the generalized eigenspaces of f, we get a decomposition $E = \bigoplus E^p$ such that $\varphi : E^p \to E^{p-1} \otimes \Omega^1_X$. By the analytic results of [Si5], the points of $\mathbf{M}_{\mathrm{Dol}}(X/S, n)^{\mathbf{C}^*}$ correspond to Higgs bundles which come from complex variations of Hodge structure.

The second construction

The idea behind this construction is that a Higgs bundle on X can be thought of as a coherent sheaf $\mathscr{E}$ on the relative cotangent bundle $T^*(X/S)$. Let Z denote a projective completion of $T^*(X/S)$, and let $D = Z - T^*(X/S)$ be the divisor at infinity. Choose Z so that the projection extends to a map $\pi : Z \to X$.

Lemma **6.8.** — *A Higgs sheaf* E *on* X *over* S *is the same thing as a coherent sheaf* $\mathscr{E}$ *on* Z *such that* $\mathrm{supp}(E) \cap D = \emptyset$. *This identification is compatible with morphisms, giving an equivalence of categories. The conditions of flatness over* S *are the same. For* $s \in S$, *the condition that* E_s *is torsion-free is the same as the condition that* $\mathscr{E}_s$ *is of pure dimension* $d = \dim(X_s)$.

Proof. — The projection $\pi : T^*(X/S) \to X$ is an affine morphism, in other words $T^*(X/S)$ is the sheafified spectrum of the sheaf of rings $\pi_* \mathcal{O}_{T^*(X/S)}$ on X. This sheaf of rings is naturally isomorphic to the symmetric algebra on the tangent bundle $\mathrm{Sym}^\cdot T(X/S)$, so giving a quasicoherent sheaf $\mathcal{E}$ on $T^*(X/S)$ it is equivalent to giving a quasicoherent sheaf $E = \pi_* \mathcal{E}$ on X together with an action of $\mathrm{Sym}^\cdot T(X/S)$. But $\mathrm{Sym}^\cdot T(X/S) = \Lambda^{\mathrm{Higgs}}$, so by the discussion in § 2, this action is the same as the data of a map $\varphi : E \to E \otimes \Omega^1_X$ such that $\varphi \wedge \varphi = 0$. A coherent Higgs sheaf E is the same thing as a sheaf $\mathcal{E}$ on $T^*(X/S)$ such that $\pi_* \mathcal{E}$ is coherent. This condition of coherence means that $\mathcal{E}$ is coherent on $T^*(X/S)$ and the closure of the support of $\mathcal{E}$ in Z does not meet the divisor at infinity D. A morphism of coherent sheaves $\mathcal{E} \to \mathcal{F}$ is the same thing as a morphism $\pi_*(\mathcal{E}) \to \pi_*(\mathcal{F})$ compatible with the action of the symmetric algebra, or equivalently compatible with φ. Since $\pi|_{T^*(X/S)}$ is an affine map, flatness of $\mathcal{E}$ over S is equivalent to flatness of $\pi_*(\mathcal{E})$ over S. Finally note that the dimension of support of any subsheaf of E is the same on X_s as it is on Z_s, because of the condition that the support doesn't meet D_s. Therefore the conditions of pure dimension $d = \dim(X_s)$ on X_s and Z_s are the same. On X_s, the condition that a sheaf has pure dimension $d = \dim(X_s)$ is the same as the condition that it is torsion free. □

Choose k so that $\mathcal{O}_Z(1) \overset{\mathrm{def}}{=} \pi^* \mathcal{O}_X(k) \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(D)$ is ample on Z (here we suppose that Z is the standard completion of the cotangent bundle to a projective space bundle). In particular, $\mathcal{O}_{T^*(X/S)}(1) = \pi^* \mathcal{O}_X(k)$. Thus, for any coherent sheaf $\mathcal{E}$ on Z with support not meeting D, the Hilbert polynomials of $\mathcal{E}$ and $\pi_* \mathcal{E}$ differ by scaling: $p(\mathcal{E}, m) = p(\pi_* \mathcal{E}, km)$.

Corollary **6.9.** — *The notions of p-semistability, p-stability, μ-semistability, and μ-stability for a Higgs sheaf* E *on* X *over* S *are the same as the corresponding notions for the coherent sheaf* $\mathcal{E}$ *on* Z *associated to* E *in the previous lemma.*

Proof. — The sub-Higgs sheaves of E correspond to the coherent subsheaves of $\mathcal{E}$, since a subsheaf of $\mathcal{E}$ is the same thing as a subsheaf of $\pi_* \mathcal{E}$ preserved by the action of the symmetric algebra. Scaling the Hilbert polynomials preserves the ordering and scales the slope. □

Fix a polynomial P of degree $d = \dim(X/S)$, and put $k^* P(m) = P(km)$. Fix a large N as required by the constructions of § 1 for sheaves on Z. Put $\mathcal{W} = \mathcal{O}_Z(-N)$ and $V = \mathbf{C}^{k^*(P(N))}$, and let $\mathrm{Hilb}(V \otimes \mathcal{W}, k^* P)$ denote the Hilbert scheme of quotients $V \otimes \mathcal{W} \to \mathcal{E} \to 0$ on Z, flat over S, with Hilbert polynomial $k^* P$. Let Q_1 and Q_2 denote the subsets defined in § 1 (not those defined in § 3), and let $Q_3 \subset Q_2$ denote the open subset parametrizing quotient sheaves $\mathcal{E}$ whose support does not meet D. By Theorem 1.19, Part I, and [Mu], a good quotient $\mathbf{M}(\mathcal{O}_Z, k^* P) = Q_2/\mathrm{Sl}(V)$ exists. The open set Q_3 is $\mathrm{Sl}(V)$-invariant and is set-theoretically the inverse image of a subset of $\mathbf{M}(\mathcal{O}_Z, k^* P)$ (since the support of $\mathcal{E}_s$ is the same as the support of $\mathrm{gr}(\mathcal{E}_s)$). Therefore a good quotient $Q_3/\mathrm{Sl}(V)$ exists and it is equal to an open subset which we denote

$\mathbf{M}(\mathcal{O}_{T^*(X/S)}, k^* P)$ of $\mathbf{M}(\mathcal{O}_Z, k^* P)$. Theorem 1.21, Part I, Lemma 6.8, and Corollary 6.9 imply that $\mathbf{M}(\mathcal{O}_{T^*(X/S)}, k^* P)$ universally co-represents the functor $\mathbf{M}^{\natural}_{Higgs}(X/S, P)$, and we have all of the properties of Theorem 1.21, Part I. We may put

$$\mathbf{M}_{Higgs}(X/S, P) = \mathbf{M}(\mathcal{O}_{T^*(X/S)}, k^* P).$$

Define the subset $\mathbf{M}_{Dol}(X/S, n) \subset \mathbf{M}_{Higgs}(X/S, nP_0)$ as before. These moduli spaces are the same as those constructed previously, because they co-represent the same functors.

Hitchin's proper map

We will now define a map from the space of Higgs bundles to the space of possible characteristic polynomials for θ. This map is the generalization of the determinant map that Hitchin studied in [Hi1]. In Hitchin's case it turned out that this map was proper ([Hi1] Theorem 8.1), and we will prove the same here also. Roughly speaking this means that the only way for a Higgs bundle to " go to infinity " is for the characteristic polynomial to become singular.

For any n let $\mathscr{V}(X/S, n) \to S$ be the scheme representing the functor which to an S-scheme S′ associates

$$\bigoplus_{i=1}^{n} H^0(X'/S', \mathrm{Sym}^i\, \Omega^1_{X'/S'})$$

[Grl] [Mu]. We consider the points of $\mathscr{V}(X/S, n)$ as polynomials written $t^n + a_1 t^{n-1} + \ldots + a_n$ with $a_i \in H^0(X'/S', \mathrm{Sym}^i\, \Omega^1_{X'/S'})$.

Let $\sigma_1, \ldots, \sigma_r$ denote the symmetric polynomials in an $r \times r$ matrix variable A such that

$$\det(t - A) = t^r + \sigma_1(A)\, t^{r-1} + \ldots + \sigma_r(A).$$

For example, $\sigma_1(A) = -\mathrm{Tr}(A)$ and $\sigma_r(A) = (-1)^r \det(A)$.

Let P be a polynomial of degree $d = \dim(X/S)$ and rank $r = \deg(X)\, n$, so that sheaves of pure dimension d and Hilbert polynomial d on the fibers X_s are torsion-free with usual rank equal to n. Suppose S′ is an S-scheme and (E, φ) is a p-semistable Higgs sheaf with Hilbert polynomial P on X′ over S′. Then there is an open subset $U' \subset X'$ such that the intersection of $X' - U'$ with any fiber has codimension at least 2, and such that E is locally free over U′. Over U′, φ is an $\Omega^1_{X'/S'}$-valued endomorphism of a rank n-vector bundle. Furthermore, the endomorphisms obtained by contracting φ with different sections of $T(X'/S')$ commute with each other. Thus we can evaluate the elementary symmetric polynomials to obtain

$$\sigma_i(\varphi|_{U'}) \in H^0(U', \mathrm{Sym}^i(\Omega^1_{U'/S'})).$$

Since $\mathrm{Sym}^i(\Omega^1_{X'/S'})$ is a locally free sheaf, Hartog's theorem applied over artinian subschemes of S′, coupled with the theorem on formal functions and Artin approximation, imply that $\sigma_i(\varphi|_{U'})$ extend uniquely to sections which we denote

$$\sigma_i(\varphi) \in \mathrm{H}^0(\mathrm{X}', \mathrm{Sym}^i(\Omega^1_{\mathrm{X}'/\mathrm{S}'})).$$

Define $\sigma(\mathrm{E}, \varphi) \in \mathscr{V}(\mathrm{X}/\mathrm{S}, n)(\mathrm{S}')$ to be the point corresponding to $(\sigma_1(\varphi), \ldots, \sigma_n(\varphi))$. This construction defines a morphism from the functor $\mathbf{M}^{\natural}_{\mathrm{Higgs}}(\mathrm{X}/\mathrm{S}, \mathrm{P})$ to $\mathscr{V}(\mathrm{X}/\mathrm{S}, n)$, and hence a morphism of schemes $\sigma : \mathbf{M}_{\mathrm{Higgs}}(\mathrm{X}/\mathrm{S}, \mathrm{P}) \to \mathscr{V}(\mathrm{X}/\mathrm{S}, n)$. We call $\sigma(\mathrm{E}, \varphi)$ the *characteristic polynomial* of (E, θ). The morphism σ was introduced by Hitchin for Higgs bundles on curves in [Hi2] and [Hi1].

There are universal sections

$$a_i^{\mathrm{univ}} : \mathrm{X} \times_{\mathrm{S}} \mathscr{V}(\mathrm{X}/\mathrm{S}, n) \to \mathrm{Sym}^i\, \mathrm{T}^*(\mathrm{X}/\mathrm{S}) \times_{\mathrm{S}} \mathscr{V}(\mathrm{X}/\mathrm{S}, n).$$

Here $\mathrm{Sym}^i\,\mathrm{T}^*(\mathrm{X}/\mathrm{S})$ denotes the total space of the i-th symmetric power of the relative cotangent bundle. There is a multiplication map

$$\mathrm{Sym}^i\, \mathrm{T}^*(\mathrm{X}/\mathrm{S}) \times_{\mathrm{X}} \mathrm{Sym}^j\, \mathrm{T}^*(\mathrm{X}/\mathrm{S}) \to \mathrm{Sym}^{i+j}\, \mathrm{T}^*(\mathrm{X}/\mathrm{S})$$

as well as a map corresponding to addition. There is a closed subscheme

$$\mathscr{W}(\mathrm{X}/\mathrm{S}, n) \subset \mathrm{T}^*(\mathrm{X}/\mathrm{S}) \times_{\mathrm{S}} \mathscr{V}(\mathrm{X}/\mathrm{S}, n)$$

defined by the equation

$$t^n + a_1^{\mathrm{univ}} t^{n-1} + \ldots + a_n^{\mathrm{univ}} = 0.$$

This represents a functor which can be seen by considering t^i and a_{n-1}^{univ} as points in the total spaces of the symmetric powers, then multiplying them together and adding to get a point in $\mathrm{Sym}^n\,\mathrm{T}^*(\mathrm{X}/\mathrm{S})$ which is required to be in the zero section.

Lemma **6.10.** — *Suppose* S′ *is an* S*-scheme and* (E, φ) *is a p-semistable Higgs sheaf with Hilbert polynomial* P *on* X′ *over* S′, *corresponding to a coherent sheaf* $\mathscr{E}$ *on* $\mathrm{T}^*(\mathrm{X}'/\mathrm{S}')$. *Let* $\sigma(\mathrm{E}, \theta) : \mathrm{S}' \to \mathscr{V}(\mathrm{X}/\mathrm{S}, n)$ *be the characteristic polynomial defined above. Then* $\mathscr{E}$ *is supported set theoretically on*

$$\mathscr{W}(\mathrm{X}/\mathrm{S}, n) \times_{\mathscr{V}(\mathrm{X}/\mathrm{S}, n),\, \sigma(\mathrm{E}, \varphi)} \mathrm{S}' \subset \mathrm{T}^*(\mathrm{X}'/\mathrm{S}').$$

Proof. — In general, if A is a vector space and $\mathrm{Sym}^{\bullet}(\mathrm{A})$ acts on a vector space B, then the support of B considered as a coherent sheaf on A^* is equal to the set of eigenforms of the action of A. Now let $\mathrm{U}' \subset \mathrm{X}'$ be the open set used above in the construction of $\sigma(\mathrm{E}, \varphi)$. Then the zeros of the characteristic polynomial $\sigma(\mathrm{E}, \theta)$ over U′ are the eigenforms of $\varphi|_{\mathrm{U}'}$. Thus $\mathscr{W}(\mathrm{X}/\mathrm{S}, n) \times_{\mathscr{V}(\mathrm{X}/\mathrm{S}, n),\, \sigma(\mathrm{E}, \varphi)} \mathrm{S}'$ is the spectral variety of the endomorphism $\varphi|_{\mathrm{U}'}$. From the above general principle, $\mathscr{E}|_{\mathrm{T}^*(\mathrm{U}'/\mathrm{S}')}$ is supported on this spectral variety. On the other hand, any section of $\mathscr{E}$ whose support is contained in $\mathrm{T}^*(\mathrm{X}'/\mathrm{S}') - \mathrm{T}^*(\mathrm{U}'/\mathrm{S}')$ restricts to a section of the fiber $\mathscr{E}_s$ supported over the complement $X'_s - U'_s$ (which has dimension less than or equal to $d-2$). The $\mathscr{E}_s$ are finite over X'_s, so such a section is supported in dimension less than or equal to $d-2$. By the hypothesis that $\mathscr{E}_s$ are of pure dimension d, such a section is zero. Hence any section of $\mathscr{E}$ supported in $T^*(X'/S') - T^*(U'/S')$ restricts to zero in all the fibers, so it is zero. Since the spectral variety is closed, this implies that all the sections of $\mathscr{E}$ are supported in $\mathscr{W}(X/S, n) \times_{\mathscr{V}(X/S, n), \sigma(E, \varphi)} S'$. □

Remark. — Using Cayley's theorem, one can see that the support is scheme-theoretically contained in the spectral scheme $\mathscr{W}(X/S, n) \times_{\mathscr{V}(X/S, n), \sigma(E, \varphi)} S'$.

Theorem **6.11.** — *The map* $\sigma : \mathbf{M}_{\mathrm{Higgs}}(X/S, P) \to \mathscr{V}(X/S, n)$ *is proper.*

Proof. — Note, first of all, that all schemes involved are separated. Suppose S' is a curve, $s \in S'$ is a closed point, and put $S'' = S' - \{s\}$. Suppose that

$$g : S'' \to \mathbf{M}_{\mathrm{Higgs}}(X/S, P)$$

is a map such that the composed map σg extends to a map $h : S' \to \mathscr{V}(X/S, n)$. Recall that $\mathbf{M}_{\mathrm{Higgs}}(X/S, P)$ is an open set in $\mathbf{M}(\mathcal{O}_Z, k^* P)$, and that $\mathbf{M}(\mathcal{O}_Z, k^* P)$ is projective over S. The map g extends to a map $g' : S' \to \mathbf{M}(\mathcal{O}_Z, k^* P)$. Let $\varphi : Q_2 \to \mathbf{M}(\mathcal{O}_Z, k^* P)$ be the good quotient of the parameter scheme Q_2 for sheaves on Z used in the construction of § 1. Then $Q_2 \times_{\mathbf{M}(\mathcal{O}_Z, k^* P)} S' \to S'$ is a categorical quotient. Thus there is a quasi-finite morphism of curves $Y \to S'$ such that s is the image of a point $y \in Y$, such that $Y' \stackrel{\mathrm{def}}{=} Y - \{y\}$ maps to $S'' \subset S'$, and such that the resulting map $Y \to \mathbf{M}(\mathcal{O}_Z, k^* P)$ lifts to a map $Y \to Q_2$. Let $\mathscr{E}$ be the resulting sheaf on $Z \times_S Y$. If $w \in Y'$ then $\mathscr{E}_w$ is a sheaf corresponding to a point in $\mathbf{M}_{\mathrm{Higgs}}(X/S, P)$. Thus $\mathscr{E}_w$ has support contained in $T^*(X_w)$. This implies that the support of $\mathscr{E}|_{Y'}$ is contained in $T^*(X/S) \times_S (Y')$, so it corresponds to a Higgs sheaf (E', φ') on $X \times_S (Y')$ over Y'. The map $Y' \to \mathbf{M}_{\mathrm{Higgs}}(X/S, P)$ corresponding to (E', φ') is equal to the map obtained by composing $Y' \to S''$ with g. In particular, the characteristic polynomial $\sigma(E', \varphi') : Y' \to \mathscr{V}(X/S, n)$ is the composition of $Y' \to S''$ with σg. Thus $\sigma(E', \varphi')$ extends to a map $f : Y \to \mathscr{V}(X/S, n)$, equal to the composition of $Y \to S'$ with h. By the previous lemma, $\mathscr{E}|_{Y'}$ is supported in $f^* \mathscr{W}(X/S, n) \subset T^*(X/S) \times_S Y$. Since $\mathscr{E}$ is flat over Y, it has no local sections supported on Z_y. Thus $\mathscr{E}$ is supported on the closure of $f^* \mathscr{W}(X/S, n)$ in $Z \times_S Y$. But since the equation defining $\mathscr{W}(X/S, n)$ is monic, the subscheme $\mathscr{W}(X/S, n)$ is closed in $Z \times_S \mathscr{V}(X/S, n)$. Therefore $f^* \mathscr{W}(X/S, n)$ is closed in $Z \times_S Y$, and $\mathscr{E}$ is supported in $f^* \mathscr{W}(X/S, n) \subset T^*(X/S) \times_S Y$. Hence $\mathscr{E}$ corresponds to a Higgs sheaf (E, φ) on $X \times_S Y$ over Y, restricting to (E', φ') over Y'. As $\mathscr{E}$ is a p-semistable sheaf, (E, φ) is a p-semistable Higgs sheaf by Corollary 6.9. We obtain a map $Y \to \mathbf{M}_{\mathrm{Higgs}}(X/S, P)$ extending the composition $Y' \to S'' \to \mathbf{M}_{\mathrm{Higgs}}(X/S, P)$. But this map is also equal to the composition $Y \to S' \to \mathbf{M}(\mathcal{O}_Z, k^* P)$ since the moduli space is separated. In this last map, the image of y is equal to the image $g'(s)$. From the fact that Y maps into

$\mathbf{M}_{\text{Higgs}}(X/S, P)$ we obtain $g'(s) \in \mathbf{M}_{\text{Higgs}}(X/S, P)$. Thus g' maps S' into $\mathbf{M}_{\text{Higgs}}(X/S, P)$. This is the extended map required to prove properness of σ. □

Remark. — Hitchin gives an analytic proof of the properness of σ in the case when X is a curve [Hi1].

Corollary **6.12.** — *Any p-semistable torsion-free Higgs sheaf on a fiber* X_s *can be deformed to one which is fixed by the action of* $\mathbf{C}^*$. *A Higgs bundle of semiharmonic type can be deformed to a Higgs bundle of semiharmonic type which is fixed by* $\mathbf{C}^*$, *through a family of Higgs bundles of semiharmonic type.*

Proof. — Suppose (E, φ) is a p-semistable torsion-free Higgs sheaf on X_s. Write the characteristic polynomial as

$$\sigma(E, \varphi) = t^n + a_1 t^{n-1} + \ldots + a_n.$$

For $z \in \mathbf{C}^*$, the characteristic polynomial of $z\varphi$ is

$$\sigma(E, z\varphi) = t^n + z a_1 t^{n-1} + \ldots + z^n a_n.$$

As $z \to 0$ these polynomials approach the limit t^n in $\mathscr{V}(X_s, n)$. The orbit of (E, φ) is a map $\mathbf{C}^* \to \mathbf{M}_{\text{Higgs}}(X_s, P)$ such that the composed map $\mathbf{C}^* \to \mathscr{V}(X_s, n)$ extends to a map $\mathbf{A}^1 \to \mathscr{V}(X_s, n)$. By the theorem, the orbit extends to a map $\mathbf{A}^1 \to \mathbf{M}_{\text{Higgs}}(X/S, P)$. Let Q_2 be the parameter scheme used to construct $\mathbf{M}_{\text{Higgs}}(X/S, P)$. Then

$$Q_2 \times_{\mathbf{M}_{\text{Higgs}}(X_s, P)} \mathbf{A}^1 \to \mathbf{A}^1$$

is a good quotient. In this situation, the unique closed orbit lying over $0 \in \mathbf{A}^1$ is contained in the closure of the union of orbits corresponding to $(E, z\varphi)$. Thus (E, φ) may be deformed to a Higgs sheaf corresponding to a closed orbit over $0 \in \mathbf{A}^1$. The map $\mathbf{A}^1 \to \mathbf{M}_{\text{Higgs}}(X/S, P)$ is equivariant under the action of $\mathbf{C}^*$, so the image of the origin is a fixed point. Since there is a unique closed orbit lying over the origin, this closed orbit is preserved by $\mathbf{C}^*$. Thus the Higgs sheaf corresponding to the closed orbit over the origin is fixed up to isomorphism by the action of $\mathbf{C}^*$. This proves the first statement. For the second statement, note that by Proposition 6.6, if the rational Chern classes of E vanish, then all of the Higgs sheaves involved are Higgs bundles. □

When X is a curve, Hitchin has a beautiful description of the generic fiber of the map σ [Hi2]. In terms of our description of the moduli space, the idea is as follows. For a generic point $s \in S$, the corresponding polynomial defines a smooth curve in the cotangent bundle, counted with multiplicity one. A Higgs bundle E in the fiber over that point is a coherent torsion free sheaf on the curve, of rank one. In other words, it is a line bundle. Furthermore, all line bundles of the appropriate degree occur. Thus the fiber is the Jacobian of the curve. See also [Ox].

Vector bundles with integrable connections

In this section we will apply the results of § 4, Part I, to construct a moduli space of vector bundles with integrable algebraic connection. For greatest generality, suppose that S is a base scheme of finite type over **C**, and that X is smooth and projective over S. A *vector bundle with connection on* X/S is a vector bundle or locally free sheaf E on X, together with a map of sheaves

$$\nabla : E \to E \otimes \Omega^1_{X/S}$$

such that Leibniz's rule $\nabla(ae) = d_{X/S}(a)\, e + a\nabla(e)$ is satisfied for any sections a of $\mathcal{O}_X$ and e of E. Here $d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$ is the relative exterior derivative. Given a vector bundle with connection, we can extend ∇ to an operator

$$\nabla : E \otimes \Omega^i_{X/S} \to E \otimes \Omega^{i+1}_{X/S}$$

by enforcing Leibniz's rule for forms a, using the usual sign conventions. In particular, the square of ∇ is an operator

$$\nabla^2 : E \to E \otimes \Omega^2_{X/S}.$$

Using Leibniz's rule and the fact that $(d_{X/S})^2 = 0$, it is easy to see that ∇^2 is $\mathcal{O}_X$-linear. Thus it is given by a section $\nabla^2 \in H^0(\mathrm{End}(E) \otimes \Omega^2_{X/S})$ called the *curvature* of ∇. A *vector bundle with integrable connection* is a vector bundle with connection (E, ∇), such that the curvature vanishes, $\nabla^2 = 0$.

We could make a similar definition of coherent sheaf with integrable connection. However, it is a well known fact (which we will not prove here) that if E_s is a coherent sheaf with integrable connection on X_s, then E_s is locally free, and the Chern classes of E_s vanish (hence the normalized Hilbert polynomial of E_s is the same as that of $\mathcal{O}_X$). If E is a coherent sheaf on X with integrable relative connection, such that E is flat over S, then E is locally free by Lemma 1.27, Part I. Because of this, we may as well assume that the pure dimension d is equal to the relative dimension of X/S, and that the normalized Hilbert polynomial p_0 is equal to that of $\mathcal{O}_X$—otherwise the moduli spaces are empty. Furthermore, any subsheaf of E_s preserved by ∇ is again a vector bundle with integrable connection, with the same normalized Hilbert polynomial p_0.

Theorem **6.13.** — *Suppose* X *is smooth and projective over* S. *There is a scheme* $\mathbf{M}_{DR}(X/S, n)$ *quasi-projective over* S, *universally co-representing the functor* $\mathbf{M}^{\natural}_{DR}(X/S, n)$ *which assigns to an* S-*scheme* S′ *the set of isomorphism classes of vector bundles with integrable connection* (E, ∇) *on* X′/S′ *of a given rank n.*

Suppose X *is smooth and projective with connected fibers over* S, *and suppose* $x : S \to X$ *is a section. There is a scheme* $\mathbf{R}_{DR}(X/S, \xi, n)$ *quasi-projective over* S, *representing the functor which assigns to an* S-*scheme* S′ *the set of isomorphism classes of* (E, ∇, α) *where* (E, ∇) *is a vector bundle with integrable connection on* X/S, *and* $\alpha : E|_{x(S')} \xrightarrow{\cong} \mathcal{O}^n_{S'}$ *is a frame along the section.*

Furthermore, with respect to an appropriate line bundle all points of $\mathbf{R}_{DR}(X/S, \xi, n)$ *are semistable for the natural action of* $Gl(n, \mathbf{C})$, *and the universal categorical quotient is naturally identified with* $\mathbf{M}_{DR}(X/S, n)$.

Proof. — Let Λ^{DR} be the sheaf of rings of all relative differential operators on X over S. It is split almost polynomial, and the sheaf H arising in the description of § 2, Part I, is equal to $\Omega^1_{X/S}$; its dual H^* is the relative tangent bundle $T(X/S)$. The derivation δ is the standard one, and the bracket $\{\ ,\ \}_Y$ is given by commutator of vector fields. The description of Λ^{DR}-modules given in Lemma 2.13, Part I, coincides with the above definition of vector bundle (or sheaf) with integrable connection. If E is a vector bundle with integrable relative connection on X over S, then any subsheaf of E_s preserved by the connection has the same normalized Hilbert polynomial p_0, so E_s is p-semistable as a Λ^{DR}-module. If E is flat over S then E is a p-semistable Λ^{DR}-module. The theorem follows from the general construction of moduli spaces given in Theorem 4.7, Part I. For the second paragraph, note that any vector bundle with relative integrable connection automatically satisfies condition $LF(\xi)$. Hence we may apply Theorem 4.10, Part I, to obtain $\mathbf{R}_{DR}(X/S, \xi, n)$. □

Dependence on the base point

In the Betti case, given two different base points x and y, and a choice of path γ from x to y, we obtain an isomorphism $T_\gamma : \mathbf{R}_B(X, x, n) \cong \mathbf{R}_B(X, y, n)$. This projects to a canonical isomorphism of $\mathbf{M}_B(X, n)$, justifying dropping the basepoint from the notation for the moduli space. On the other hand, there is no natural isomorphism between $\mathbf{R}_{DR}(X, x, n)$ and $\mathbf{R}_{DR}(X, y, n)$. Our construction began with a construction of $\mathbf{M}_{DR}(X, n)$ independent of the base point. On the complex analytic spaces, the isomorphism T_γ gives a complex analytic isomorphism (cf. § 7 below). This projects to an algebraic isomorphism (the identity) in the quotient, and on each orbit of the group $Gl(n, \mathbf{C})$, it comes from an algebraic automorphism of groups; however T_γ doesn't seem to be algebraic. One might conjecture that, in good cases, the isomorphism class of $\mathbf{R}_{DR}(X, x, n)$ is a distinguishing invariant of the point x.

7. Identifications between the moduli spaces

The analytic isomorphism between the de Rham and Betti spaces

Suppose $f: X \to S$ is a smooth projective morphism with connected fibers, and suppose $\xi : S \to X$ is a section. Recall that $\mathbf{R}_B^{(an)}(X/S, \xi, n)$ denotes the complex analytic space over S^{an} associated to the local system of complex analytic spaces $\mathbf{R}_B^{an}(X/S, \xi, n)$. Let $\mathbf{R}_{DR}^{an}(X/S, \xi, n)$ denote the complex analytic space associated to the de Rham representation scheme.

Theorem **7.1.** — (The framed Riemann-Hilbert correspondence.) *There is a natural isomorphism of complex analytic spaces*

$$\mathbf{R}_{\mathrm{B}}^{(\mathrm{an})}(\mathrm{X/S}, \xi, n) \cong \mathbf{R}_{\mathrm{DR}}^{\mathrm{an}}(\mathrm{X/S}, \xi, n),$$

compatible with the action of $\mathrm{Gl}(n, \mathbf{C})$.

We will prove this by showing that both spaces represent the same functor from the category of complex analytic spaces over S^{an} to the category of sets.

Lemma **7.2.** — *Suppose* Y *and* Z *are topological spaces which can be exhausted by relatively compact open subsets, and suppose* Y *is locally simply connected. Suppose* A *is a sheaf of rings on* Z. *Suppose* F *is a locally free sheaf of* $p_2^{-1}(\mathrm{A})$*-modules on* $\mathrm{Y} \times \mathrm{Z}$. *Then* $\mathrm{G} = p_{1,*}(\mathrm{F})$ *is a locally constant sheaf of* $\mathrm{H}^0(\mathrm{Z}, \mathrm{A})$*-modules on* Y, *and the stalk at* $y \in \mathrm{Y}$ *is given by*

$$\mathrm{G}_y = \mathrm{H}^0(\{y\} \times \mathrm{Z}, \mathrm{F}|_{\{y\} \times \mathrm{Z}}).$$

Proof. — First of all, suppose that U is a connected open subset of Y. Then $p_{2,*}(p_2^{-1}(\mathrm{A})|_{\mathrm{U} \times \mathrm{Z}}) = \mathrm{A}$. This implies that if F is free over $p_2^{-1}(\mathrm{A})$, $\mathrm{U} \subset \mathrm{Y}$ is a connected open subset, and $y \in \mathrm{U}$, then

$$p_{2,*}(\mathrm{F}|_{\mathrm{U} \times \mathrm{Z}}) \to p_{2,*}(\mathrm{F}|_{\{y\} \times \mathrm{Z}})$$

is an isomorphism of free sheaves of A-modules. Suppose that there exists an open covering $\mathrm{Z} = \bigcup_\alpha \mathrm{Z}_\alpha$ such that F is free of rank n on $\mathrm{Y} \times \mathrm{Z}_\alpha$. The previous result implies that if $y \in \mathrm{U} \subset \mathrm{Y}$ and U is connected, then

$$p_{2,*}(\mathrm{F}|_{\mathrm{U} \times \mathrm{Z}}) \to p_{2,*}(\mathrm{F}|_{\{y\} \times \mathrm{Z}})$$

is an isomorphism of locally free sheaves on Z. This gives an isomorphism of spaces of global sections, in other words the restriction maps

$$\mathrm{H}^0(\mathrm{U} \times \mathrm{Z}, \mathrm{F}|_{\mathrm{U} \times \mathrm{Z}}) \to \mathrm{H}^0(\{y\} \times \mathrm{Z}, \mathrm{F}|_{\{y\} \times \mathrm{Z}})$$

are isomorphisms. This implies the lemma in this case. It follows that the lemma is true for sheaves F such that there exist open coverings $\mathrm{Y} = \bigcup_\mu \mathrm{Y}_\mu$ and $\mathrm{Z} = \bigcup_\alpha \mathrm{Z}_\alpha$ with F free on each $\mathrm{Y}_\mu \times \mathrm{Z}_\alpha$.

Finally we treat the general case of a locally free F. Suppose that Z^i is an increasing sequence of relatively compact open subsets exhausting Z. Then, for any relatively compact open set $\mathrm{Y}' \subset \mathrm{Y}$, $\mathrm{F}|_{\mathrm{Y}' \times \mathrm{Z}^i}$ satisfies the hypotheses of the previous paragraph. Thus $\mathrm{G}^i \stackrel{\mathrm{def}}{=} p_{1,*}(\mathrm{F}|_{\mathrm{Y} \times \mathrm{Z}^i})$ is locally constant when restricted to any Y'. This implies that G^i is locally constant. The stalk at $y \in \mathrm{Y}$ is

$$\mathrm{G}^i_y = \mathrm{H}^0(\{y\} \times \mathrm{Z}^i, |_{\{y\} \times \mathrm{Z}^i}).$$

Finally, we have

$$G = \varprojlim G^i,$$

and since Y′ is locally simply connected, the inverse limit of a system of locally constant sheaves is again locally constant. This proves that G is locally constant. The stalk G_y is the inverse limit of the stalks G^i_y, hence equal to the desired space of global sections. □

Lemma **7.3**. — *The complex analytic space* $\mathbf{R}^{(an)}_B(X/S, \xi, n)$ *over* S^{an} *represents the functor which to each morphism* $S' \to S^{an}$ *of complex analytic spaces associates the set of isomorphism classes of pairs* $(\mathcal{F}, \beta)$ *where* $\mathcal{F}$ *is a locally free sheaf of* $f^{-1}(\mathcal{O}_{S'})$*-modules of rank n on* $X' = X^{an} \times_{S^{an}} S'$, *and* $\beta : \xi^{-1}(\mathcal{F}) \cong \mathcal{O}^n_{S'}$ *is a frame over the section* ξ.

Proof. — Let X^{top} denote the topological space underlying X^{an}. Note that the quantities appearing in the statement of the lemma are local over S^{an} and depend only on the structure of $f : X^{top} \to S^{an}$ as a fibration of topological spaces with a section over S^{an}. Thus we may suppose that $X^{top} = X_0 \times S^{an}$ and $\xi(s) = (x, s)$ for $x \in X_0$. Let $\Gamma = \pi_1(X_0, x)$. Then

$$\mathbf{R}^{(an)}_B(X/S, \xi, n) = R^{an}(\Gamma, n) \times S^{an},$$

so the set of S^{an}-morphisms from S′ to $\mathbf{R}^{(an)}_B(X/S, \xi, n)$ is equal to the set of morphisms from S′ to $\mathbf{R}^{an}(\Gamma, n)$.

The scheme $\mathbf{R}(\Gamma, n)$ is affine. It follows that the complex analytic morphisms from S′ to the associated analytic space $\mathbf{R}^{an}(\Gamma, n)$ are given by the homomorphisms of **C**-algebras

$$H^0(\mathbf{R}(\Gamma, n), \mathcal{O}_{\mathbf{R}(\Gamma, n)}) \to H^0(S', \mathcal{O}_{S'})$$

(this can be seen by embedding $\mathbf{R}(\Gamma, n)$ in an affine space). In particular,

$$\begin{aligned} \mathbf{R}^{(an)}_B(X/S, \xi, n)(S') &= \mathbf{R}^{an}(\Gamma, n)(S') \\ &= \mathbf{R}(\Gamma, n)(\mathrm{Spec}(H^0(S', \mathcal{O}_{S'}))) \\ &= \mathrm{Hom}(\Gamma, \mathrm{Gl}(n, H^0(S', \mathcal{O}_{S'}))). \end{aligned}$$

Let p_1 and p_2 denote the first and second projections on $X_0 \times S'$. Suppose $\mathcal{F}$ is a locally free sheaf of $p_2^{-1}(\mathcal{O}_{S'})$-modules on $X_0 \times S'$. Let $\mathcal{G} = p_{1,*}(\mathcal{F})$. This is a sheaf of $H^0(S', \mathcal{O}_{S'})$-modules on X_0. Lemma 7.2 implies that $\mathcal{G}$ is locally constant, with fiber $\mathcal{G}_x = \mathcal{F}|_{\{x\} \times S'}$ over $x \in X_0$.

The monodromy of the locally constant sheaf $\mathcal{G}$ is a representation

$$\Gamma \to \mathrm{Aut}_{H^0(S', \mathcal{O}_{S'})}(\mathcal{G}_x).$$

If $\beta : \xi^*(\mathcal{F}) \cong \mathcal{O}^n_{S'}$ then by Lemma 7.2 the fiber is $\mathcal{G}_x \cong H^0(S', \mathcal{O}_{S'})^n$, so the monodromy of $\mathcal{G}$ gives a representation

$$\Gamma \to \mathrm{Gl}(n, H^0(S', \mathcal{O}_{S'})).$$

This gives a map from the set of isomorphism classes of $(\mathcal{F}, \beta)$ to $\mathrm{Hom}(\Gamma, \mathrm{Gl}(n, H^0(S', \mathcal{O}_{S'})))$.

For the inverse map, note that X_0 is locally simply connected, so a universal covering $\widetilde{X}_0$ exists. Choose a base point $\tilde{x}$ over $x \in X_0$. Set $\widetilde{\mathscr{F}} = p_2^{-1}(\mathcal{O}^n_{S'})$ on $\widetilde{X}_0 \times S'$. The identity gives $\widetilde{\beta} : \widetilde{\mathscr{F}}|_{\{\tilde{x}\} \times S'} \cong \mathcal{O}^n_{S'}$. A representation

$$\Gamma \to \mathrm{Gl}(n, H^0(S', \mathcal{O}_{S'}))$$

gives an action of Γ on $\widetilde{\mathscr{F}}$ over the action on $\widetilde{X}_0 \times S'$. We can use this to descend $\widetilde{\mathscr{F}}$ to a locally free sheaf of $p_2^{-1}(\mathcal{O}_{S'})$-modules $\mathscr{F}$ on $X_0 \times S'$, with the required frame β. This is the inverse of the previous construction. We obtain an isomorphism between the set of S'-valued points of $\mathbf{R}_B^{(an)}(X/S, \xi, n)$, and the set of $(\mathscr{F}, \beta)$ on X' over S' as desired. □

This lemma provides half of the proof of the theorem. For the other half, we begin with a lemma and some corollaries.

Lemma **7.4.** — *Suppose that* $V \subset \mathbf{C}^N$ *is an open disc and* $S' \subset V$ *is a complex analytic subspace, such that all embedded components pass through a single point* $s \in S'$. *Suppose* U *is an open disc centered at the origin in* $\mathbf{C}^k$. *If* (E, ∇) *is a holomorphic vector bundle with integrable connection on* $U \times S'$ *over* S', *then the map*

$$\nu : H^0(U \times S', E)^\nabla \to H^0(\{0\} \times S', E|_{\{0\} \times S'})$$

is an isomorphism (here the exponent ∇ *means the space of covariant constant sections).*

Proof. — This is well known if S' is a point. It follows that it is true if S' is an artinian complex analytic space (the same as an artinian scheme), for the result in that case follows from the same result for $p_{1,*}(E, \nabla)$ on U. It is also well known if S' is a smooth complex analytic manifold. We show how to deduce the theorem when S' may be nonreduced, for example.

Choose a point $s \in S'$ containing all irreducible embedded components of S. Let S'_m be the m-th infinitesimal neighborhood of s in S'. The map ν is injective: suppose e is a section with $\nabla(e) = 0$ and $\nu(e) = 0$; then from the result for artinian spaces, $e|_{S'_m} = 0$ for all m, and this implies that $e = 0$.

With the same hypotheses on S', suppose also that $\dim(U) = 1$. We show that ν is surjective in this case. The holomorphic bundle E is trivial, so it has an extension to a trivial bundle E^{ex} on $U \times V$. Let d denote the constant connection on E^{ex}. We may write

$$\nabla = d|_{U \times S'} + A(u, t)\, du$$

where $A(u, t)$ is a holomorphic section of $\mathrm{End}(E)$ over $U \times S$. There exists an extension of A to a holomorphic section A^{ex} of $\mathrm{End}(E^{ex})$ on $U \times V$, and we can then put

$$\nabla^{ex} = d + A^{ex}(u, v)\, du.$$

This is an integrable holomorphic connection on E^{ex} relative to V (it is integrable because $\dim(U) = 1$ implies $\Omega^2_{U \times V/V} = 0$). Suppose $e_0 \in H^0(\{0\} \times S', E|_{\{0\} \times S'})$, and choose an extension to $e_0^{ex} \in H^0(\{0\} \times V, E|_{\{0\} \times V})$. By the result for smooth base spaces, there exists $e^{ex} \in H^0(U \times V, E^{ex})^{\nabla^{ex}}$ with $\nu(e^{ex}) = e_0^{ex}$. Putting $e = e^{ex}|_{U \times S'}$ gives a section with $\nabla(e) = 0$ and $\nu(e) = e_0$. This proves that ν is an isomorphism in the case of relative dimension 1.

Now proceed by induction on the relative dimension k, assuming that the theorem is known for relative dimension $k - 1$. Let U_1 denote the disc of dimension $k - 1$ obtained by intersecting U with one of the coordinate planes. By the inductive hypothesis, there exists a section e_1 in $H^0(U_1 \times S', E|_{U_1 \times S'_m})^{\nabla}$ with e_1 restricting to e_0 on $\{0\} \times S'$. Let $h : U \times S' \to U_1 \times S'$ denote the vertical projection. Let ∇_1 denote the projection of ∇ into a relative connection for the map h. The map h is smooth of relative dimension 1, so by the previous result, there exists a section e in $H^0(U \times S', E)^{\nabla_1}$ restricting to e_1 on $U_1 \times S'$. In order to show that $\nabla(e) = 0$ we use the infinitesimal neighborhoods S'_m. There exist sections e^m in $H^0(U \times S'_m, E|_{U \times S'_m})^{\nabla}$ such that $e^m|_{\{0\} \times S'_m} = e_0|_{\{0\} \times S'_m}$. By the uniqueness result for $U_1 \times S'_m$ over S'_m, e^m is equal to e_1 on the subspace $U_1 \times S'_m$. By the uniqueness for $U \times S'_m$ over $U_1 \times S'_m$, $e|_{U \times S'_m} = e_m$. But ∇ is $\mathcal{O}_{S'}$-linear, so $\nabla(e)|_{U \times S'_m} = \nabla(e_m) = 0$. This is true for any infinitesimal neighborhood, so $\nabla(e) = 0$. This shows that ν is surjective in relative dimension k. □

Keep the same hypotheses as in this lemma. Suppose E is a trivial bundle of rank n. We can choose n sections $e_1, \ldots, e_n$ in $H^0(U \times S', E)^{\nabla}$ such that $\nu(e_i)$ form a frame for $E|_{\{0\} \times S'}$. The lemma implies that

$$H^0(U \times S', E)^{\nabla} \cong H^0(S', \mathcal{O}_{S'}) \otimes_{\mathbf{C}} (\mathbf{C}e_1 \oplus \ldots \oplus \mathbf{C}e_n).$$

Conversely, if the e_i are a collection of sections such that this formula holds, then the $\nu(e_i)$ form a frame for $E|_{\{0\} \times S'}$. We claim that the e_i form a frame for E over U. It suffices to show that over each closed point $(u, s) \in U \times S'$, the e_i are a basis for the fiber of E. But this follows from the above statement and its converse applied to $\{u\} \times S'$ instead of $\{0\} \times S'$.

Corollary **7.5.** — *Suppose $e_1, \ldots, e_n$ are sections chosen as above. Then the map*

$$p_2^{-1}(\mathcal{O}_{S'}) \otimes_{\mathbf{C}} (\mathbf{C}e_1 \oplus \ldots \oplus \mathbf{C}e_n) \to E^{\nabla}$$

is an isomorphism of sheaves on $U \times S'$.

Proof. — This is injective, because the e_i are a frame for the holomorphic bundle E. For surjectivity, suppose e is a section of E^{∇} over an open set $V \subset U \times S'$. Then V can be covered by subsets $U' \times S''$ of the form considered above. The restriction of $\{e_i\}$ to a section $\{u\} \times S''$ is a frame for the restriction of the bundle E, so the previous argument applies. There exist $a_i \in H^0(S'', \mathcal{O}_{S''})$ with $\sum a_i e_i = e$ on $U' \times S''$. This shows that the map of sheaves is surjective. □

Corollary **7.6.** — *Suppose* $f\colon X' \to S'$ *is a smooth morphism of complex analytic spaces. Suppose* (E, ∇) *is a vector bundle with integrable holomorphic connection relative to* S'. *Let* $\mathscr{F} = E^{\nabla}$ *denote the sheaf of sections* e *of* E *such that* $\nabla(e) = 0$. *Then* $\mathscr{F}$ *is a locally free sheaf of* $f^{-1}(\mathcal{O}_{S'})$-*modules.*

Proof. — Let k denote the relative dimension of X' over S'. We can cover X' by a collection of open subsets of the form $U \times S''$ where $U \subset \mathbf{C}^k$ is an open disc, and $S'' \subset S'$ is a subset satisfying the hypotheses of the lemma. By the corollary, $\mathscr{F}|_{U \times S''}$ is free over $f^{-1}(\mathcal{O}_{S'})$. Thus $\mathscr{F}$ is locally free. □

Lemma **7.7.** — *The complex analytic space* $\mathbf{R}^{an}_{DR}(X/S, \xi, n)$ *over* S^{an} *represents the functor which to each morphism* $S' \to S^{an}$ *of complex analytic spaces associates the set of isomorphism classes of pairs* $(\mathscr{F}, \beta)$ *where* $\mathscr{F}$ *is a locally free sheaf of* $f^{-1}(\mathcal{O}_{S'})$-*modules of rank* n *on* $X' = X^{an} \times_{S^{an}} S'$, *and* $\beta : \xi^{-1}(\mathscr{F}) \cong \mathcal{O}^n_{S'}$ *is a frame over the section* ξ.

Proof. — Note that by Lemma 5.7, Part I, the argument of Theorem 6.13, and the analogue of Lemma 2.13, Part I, for the complex analytic case, $\mathbf{R}^{an}_{DR}(X/S, \xi, n)$ represents the functor which to each morphism $S' \to S^{an}$ of complex analytic spaces, associates the set of isomorphism classes of triples (E, ∇, α) where E is a holomorphic vector bundle over $X' = X^{an} \times_{S^{an}} S'$, ∇ is a holomorphic integrable connection on E relative to S', and $\alpha : \xi^*(E) \cong \mathcal{O}^n_{S'}$ is a frame. We have to identify this functor with the functor given in the lemma. First, note that the trivial bundle $\mathcal{O}_{X'}$ has a natural connection $d_{X'/S'}$, the exterior derivative with values projected into $\Omega^1_{X'/S'}$. This connection is $f^{-1}(\mathcal{O}_{S'})$-linear. If $\mathscr{F}$ is a locally free sheaf of $f^{-1}(\mathcal{O}_{S'})$-modules on X', then

$$E = \mathscr{F} \otimes_{f^{-1}(\mathcal{O}_{S'})} \mathcal{O}_{X'}$$

is a locally free sheaf of $\mathcal{O}_{X'}$-modules, and it has a relative holomorphic integrable connection $\nabla = 1 \otimes d_{X'/S'}$. A frame $\beta : \xi^{-1}(\mathscr{F}) \cong \mathcal{O}^n_{S'}$ yields $\alpha : \xi^*(E) \cong \mathcal{O}^n_{S'}$. This gives a map from the set $\mathbf{R}^{(an)}_{B}(X/S, \xi, n)(S')$ to the set $\mathbf{R}^{an}_{DR}(X/S, \xi, n)(S')$.

Suppose (E, ∇) is a holomorphic vector bundle with integrable connection on X' over S'. By Corollary 7.6, the sheaf $\mathscr{F} = E^{\nabla}$ is a locally free sheaf of $f^{-1}(\mathcal{O}_{S'})$-modules. Suppose α is a frame for E along ξ. Let β denote the composed map

$$\xi^{-1}(E^{\nabla}) \to \xi^{-1}(E) \to \xi^*(E) \to \mathcal{O}^n_{S'}.$$

The arguments from the proof of Lemma 7.4 show that this is an isomorphism. This completes the construction of the inverse to our previous map, so we obtain an isomorphism between the set of (E, ∇, α) and the set of $(\mathscr{F}, \beta)$. □

Proof of Theorem 7.1. — Lemmas 7.3 and 7.7 show that the spaces $\mathbf{R}^{(an)}_{B}(X/S, \xi, n)$ and $\mathbf{R}^{an}_{DR}(X/S, \xi, n)$ both represent the same functor. Thus they are naturally isomorphic. The isomorphism between functors is compatible with the group action, so the isomorphism between spaces is too. □

Proposition **7.8.** — (The Riemann-Hilbert correspondence.) *Suppose* $f\colon X \to S$ *is a smooth projective morphism. Then there is a natural isomorphism*

$$\mathbf{M}_{\mathrm{B}}^{(\mathrm{an})}(X/S, n) \cong \mathbf{M}_{\mathrm{DR}}^{\mathrm{an}}(X/S, n).$$

If f has connected fibers and ξ is a section, then this isomorphism is compatible with the isomorphism given by Theorem 7.1.

Proof. — Suppose first of all that f has connected fibers and a section ξ exists. Then $\mathbf{M}_{\mathrm{DR}}(X/S, n)$ is a good quotient of $\mathbf{R}_{\mathrm{DR}}(X/S, \xi, n)$ by the action of $\mathrm{Gl}(n, \mathbf{C})$. Proposition 5.5, Part I, implies that $\mathbf{M}_{\mathrm{DR}}^{\mathrm{an}}(X/S, n)$ is a universal categorical quotient of $\mathbf{R}_{\mathrm{DR}}^{\mathrm{an}}(X/S, \xi, n)$ in the category of complex analytic spaces over S^{an}. On the other hand, $\mathbf{M}_{\mathrm{B}}(X_s, n)$ is the good quotient of $\mathbf{R}_{\mathrm{B}}(X_s, \xi(s), n)$ by the action of $\mathrm{Gl}(n, \mathbf{C})$, so again by Proposition 5.5, $\mathbf{M}_{\mathrm{B}}^{\mathrm{an}}(X_s, n)$ is a universal categorical quotient of $\mathbf{R}_{\mathrm{B}}^{\mathrm{an}}(X_s, \xi(s), n)$. The space $\mathbf{M}_{\mathrm{B}}^{(\mathrm{an})}(X/S, n)$ is, locally over S^{an}, of the form $\mathbf{M}_{\mathrm{B}}^{\mathrm{an}}(X_s, n) \times S^{\mathrm{an}}$. The property of being a universal categorical quotient is preserved under taking the product with another space, as well as localization in the quotient space (hence by localization in S^{an}), so $\mathbf{M}_{\mathrm{B}}^{\mathrm{an}}(X_s, n) \times S^{\mathrm{an}}$ is a universal categorical quotient of $\mathbf{R}_{\mathrm{B}}^{\mathrm{an}}(X_s, \xi(s), n) \times S^{\mathrm{an}}$. The isomorphism of Theorem 7.1 is compatible with the action of $\mathrm{Gl}(n, \mathbf{C})$, so it induces an isomorphism of universal categorical quotients

$$\mathbf{M}_{\mathrm{B}}^{(\mathrm{an})}(X/S, n) \cong \mathbf{M}_{\mathrm{DR}}^{\mathrm{an}}(X/S, n).$$

Suppose that X is a disjoint union of components, each of which has connected fibers over S and admits a section. The resulting moduli spaces $\mathbf{M}_{\mathrm{B}}^{(\mathrm{an})}(X/S, n)$ and $\mathbf{M}_{\mathrm{DR}}^{\mathrm{an}}(X/S, n)$ are then products of spaces obtained by taking quotients of representation spaces. The above isomorphism for each factor gives the desired isomorphism.

In general, we can make a surjective étale base change $S' \to S$ such that X'/S' satisfies the hypotheses of the previous paragraph. The isomorphism

$$\mathbf{M}_{\mathrm{B}}^{(\mathrm{an})}(X'/S', n) \cong \mathbf{M}_{\mathrm{DR}}^{\mathrm{an}}(X'/S', n)$$

descends to give the desired isomorphism. □

The homeomorphism between the de Rham and Dolbeault spaces

Recall some facts from [Si5]. These results are based on non-linear partial differential equations, and in particular on the works [NS] [Co] [Do1] [Do2] [Do3] [Hi1] [Si2] [UY]. There is a notion of harmonic metric for a vector bundle with integrable connection (flat bundle) or a Higgs bundle on a smooth projective variety X. Given a flat bundle and a harmonic metric, one obtains a Higgs bundle, and vice versa. The structures of Higgs or flat bundles obtained from the harmonic metric do not depend on the choice of harmonic metric. The conditions for the existence of a harmonic metric are as follows. A flat bundle has a harmonic metric if and only if it is semisimple [Co] [Do3]. A Higgs bundle has a harmonic metric if and only if it is a direct sum of μ-stable

Higgs bundles with vanishing rational Chern classes [Hi1] [Si2]. A *harmonic bundle* consists of a flat bundle and a Higgs bundle related by a C^∞ isomorphism such that there exists a common harmonic metric relating the structures. The set of isomorphism classes of harmonic bundles is exactly the same as the set of flat bundles parametrized by points of the moduli space $\mathbf{M}_{DR}(X, n)$. It is also the same as the set of Higgs bundles parametrized by points of the moduli space $\mathbf{M}_{Dol}(X, n)$. We obtain an isomorphism of sets of closed points between these two moduli spaces [Si5].

If $X \to S$ is a smooth projective morphism, we can take the isomorphisms of sets given in each fiber all together to get an isomorphism between the sets of closed points of $\mathbf{M}_{DR}(X/S, n)$ and $\mathbf{M}_{Dol}(X/S, n)$. Recall that the superscript M^{top} denotes the topological space underlying the complex analytic space M^{an}. We will show that our isomorphism of sets gives a homeomorphism of topological spaces $\mathbf{M}^{top}_{DR}(X/S, n) \cong \mathbf{M}^{top}_{Dol}(X/S, n)$.

We recall a weak compactness property for harmonic bundles, following the notation of [Si5] (except that the Higgs field which was denoted by θ there is denoted by φ here, to conform with Hitchin's original notation). Suppose $X \to S$ is smooth with connected fibers, and suppose $\xi : S \to X$ is a section. Suppose $\{ s_i \}$ is a sequence of points converging to t in S. Choose a standardized sequence of diffeomorphisms $\Psi_i : X_{s_i} \cong X_t$, such that $\Psi_i(\xi(s_i)) = \xi(t)$. Choose a family of metrics on X_{s_i} which, when transported via Ψ_i, are uniformly bounded in any norm with respect to a metric on X_t. Use these metrics to measure forms on X_{s_i}.

Proposition **7.9.** — *Fix* $q > 1$. *Suppose* V_i *is a harmonic bundle on* X_{s_i} *with harmonic metric* K_i *for each* i, *such that the coefficients of the characteristic polynomials of the Higgs fields* φ_i *are uniformly bounded in* L^1 *norm. Then there is a harmonic bundle* V, *a subsequence* $\{ i' \}$, *and isomorphisms* $\eta_{i'} : \Psi_{i',*}(V_{i'}) \cong V$ *of* C^∞ *bundles satisfying the following properties. There is a harmonic metric* K *for* V *with* $\eta_{i'}(K_i) = K$, *and if* O *represents any of the operators* ∂, $\bar\partial$, φ, $\bar\varphi$ *or combinations thereof, the differences* $\mathrm{dif}(O, i') \stackrel{\text{def}}{=} \eta_{i',*}(O_{i'}) - O$ *converge to zero strongly in the operator norm for operators from* L^q_1 *to* L^q.

Proof. — This is essentially the same as Lemma 2.8 of [Si5], which is based in turn on Uhlenbeck's weak compactness property [Uh] and the properness of Hitchin's map [Hi1]. There are a few new twists. The main difference is that the underlying spaces X_{s_i} are varying. In particular, the differences $\mathrm{dif}(O, i)$ are differential operators, so they must be measured with respect to operator norms. We recall the proof with this in mind.

First of all, the hypothesis that the characteristic polynomials are bounded in L^1 norm implies that they are bounded in C^0 norm, since the coefficients are holomorphic sections of certain bundles on X_{s_i}.

Fix p large. The bound for the coefficients of the characteristic polynomials implies that $| \varphi_i |_{K_i}$ are uniformly bounded [Si5], Lemma 2.7. The curvatures

$$F_{\partial_i + \bar\partial_i} = - \varphi_i \bar\varphi_i - \bar\varphi_i \varphi_i$$

are therefore uniformly bounded in C^0. Uhlenbeck's weak compactness theorem [Uh] gives a unitary bundle (V, K) with unitary connection $\partial + \bar{\partial}$ (of type L_1^p) and a sequence of unitary isomorphisms $\eta_i : \Psi_{i,*}(V_i) \cong V$ such that

$$\mathrm{dif}(\partial + \bar{\partial}, i) = \eta_{i,*}(\partial_i + \bar{\partial}_i) - \partial - \bar{\partial}$$

converges to zero weakly in L_1^p. In particular, for p big enough this converges strongly to zero in C^0 (note that $\mathrm{dif}(\partial + \bar{\partial}, i)$ is a 0-th order operator), and $\mathrm{dif}(\partial + \bar{\partial}, i) \to 0$ strongly in the operator norm for $\mathrm{Hom}(L_1^q, L^q)$. We have

$$(\partial_i + \bar{\partial}_i)(\varphi_i) = 0$$

so

$$(\partial + \bar{\partial})(\eta_{i,*}\varphi_i) = \mathrm{dif}(\partial + \bar{\partial}, i)(\eta_{i,*}\varphi_i).$$

Since η_i are unitary isomorphisms, $|\eta_{i,*}\varphi_i|_K$ are uniformly bounded. Thus $\mathrm{dif}(\partial + \bar{\partial}, i)(\eta_{i,*}\varphi_i) \to 0$ strongly in C^0, so $(\partial + \bar{\partial})(\eta_{i,*}\varphi_i) \to 0$ strongly in C^0. We have to be slightly careful, since $\eta_{i,*}\varphi_i$ are one-forms—this doesn't constitute an estimate for the full covariant derivatives. The $\eta_{i,*}\varphi_i$ are of type (1, 0) but each for a different complex structure. More precisely, let $T_i^{1,0} \subset T_{\mathbf{C}}(X_t)$ denote the subbundle of forms of type (1, 0) with respect to the complex structure of X_{s_i} as transported to X_t by Ψ_i. Then

$$\eta_{i,*}\varphi_i \in H^0(X_t, \mathrm{End}(V) \otimes_{\mathscr{C}^\infty(X_t)} T_i^{1,0}).$$

We can choose an open subset $U \subset X_t$ and a sequence of open immersions $\Phi_i : U \to X_t$ which converge to the identity in any norm, but such that $\Phi_i^*(T_i^{1,0}) = T^{1,0}U$ is the subbundle of forms on U of type (1, 0) with respect to the holomorphic structure of X_t. We may also choose a sequence of unitary isomorphisms $\zeta_i : \Phi_i^*(V) \cong V$ converging to the identity in any norm, so for example $\zeta_i \Phi_i^*(\partial + \bar{\partial}) - \partial - \bar{\partial}$ converges to zero in L_1^p. Then $\zeta_i \Phi_i^* \eta_{i,*}\varphi_i$ are End(V)-valued (1, 0)-forms on U with

$$(\partial + \bar{\partial})(\zeta_i \Phi_i^* \eta_{i,*}\varphi_i)$$

uniformly bounded in C^0. Now we can conclude (from the elliptic estimates for $\bar{\partial}$) that $\zeta_i \Phi_i^* \eta_{i,*}\varphi_i$ are uniformly bounded in L_1^p on any relatively compact subset of U. This argument, done for a collection of open sets U covering X_t, implies that $\eta_{i,*}(\varphi_i)$ are uniformly bounded in L_1^p. By going to a subsequence, we may suppose that $\eta_{i,*}\varphi_i$ approach a limit φ weakly in L_1^p (hence strongly in C^0). The limit satisfies $(\partial + \bar{\partial})(\varphi) = 0$ and $\varphi \wedge \varphi = 0$. Finally, in the conclusion of Uhlenbeck's weak compactness theorem, $\eta_{i,*}(F_{\partial_i + \bar{\partial}_i})$ approach $F_{\partial + \bar{\partial}}$ weakly in L^p. In particular (even taking into consideration the change of complex structure), the component of type (0, 2) is the weak limit of the components of type (0, 2), which are zero. Therefore $\bar{\partial}^2 = 0$, so $(V, \bar{\partial}, \varphi)$ is a Higgs bundle. The $\eta_{i,*}(\bar{\varphi}_i)$ approach the K-complex conjugate $\bar{\varphi}$ weakly in L_1^p, and the operator ∂ is the one associated to $\bar{\partial}$ by the metric K. If we set $D = \partial + \bar{\partial} + \varphi + \bar{\varphi}$ then the differences

$$\mathrm{dif}(D, i) = \eta_{i,*}(D_i) - D$$

are 0-th order operators converging to zero weakly in L_1^p. In particular, they converge to zero strongly in C^0 and hence they converge to zero in the operator norm for operators from L_1^q to L^q. The weak convergence in L_1^p implies that the curvature D^2 is the weak limit of the curvatures $\eta_{i,*}(D_i^2)$, which are zero. Thus $D^2 = 0$, proving that V together with all of its operators and its metric, is a harmonic bundle.

We know from above that $\mathrm{dif}(O, i) \to 0$ strongly in C^0 for $O = \varphi$ and $O = \partial + \bar\partial$. The same argument as for φ works for $O = \bar\varphi$. We have to extract the cases of $O = \partial$ and $O = \bar\partial$ from the case $O = \partial + \bar\partial$. Let $P_i^{1,0}$ denote the projection onto $T_i^{1,0}$, and let $P^{1,0}$ denote the projection for the complex structure of X_t. Then $P_i^{1,0} \to P^{1,0}$ in any norm. We have

$$\eta_{i,*}(\partial_i) = P_i^{1,0}\, \eta_{i,*}(\partial_i + \bar\partial_i).$$

Thus

$$\mathrm{dif}(\partial, i) = P_i^{1,0}\, \mathrm{dif}(\partial + \bar\partial, i) + (P_i^{1,0} - P^{1,0})\, (\partial + \bar\partial).$$

Since $\mathrm{dif}(\partial + \bar\partial, i) \to 0$ in C^0 and the $P_i^{1,0}$ are bounded, the first term converges to zero in the operator norm. Now $\partial + \bar\partial$ is a bounded operator from L_1^q to L^q, and $P_i^{1,0} - P^{1,0}$ converges to zero in C^0, hence in the operator norm of $\mathrm{Hom}(L^q, L^q)$. Therefore their composition, the second term, converges to zero in the operator norm of $\mathrm{Hom}(L_1^q, L^q)$. The same argument works for $\mathrm{dif}(\bar\partial, i)$. This proves the proposition. □

Let J denote the standard unitary metric on $\mathbf{C}^n$. Let

$$\mathbf{R}_{\mathrm{Dol}}^{\mathrm{J}}(X/S, \xi, n) \subset \mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n)$$

denote the subset consisting of triples (s, E, β) where $s \in S$, E is a Higgs bundle of harmonic type on X_s, and $\beta : E_{\xi(s)} \cong \mathbf{C}^n$ is a frame, such that there exists a harmonic metric K for E with $\beta(K_{\xi(s)}) = J$. Note that the harmonic metric K is uniquely determined once it is fixed at one point $\xi(s)$ [Si2]—we call this K the *chosen harmonic metric*. Endow $\mathbf{R}_{\mathrm{Dol}}^{\mathrm{J}}(X/S, \xi, n)$ with the topology induced by the analytic topology of $\mathbf{R}_{\mathrm{Dol}}^{\mathrm{an}}(X/S, \xi, n)$.

Suppose s_i is a sequence of points approaching t in S. Choose a standardized sequence of diffeomorphisms $\Psi_i : X_{s_i} \cong X_t$ such that $\Psi_i(\xi(s_i)) = \xi(t)$.

Corollary **7.10.** — *Suppose* (E_i, β_i) *are points in* $\mathbf{R}_{\mathrm{Dol}}^{\mathrm{J}}(X_{s_i}, \xi(s_i), n)$ *which remain inside the inverse image of a compact subset of* $\mathbf{M}_{\mathrm{Dol}}^{\mathrm{an}}(X/S, \xi, n)$. *Then after going to a subsequence, there is a point* (E, β) *in* $\mathbf{R}_{\mathrm{Dol}}^{\mathrm{J}}(X_t, \xi(t), n)$ *and a sequence of bundle isomorphisms* $\eta_i : \Psi_{i,*}(E_i) \cong E$ *such that: the* η_i *preserve the chosen harmonic metrics; the operators* $\eta_{i,*}(\bar\partial_i)$ *and* $\eta_{i,*}(\varphi_i)$ *converge to* $\bar\partial$ *and* φ *in the operator norm for operators from* L_1^q *to* L^q; *and finally, the frames* $\eta_i(\beta_i)$ *converge to* β.

Proof. — Since the points remain in the inverse image of a compact subset of $\mathbf{M}_{\mathrm{Dol}}(X/S, \xi, n)$, the eigenforms of the Higgs fields φ_i for E_i are uniformly bounded (this is because of the existence of the map σ sending (E_i, φ_i) to the characteristic polynomial of φ_i—see the discussion above Lemma 6.10). From the previous proposition, we can go to a subsequence and obtain a Higgs bundle E with harmonic metric K and a sequence of bundle isomorphisms η_i with the desired convergence properties. Since the unitary group is compact, we may, by going to a further subsequence, assume that the frames $\eta_i(\beta_i)$ converge to a unitary frame β. Then (E, β) is a point in $\mathbf{R}^{J}_{Dol}(X_t, \xi(s_i), n)$. □

Corollary **7.11.** — *The subset* $\mathbf{R}^{J}_{Dol}(X/S, \xi, n) \subset \mathbf{R}^{an}_{Dol}(X/S, \xi, n)$ *is closed.*

Proof. — Suppose (s_i, E_i, β_i) is a sequence of points in $\mathbf{R}^{J}_{Dol}(X/S, \xi, n)$, converging to a point (t, E', β) in $\mathbf{R}^{an}_{Dol}(X_t, \xi(s_i), n)$. The images in $\mathbf{M}^{an}_{Dol}(X_t, \xi(s_i), n)$ converge, so they lie in a compact set. Apply the previous corollary to obtain a point (t, E, β) in $\mathbf{R}^{J}_{Dol}(X/S, \xi, n)$ and a sequence of bundle isomorphisms η_i. By Theorem 5.12, Part I, for the case of Λ^{Dol}, the points (s_i, E_i, β_i) converge to (t, E, β). Since $\mathbf{R}^{an}_{Dol}(X/S, \xi, n)$ is separated, $(t, E', \beta') = (t, E, \beta)$. Thus the limit is in $\mathbf{R}^{J}_{Dol}(X/S, \xi, n)$. □

Corollary **7.12.** — *The subset* $\mathbf{R}^{J}_{Dol}(X/S, \xi, n)$ *is proper over* $\mathbf{M}^{an}_{Dol}(X/S, n)$.

Proof. — Suppose (s_i, E_i, β_i) is a sequence of points in $\mathbf{R}^{J}_{Dol}(X/S, \xi, n)$ lying over a compact subset of $\mathbf{M}^{an}_{Dol}(X/S, \xi, n)$. First, we may choose a subsequence so that the points s_i converge to a point t. Then we can apply Corollary 7.10 and Theorem 5.12, Part I, for the case of Λ^{Dol}, to obtain a subsequence which has a limit (t, E, β) in $\mathbf{R}^{J}_{Dol}(X/S, \xi, n)$. □

We do the same thing for the de Rham spaces. Let

$$\mathbf{R}^{J}_{DR}(X/S, \xi, n) \subset \mathbf{R}_{DR}(X/S, \xi, n)$$

denote the subset consisting of triples (s, E, β) where $s \in S$, E is a semisimple vector bundle with integrable connection on X_s, and $\beta : E_{\xi(s)} \cong \mathbf{C}^n$ is a frame, such that there exists a harmonic metric K for E with $\beta(K_{\xi(s)}) = J$. The harmonic metric K is uniquely determined once it is fixed at the point $\xi(s)$ [Co], and we again call K the *chosen harmonic metric*. Endow $\mathbf{R}^{J}_{DR}(X/S, \xi, n)$ with the topology induced by the analytic topology of $\mathbf{R}^{an}_{DR}(X/S, \xi, n)$.

Suppose s_i is a sequence of points approaching t in S. Choose a standardized sequence of diffeomorphisms $\Psi_i : X_{s_i} \cong X_t$ such that $\Psi_i(\xi(s_i)) = \xi(t)$.

Lemma **7.13.** — *Suppose* (E_i, β_i) *are points in* $\mathbf{R}^{J}_{DR}(X_{s_i}, \xi(s_i), n)$ *which remain inside the inverse image of a compact subset of* $\mathbf{M}^{an}_{DR}(X/S, \xi, n)$. *Then after going to a subsequence, there is a point* $(E, \beta) \in \mathbf{R}^{J}_{DR}(X_t, \xi(s_i), n)$, *and a sequence of bundle isomorphisms* $\eta_i : \Psi_{i,*}(E_i) \cong E$ *such that: the* η_i *preserve the chosen harmonic metrics; the operators* $\eta_{i,*}(\bar\partial_i)$ *and* $\eta_{i,*}(\nabla_i)$ *converge to* $\bar\partial$ *and* ∇ *in the operator norm for operators from* L^q_1 *to* L^q; *and the frames* $\eta_i(\beta_i)$ *converge to* β.

Proof. — This is the same as for Corollary 7.10, except that we have to show that the characteristic polynomials of the Higgs fields φ_i of the harmonic bundles corresponding to E_i, are bounded. We follow the argument of ([Si4], Lemmas 3 and 5).

Let $\Gamma = \pi_1(X_t, \xi(t))$, which is also equal to $\pi_1(X_{s_i}, \xi(s_i))$ via the diffeomorphisms Ψ_i. The condition that the points lie over a compact subset of $\mathbf{M}^{an}_{DR}(X/S, \xi, n)$ implies that the monodromy representations of $\Psi_{i,*}(E_i, \nabla_i)$ lie over a compact subset of $\mathbf{M}(\Gamma, n)$ (by Theorem 7.1).

The first thing we note is that it is possible to choose frames β'_i for E_i such that the monodromy representations of (E_i, β'_i) lie in a compact subset of $\mathbf{R}(\Gamma, n)$. The argument (from [Si4], Lemma 3) is that the subset of zeros of the moment map in $\mathbf{R}(\Gamma, n)$ is proper over $\mathbf{M}(\Gamma, n)$ [Ki] [KN] [GS]; our monodromy representations come from harmonic bundles, so they are semisimple—lying in the closed orbits—thus by appropriate choice of frames we can assume they correspond to points in the set of zeros of the moment map.

Let ρ_i denote the monodromy representations corresponding to (E_i, β'_i). Since they are bounded, it is possible to choose initial ρ_i-equivariant maps from the universal covers $\widetilde{X}_{s_i}$ to $Gl(n, \mathbf{C})/U(n)$, which have uniformly bounded energy (note that the diffeomorphisms Ψ_i are uniformly bounded in any norm). See [Si4], Lemma 5, for a description of how to do this (the process described there works the same way for any rank). Finally, the harmonic equivariant map has lower energy, and the energy is equal to the L^2 norm of φ_i. Thus $\| \varphi_i \|_{L^2(X_{s_i})}$ are uniformly bounded. This implies that the eigenforms of φ_i are uniformly bounded in L^2 norm. The eigenforms of φ_i are multi-valued holomorphic sections of $\Omega^1_{X_{s_i}}$ which do not depend on our choices of frame β'_i. The maximum norm of an eigenvalue of a holomorphic matrix is a subharmonic function, so the eigenforms of φ_i are uniformly bounded in C^0. Thus the characteristic polynomials of the Higgs fields φ_i are uniformly bounded in C^0.

The rest of the proof is the same as that of Corollary 7.10. □

Corollary **7.14.** — *The subset* $\mathbf{R}^J_{DR}(X/S, \xi, n) \subset \mathbf{R}^{an}_{DR}(X/S, \xi, n)$ *is closed.*

Proof. — The same as for Corollary 7.11, but using Theorem 5.12, Part I, for the case of Λ^{DR}. □

Corollary **7.15.** — *The subset* $\mathbf{R}^J_{DR}(X/S, \xi, n)$ *is proper over* $\mathbf{M}^{an}_{DR}(X/S, n)$.

Proof. — The same as for Corollary 7.12, but using Theorem 5.12, Part I, for the case of Λ^{DR}. □

There is an isomorphism of sets $\mathbf{R}_{Dol}(X/S, \xi, n) \cong \mathbf{R}_{DR}(X/S, \xi, n)$. Over each fiber X_s, this comes from the equivalence between the category of semistable Higgs bundles with vanishing Chern classes, and the category of flat bundles, constructed in [Si5]. This equivalence of categories is compatible with pullback to a point $\xi(s) \to X_s$, so it gives an isomorphism between the sets of isomorphism classes of framed objects in the two categories. In other words we get an isomorphism between the set of points in $\mathbf{R}_{Dol}(X_s, \xi(s), n)$ and $\mathbf{R}_{DR}(X_s, \xi(s), n)$. Putting these together for all s we obtain the isomorphism of sets stated above.

Lemma **7.16.** — *This isomorphism of sets induces a homeomorphism between the subsets* $\mathbf{R}^{J}_{\mathrm{Dol}}(X/S, \xi, n)$ *and* $\mathbf{R}^{J}_{\mathrm{DR}}(X/S, \xi, n)$.

Proof. — We prove continuity of the map from the Dolbeault space to the de Rham space. Suppose s_i is a sequence of points approaching t in S. Choose a standardized sequence of diffeomorphisms $\Psi_i : X_{s_i} \cong X_t$ such that $\Psi_i(\xi(s_i)) = \xi(t)$. Suppose (s_i, E_i, β_i) are points in $\mathbf{R}^{J}_{\mathrm{Dol}}(X_{s_i}, \xi(s_i), n)$, converging to a point (t, E', β') in $\mathbf{R}^{J}_{\mathrm{Dol}}(X_t, \xi(s_i), n)$. The points lie over a compact set in $\mathbf{M}^{\mathrm{an}}_{\mathrm{Dol}}(X_t, \xi(s_i), n)$. Choose any subsequence. Apply Proposition 7.9 to obtain a harmonic bundle V over X_t and (after going to a further subsequence) a sequence of bundle isomorphisms $\eta_i : \Psi_{i,*}(E_i) \cong V$, such that the transported structures of harmonic bundle on E_i converge to the structure of harmonic bundle on V. Then the convergence statements of Lemma 7.13 hold for the operators d'' and ∇ giving the structures of vector bundle with integrable connection: the $\eta_{i,*}(d_i'')$ and $\eta_{i,*}(\nabla_i)$ converge to d'' and ∇ in the operator norm for operators from L^q_1 to L^q, and the frames $\eta_i(\beta_i)$ converge to a frame β. By Theorem 5.12, Part I, for the case Λ^{DR}, the points $(s_i, (E_i, d_i'', \nabla_i), \beta_i)$ converge in $\mathbf{R}^{J}_{\mathrm{DR}}(X/S, \xi, n)$ to the point $(t, (V, d'', \nabla), \beta)$. Similarly, $(s_i, (E_i, \overline{\partial}_i, \varphi_i), \beta_i)$ converge to $(t, (V, \overline{\partial}, \varphi), \beta)$ in $\mathbf{R}^{J}_{\mathrm{Dol}}(X/S, \xi, n)$. But this implies that $(t, (V, \overline{\partial}, \varphi), \beta) = (t, E', \beta')$, so $(t, (V, d'', \nabla), \beta)$ is the point in $\mathbf{R}^{J}_{\mathrm{DR}}(X/S, \xi, n)$ corresponding to (t, E', β').

We have shown that every subsequence has a further subsequence where the corresponding points converge to the correct limit. This proves that the sequence of points in $\mathbf{R}^{J}_{\mathrm{DR}}(X/S, \xi, n)$ corresponding to the original sequence of points (s_i, E_i, β_i) converges to the point corresponding to (t, E', β). Thus the map from the Dolbeault space to the de Rham space is continuous. The proof of continuity of the map from the de Rham space to the Dolbeault space is exactly the same. □

Note that the unitary group $U(n) = \mathrm{Aut}(\mathbf{C}^n, J)$ acts on the representation spaces, and preserves the subsets $\mathbf{R}^{J}_{\mathrm{Dol}}(X/S, \xi, n)$ and $\mathbf{R}^{J}_{\mathrm{DR}}(X/S, \xi, n)$. This action is continuous in the analytic topology.

Lemma **7.17.** — *The moduli spaces* $\mathbf{M}^{\mathrm{top}}_{\mathrm{Dol}}(X/S, \xi, n)$ *and* $\mathbf{M}^{\mathrm{top}}_{\mathrm{DR}}(X/S, \xi, n)$ *are the topological quotients of the representation spaces* $\mathbf{R}^{J}_{\mathrm{Dol}}(X/S, \xi, n)$ *and* $\mathbf{R}^{J}_{\mathrm{DR}}(X/S, \xi, n)$ *by the action of* $U(n)$.

Proof. — Let

$$N = \mathbf{R}^{J}_{\mathrm{Dol}}(X/S, \xi, n)/U(n)$$

denote the topological quotient. Since $U(n)$ is compact, N is separated (Hausdorff). Two points in $\mathbf{R}^{J}_{\mathrm{Dol}}(X/S, \xi, n)$ map to the same point in $\mathbf{M}_{\mathrm{Dol}}(X/S, \xi, n)$ if and only if the underlying harmonic bundles are isomorphic, thus if and only if the points are related by a unitary change of frame. Thus the map $f : N \to \mathbf{M}_{\mathrm{Dol}}(X/S, \xi, n)$ is one-to-one. The map from the representation space to the moduli space is continuous and proper, and since N is the topological quotient, this implies that the map f is continuous and proper. Therfore f is a homeomorphism identifying the moduli space with the quotient. The proof for the de Rham spaces is the same. □

We state our next corollary as a theorem.

Theorem **7.18.** — *The isomorphism of sets induced by the equivalence of categories given in* [Si5] *is a homeomorphism*

$$\mathbf{M}^{\mathrm{top}}_{\mathrm{Dol}}(X/S, n) \cong \mathbf{M}^{\mathrm{top}}_{\mathrm{DR}}(X/S, n)$$

of the topological spaces underlying the usual analytic spaces.

Proof. — It is easy to reduce to the case where $X \to S$ has connected fibers and admits a section. Then we may refer to the previous discussion. The moduli spaces are identified, in the previous lemma, as topological quotients of $\mathbf{R}^{J}_{\mathrm{Dol}}(X/S, \xi, n)$ and $\mathbf{R}^{J}_{\mathrm{DR}}(X/S, \xi, n)$. But Lemma 7.16 says that the identification between the representation spaces given by the equivalence of categories of [Si5] is a homeomorphism. This gives a homeomorphism between the quotients. □

Remark. — Combining this with Proposition 7.8, we obtain a homeomorphism

$$\mathbf{M}^{\mathrm{top}}_{\mathrm{Dol}}(X/S, n) \cong \mathbf{M}^{(\mathrm{top})}_{\mathrm{B}}(X/S, n)$$

where the right hand side denotes the topological space underlying $\mathbf{M}^{(\mathrm{an})}_{\mathrm{B}}(X/S, n)$.

Corollary **7.19.** — *If* X *is a smooth projective variety, then any representation of the fundamental group of* X *can be deformed to a representation which comes from a complex variation of Hodge structure.*

Proof. — By Corollary 6.12, any point in $\mathbf{M}_{\mathrm{Dol}}(X, n)$ can be deformed to a fixed point of the action of $\mathbf{C}^*$. These fixed points correspond to representations which come from complex variations of Hodge structure [Si5]. By the continuity result of the theorem, any connected component of $\mathbf{M}_{\mathrm{B}}(X, n)$ contains a point parametrizing a complex variation of Hodge structure. But the inverse image of a connected component in $\mathbf{M}_{\mathrm{B}}(X, n)$, is connected in $\mathbf{R}_{\mathrm{B}}(X, n)$, since $\mathbf{M}_{\mathrm{B}}(X, n)$ is a universal categorical quotient of $\mathbf{R}_{\mathrm{B}}(X, n)$ by a connected group. A point in the closed orbit over a fixed point of $\mathbf{C}^*$ comes from a complex variation of Hodge structure. Thus in any connected component of the space of representations, there is a representation which comes from a complex variation of Hodge structure. □

Remark. — This has topological consequences that were explained in [Si5]. The corresponding result is also true for principal bundles (cf. § 9 below).

Counterexample

We show that the isomorphism of sets $\mathbf{R}_{\mathrm{DR}}(X, x, n) \cong \mathbf{R}_{\mathrm{Dol}}(X, x, n)$ given by the equivalence of categories constructed in ([Si5] Lemma 3.11) is not, in general, continuous.

In fact, the isomorphism on the open subset of stable points (which is continuous), has no continuous extension over the whole representation space.

Let X be an elliptic curve, with nonvanishing differential dz. Let $E = \mathcal{O}_X \oplus \mathcal{O}_X$ be the trivial bundle of rank 2, with the canonical identification $\beta : E_x \cong \mathbf{C}^2$. Let

$$\theta_t = \begin{pmatrix} 0 & dz \\ 0 & at\, dz \end{pmatrix}.$$

Suppose t is real and approaches 0. Then the point (E, θ_t, β) approaches the point (E, θ_0, β). This limit is independent of the choice of a. However, we will see that the associated representations approach a limit that depends on a. Let $g_t = \begin{pmatrix} 1 & 1/at \\ 0 & 1 \end{pmatrix}$. Then

$$\theta'_t = g_t^{-1}\, \theta_t\, g_t = \begin{pmatrix} 0 & 0 \\ 0 & at\, dz \end{pmatrix}.$$

Now the metric for (E, θ'_t) is the usual constant metric, and the associated flat connection is given by the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & at\, dz + \overline{at}\, d\bar{z} \end{pmatrix}.$$

Thus the flat connection associated to (E, θ_t) is given by the conjugate of this matrix by g_t:

$$g_t \begin{pmatrix} 0 & 0 \\ 0 & at\, dz + \overline{at}\, d\bar{z} \end{pmatrix} g_t^{-1} = \begin{pmatrix} 0 & dz + \dfrac{\overline{at}}{at}\, d\bar{z} \\ 0 & at\, dz + \overline{at}\, d\bar{z}\, \overline{at}\, d\bar{z} \end{pmatrix}$$

(note that the entries of g_t are constant so there is no need to differentiate in conjugating the connection). Since we assumed that t was real, this connection matrix approaches

$$\begin{pmatrix} 0 & dz + \dfrac{\bar{a}}{a}\, d\bar{z} \\ 0 & 0 \end{pmatrix}$$

as $t \to 0$. The limit depends on $\arg(a)$. Thus the map between the space of representations and the space of (E, β) cannot be continuous.

It might still be the case that there is a homeomorphism between the topological quotient spaces $\mathbf{R}^{\mathrm{top}}_{\mathrm{DR}}(X/S, \xi, n)/\mathrm{Gl}(n)$ and $\mathbf{R}^{\mathrm{top}}_{\mathrm{Dol}}(X/S, \xi, n)/\mathrm{Gl}(n)$, which are non-Hausdorff spaces. Philosophically it would be important because of the interpretation of the topological quotient spaces as non abelian first cohomology spaces. This is an interesting problem for further study.

8. The Gauss-Manin connection

Suppose $f: X \to S$ is a smooth projective morphism. We have constructed the relative de Rham moduli spaces $\mathbf{M}_{DR}(X/S, n)$. On the other hand, the relative Betti space is in fact a local system of schemes $\mathbf{M}_B(X/S, n)$. The associated analytic total space $\mathbf{M}_B^{(an)}(X/S, n)$ has a *connection*, namely a compatible system of trivializations over artinian subspaces of S^{an}. The isomorphism of Theorem 7.1 gives a connection on $\mathbf{M}_{DR}^{an}(X/S, n)$. We will show that this comes from an algebraic connection on $\mathbf{M}_{DR}(X/S, n)$. We will call this connection the *Gauss-Manin connection* because it is the analogue for nonabelian cohomology of the usual Gauss-Manin connection on the relative abelian de Rham cohomology. For constructing the algebraic connection, we follow the ideas of Grothendieck's construction for the case of abelian cohomology.

Crystalline interpretation of integrable connections

The first step is to give an interpretation of vector bundles with integrable connection on X/S as *crystals*. The advantage of this is that if S' is an S-scheme which contains a closed subscheme S'_0 defined by a nilpotent ideal, and we set $X'_0 = X' \times_{S'} S'_0$, then a crystal on X'/S' is canonically the same thing as a crystal on X'_0/S'. The set of crystals on X'_0/S' depends only on the restricted map $S'_0 \to S$, so the functor $\mathbf{M}^{\natural}_{DR}(X/S, n)$ is itself a crystal on S. The resulting stratifications for $\mathbf{M}_{DR}(X/S, n)$ and $\mathbf{R}_{DR}(X/S, \xi, n)$ provide the Gauss-Manin connections on these schemes over S. This argument shows that the notion of a crystal can be useful in characteristic zero too. We will begin with an intermediate interpretation of vector bundles with connection on a smooth X/S, then proceed to describe what is meant by a crystal (in the present simple case).

The contents of this discussion are based on the ideas of Grothendieck [Gr3], by now well known. We present them here for the convenience of the reader, since most of the literature on crystals has concentrated on characteristic p. Our terminology may not be completely standard.

Suppose as usual, that X/S is smooth and projective. Denote by

$$(X \times_S X)^\wedge \quad \text{and} \quad (X \times_S X \times_S X)^\wedge$$

the formal neighborhoods of the diagonals in products of X. We have projections denoted p_i or p_{ij} in an obvious manner.

Lemma **8.1.** — *Suppose* E *is a vector bundle on* X. *Then an integrable connection* ∇ *is the same thing as an isomorphism* $\varphi : p_1^* E \xrightarrow{\cong} p_2^* E$ *on* $(X \times_S X)^\wedge$, *such that the restriction of* φ *to the diagonal is the identity, and*

$$(p_{23}^* \varphi)\,(p_{12}^* \varphi) = p_{13}^* \varphi$$

on $(X \times_S X \times_S X)^\wedge$.

Proof. — Given such an identification φ, we obtain a connection ∇ as follows. Let J denote the ideal of the diagonal in $X \times_S X$. If e is a section of E, then set

$$\nabla(e) = p_2^*(e) - \varphi(p_1^*(e)) \ (\bmod J^2).$$

It is an element of $p_2^* E \otimes (J/J^2)$, and considered as a module on the diagonal X, J/J^2 is (by definition) equal to the module of relative differentials $\Omega^1_{X/S}$. Note that $p_2^* E/J = E$ on the diagonal, so $\nabla(e)$ is an element of $E \otimes \Omega^1_{X/S}$. From the discussion below, it will be clear that ∇ is an integrable connection.

We would like to see that this construction gives a correspondence between φ and ∇. This statement does not depend on the fact that X is projective. We can cover X by open sets V which are finite étale covers of open sets U in affine space $\mathbf{A}^m_S$, and it suffices (by considering the direct image from V to U, and the $\mathcal{O}_V$-module structure over U) to verify the lemma for vector bundles on U. We may further assume that E is a trivial bundle, $E \cong \mathcal{O}^n_U$. The isomorphism φ is then given by a function $g(x,y)$ with values in $Gl(n)$, defined for $x, y \in U$ with y infinitesimally close to x (more precisely it is defined on the formal scheme $(U \times U)^\wedge$). The conditions on g are that $g(x, x) = 1$, and that $g(y, z)\, g(x,y) = g(x, z)$.

Given such a function g, we can write $g(x,y) = 1 + A(x)\,(x - y) + O((x - y)^2)$, where $A(x)$ is an $n \times n$ matrix-valued one-form on U. Then

$$\nabla(e)\,(x - y) = e(y) - g(x,y)\,e(x) = e(y) - e(x) - A(x)\,e(x)\,(x - y),$$

in other words $\nabla = d - A$. This shows that ∇ is a connection. The cocycle condition for g, taken when $(y - z)$ is a first order infinitesimal, becomes a differential equation:

$$g(x,y) + A(y)\,(y - z)\,g(x,y) = g(x, z)$$

$$g(x, z) - g(x,y) = A(y)\,g(x,y)\,(y - z)$$

or

$$d_y\, g(x,y) = A_y(y)\, g(x,y).$$

The subscripts indicate that the differentials are of the form dy. This equation uniquely determines g given the initial conditions $g(x, x) = 1$, so ∇ determines φ uniquely.

To complete the proof we have to show that φ or g exists if and only if ∇ is integrable. Note that if we can solve the equation $d_y\, g(x,y) = A_y(y)\, g(x, y)$ with initial conditions $g(x, x) = 1$, then the solution will satisfy the cocycle condition. This is because both $g(y, z)\, g(x,y)$ and $g(x, z)$ satisfy the same differential equation in z, and they are equal when $z = y$, so they are equal for all values of z.

Change variables by setting $t = y - x$. Set $A(x + t) = \Sigma_i A_i(x, t)\, dt_i$. This is a formal power series in t with coefficients which are regular functions of $x \in U$. The differential equation (really a system because there are several $t_1, \ldots, t_m$) becomes

$$\frac{\partial g(x, t)}{\partial t_i} = A_i(x, t)\, g(x, t).$$

This is an ordinary differential equation for $g(x, t)$ which is a formal power series in t with coefficients which are regular functions in $x \in U$. The initial conditions are $g(x, 0) = 1$. It has a solution if and only if it satisfies the integrability condition

$$\frac{\partial A_i}{\partial t_j} + A_i A_j = \frac{\partial A_j}{\partial t_i} + A_j A_i.$$

The solution may be constructed inductively to higher and higher order in t. This integrability condition is equivalent to $(d - A)^2 = 0$, so the function g exists if and only if ∇ is integrable. This completes the proof of the lemma. □

Crystals of schemes

Suppose that X is a scheme of finite type over S, not necessarily smooth. Define a category **Inf**(X/S) as follows. Its objects are pairs $(U \subset V)$ consisting of an X-scheme $U \to X$ and an S-scheme V, with an inclusion $U \hookrightarrow V$ over S, which makes U into a closed subscheme defined by a nilpotent sheaf of ideals (by this we mean a sheaf of ideals I such that $I^k = 0$ for some k). Such a nilpotent inclusion is sometimes referred to as an *infinitesimal thickening*. A morphism

$$f: (U \subset V) \to (U' \subset V')$$

consists of a morphism $f: V \to V'$ of S-schemes, such that the restriction $f: U \to U'$ is a morphism of X-schemes.

Let $\mathbf{Inf}^r$(X/S) denote the full subcategory of **Inf**(X/S) consisting of objects $(U \subset V)$ such that there exists a morphism $V \to X$ compatible with the map from U. This morphism is not, however, considered part of the data of $(U \subset V)$.

Remark. — If X/S is smooth, then any object of **Inf**(X/S) is, locally in the Zariski topology, isomorphic to an object of $\mathbf{Inf}^r$(X/S). This is because the infinitesimal lifting property for smooth morphisms guarantees the local existence of $V \to X$.

A *crystal of schemes* F on X/S is a specification, for each $(U \subset V)$ in **Inf**(X/S), of a V-scheme $F(U \subset V) \to V$; and for each morphism $f: (U \subset V) \to (U' \subset V')$, an isomorphism

$$\psi(f) : F(U \subset V) \xrightarrow{\cong} f^*(F(U' \subset V'));$$

such that $\psi(gf) = f^*(\psi(g))\, \psi(f)$. A *crystal of vector bundles*, or just *crystal* for short, is a crystal of schemes F with structures of vector bundles for $F(U \subset V)$, such that the $\psi(f)$ are bundle maps. Equivalently, it is a specification of locally free sheaves $F(U \subset V)$ on V, with isomorphisms of locally free sheaves $\psi(f) : F(U \subset V) \xrightarrow{\cong} f^* F(U' \subset V')$.

A *stratification of schemes* F on X/S is the same sort of thing as a crystal of schemes, but with $F(U \subset V)$ defined only for $(U \subset V)$ in the restricted category $\mathbf{Inf}^r$(X/S). Similarly for a *stratification of vector bundles*. According to the above remark, if X/S is smooth then stratifications are the same as crystals. In general, a crystal gives a stratification but not necessarily vice versa.

We will also use the following terminology. If $F \to X$ is a morphism of schemes, a *relative integrable connection* for F on X over S is a stratification of schemes with F as the value over X. The corresponding notion for vector bundles is the same as the usual notion of vector bundle with relative integrable connection. This follows from Lemmas 8.1 above and 8.2 below.

Suppose F and G are crystals or stratifications of schemes on X/S. A *morphism* $u : F \to G$ consists of a specification of morphisms of schemes $u : F(U \subset V) \to G(U \subset V)$, compatible with morphisms f in $\mathbf{Inf}(X/S)$ in the sense that $\psi(f)\, u = u\psi(f)$. A morphism of crystals or stratifications of vector bundles is the same, with the condition that the u should be morphisms of vector bundles, in other words linear.

Lemma **8.2.** — *A stratification of schemes on* X/S *is the same thing as a scheme* $F(X) \to X$, *together with an isomorphism*

$$\varphi : F(X) \times_S X|_{(X \times_S X)^\wedge} \xrightarrow{\cong} X \times_S F(X)|_{(X \times_S X)^\wedge}$$

satisfying a cocycle condition. This condition says that the two resulting isomorphisms $p_{23}^*(\varphi)\, p_{12}^*(\varphi)$ *and* $p_{13}^*(\varphi)$ *between the restrictions of* $F(X) \times_S X \times_S X$ *and* $X \times_S X \times_S F(X)$, *are equal. A stratification of vector bundles on* X/S *is the same as above but where* F(X) *has a structure of vector bundle over* X *and* φ *is an isomorphism of vector bundles.*

Proof. — Suppose F is a stratification of schemes on X. This gives a scheme F(X) over X. Let $(X \times_S X)^{(n)}$ denote the n-th infinitesimal neighborhood of the diagonal in $X \times_S X$, and similarly in triple products. These are objects in the category $\mathbf{Inf}^r(X/S)$. The maps $p_{1,(n)}$ and $p_{2,(n)}$ from $(X \times_S X)^{(n)}$ to X give, by definition, isomorphisms

$$F((X \times_S X)^{(n)}) \cong p_{1,(n)}^*(F(X))$$

and

$$F((X \times_S X)^{(n)}) \cong p_{2,(n)}^*(F(X)).$$

Composing these, we get isomorphisms

$$\varphi_n : p_{1,(n)}^*(F(X)) \cong p_{2,(n)}^*(F(X)).$$

Since the pullback isomorphisms defining the stratification F are functorial and satisfy an associativity, we have

$$\varphi_m|_{(X \times_S X)^{(n)}} = \varphi_n$$

for $n \leqslant m$, so these isomorphisms fit together into an isomorphism φ between the two pullbacks to the formal scheme $(X \times_S X)^\wedge$. This provides the desired φ. The associativity rule for the pullback maps implies that on $(X \times_S X \times_S X)^\wedge$ the two isomorphisms

$$p_1^*\, F(X) \cong F((X \times_S X \times_S X)^\wedge)$$

and

$$p_1^*\, F(X) \cong p_{12}^*\, F((X \times_S X)^\wedge) \cong F((X \times_S X \times_S X)^\wedge)$$

are equal. Similarly in other combinations. Thus, all of the resulting isomorphisms between $p_1^* F(X)$ and $p_3^* F(X)$, are equal. This provides the cocycle condition.

Suppose given, on the other hand, an isomorphism φ satisfying a cocycle condition as described in the hypotheses. For every object $U \subset V$ in the category $\mathbf{Inf}^r(X/S)$, *choose* a map $i_V : V \to X$ compatible with the map $U \to X$. Define

$$F(U \subset V) = i_V^*(F(X)).$$

Suppose $f: V \to V'$. Then $i_{V'} f|_U$ is equal to $i_V|_U$, although they may not be equal on V. Since $U \subset V$ is defined by a nilpotent ideal, the pair $(i_V, i_{V'} f)$ maps V into $(X \times_S X)^\wedge$. Note that

$$(i_V, i_{V'} f)^* p_1^* F(X) = F(V).$$

while

$$(i_V, i_{V'} f)^* p_2^* F(X) = f^* F(V').$$

Our hypothesis gives $\varphi : p_1^* F(X) \cong p_2^* F(X)$. Thus we may define $\varphi(f) = (i_V, i_{V'} f)^* (\varphi)$, to obtain

$$\varphi(f) : F(V) \cong f^* F(V').$$

Given $f: V \to V'$ and $g : V' \to V''$, we obtain a map

$$(i_V, i_{V'} f, i_{V''} gf) : V \to (X \times_S X \times_S X)^\wedge.$$

The cocycle condition for φ implies that the two possible maps $i_V^* F(X) \cong (gf)^* i_{V''}^* F(X)$ are equal. In other words, $f^*(\varphi(g))\, \varphi(f) = \varphi(gf)$. This shows that we have defined a stratification of schemes. These two constructions are essential inverses, so we get an equivalence of categories. □

Corollary **8.3.** — *Suppose* X/S *is smooth. A vector bundle with integrable connection on* X/S *is the same thing as a crystal of vector bundles on* X/S.

Proof. — This follows immediately from the previous two lemmas, and the contention that crystals and stratifications are the same if X/S is smooth. This contention follows from the remark several paragraphs ago, that any object of $\mathbf{Inf}(X/S)$ is locally in $\mathbf{Inf}^r(X/S)$. In order to define $F(U \subset V)$ for $(U \subset V) \in \mathbf{Inf}(X/S)$, cover V by Zariski open sets V_α which are in $\mathbf{Inf}^r(X/S)$. Then use the isomorphisms which are provided on overlaps $V_{\alpha\beta}$, to glue together the objects $F(V_\alpha)$, forming F(V). □

Remark. — Suppose Z/S is another S-scheme, and $j : Z \to X$ is a morphism of S-schemes. We obtain a functor $j : \mathbf{Inf}(Z/S) \to \mathbf{Inf}(X/S)$ in an obvious way. In fact, $\mathbf{Inf}(Z/S)$ is a subcategory of $\mathbf{Inf}(X/S)$. If F is a crystal of schemes or vector bundles on X, then the restriction is a crystal of schemes or vector bundles $j^* F$ on Z/S. The equivalences of categories given by the preceding lemmas and corollary are compatible with pullbacks.

The following proposition was Grothendieck's main observation.

Proposition **8.4.** — *Suppose* $S_0 \subset S$ *is a closed subscheme defined by a nilpotent sheaf of ideals. Suppose* X *is an* S*-scheme. Let* $X_0 = X \times_S S_0$, *still considered as an* S*-scheme. Let* $j : X_0 \to X$ *denote the inclusion. Then the pullback functor* $F \mapsto j^* F$ *is an equivalence from the category of crystals of schemes on* X/S *to the category of crystals of schemes on* X_0/S. *The same is true for crystals of vector bundles.*

Proof. — We have a functor $a : \mathbf{Inf}(X_0/S) \to \mathbf{Inf}(X/S)$ defined by

$$a(U \subset V) = (U \subset V),$$

and a functor $b : \mathbf{Inf}(X/S) \to \mathbf{Inf}(X_0/S)$ defined by $b(U \subset V) = (U_0 \subset V)$. The composition ba is equal to the identity. On the other hand, if $(U \subset V) \in \mathbf{Inf}(X/S)$ then there is a natural map $(U_0 \subset V) \to (U \subset V)$, so we get a natural morphism $ab \to I$. The functors a and b (and this natural morphism) preserve the schemes V. We obtain functors a^*, from the category of crystals of schemes on X/S to the category of crystals of schemes on X_0/S, and b^*, from the category of crystals of schemes on X_0/S to the category of crystals of schemes on X/S. We have $a^* b^* = I$, and there is a natural morphism from $b^* a^*$ to the identity. Note that $(b^* a^* F)(U \subset V) = F(U_0 \subset V)$. The natural morphism is given by the pullback (using $j : (U_0 \subset V) \to (U \subset V)$), $\varphi(j) : (F(U_0 \subset V) \to F(U \subset V)$. But $\varphi(j)$ is an isomorphism of schemes. A morphism of crystals of schemes which is an isomorphism over each element of $\mathbf{Inf}(X/S)$, is an isomorphism of crystals of schemes—the inverse will also be a morphism. Hence $b^* a^* F \cong F$. Thus a and b give an equivalence of categories. □

Representability

One can define, in exactly the same way as before, the notions of *crystal of functors* or *stratification of functors*. These mean that for any object $(U \subset V)$, $F(U \subset V)$ is a functor of schemes $Y \to V$. The set of such functors forms a pre-stack. In fact, given any stack or pre-stack $\mathscr{C}$ over the category of schemes, one can define a notion of crystal of $\mathscr{C}$-objects. The above lemmas, done for the stacks of schemes or vector bundles, remain valid. (Any comments about glueing are valid only for stacks, not pre-stacks.)

Lemma **8.5.** — *Suppose* $F^\natural$ *is a stratification of functors on* X/S. *Suppose* $F^\natural(X)$ *is represented or universally co-represented by a scheme* F(X). *Then we obtain a stratification of schemes* F, *such that for any* $V \in \mathbf{Inf}^r(X/S)$, F(V) *represents or universally co-represents* $F^\natural(V)$.

Proof. — Use the characterization of Lemma 8.2. Note that $p_1^* F(X)$ represents or universally co-represents the functor $p_1^* F^\natural(X)$, and so forth. Hence the isomorphism φ of functors on $(X \times_S X)^\wedge$ translates into an isomorphism between the pullback schemes. The cocycle condition for the isomorphisms of functors implies the cocycle condition for isomorphisms of schemes. □

Remark. — It is in this lemma that we are forced to go from crystals to stratifications.

The Gauss-Manin connections

Suppose S is a scheme over **C**, and X/S is a smooth projective family. Suppose $\xi : S \to X$ is a section. Define crystals of functors $\mathbf{M}^{\natural}_{\rm crys}$ and $\mathbf{R}^{\natural}_{\rm crys}$ on $S/\mathbf{C}$ as follows. For $(S'_0 \subset S')$ in $\mathbf{Inf}(S/\mathbf{C})$, define $\mathbf{M}^{\natural}_{\rm crys}(S'_0 \subset S')$ to be equal to the set of isomorphism classes of crystals of vector bundles of rank n, on X'_0/S'. Define $\mathbf{R}^{\natural}_{\rm crys}(S'_0 \subset S')$ to be equal to the set of isomorphism classes of pairs (E, β) where E is a crystal of vector bundles of rank n, on X'_0/S', and

$$\beta : E\big|_{x'_0/S'} \cong \mathbf{1}^n.$$

Here $\mathbf{1}$ is the trivial crystal on $x'_0/S' \cong S'_0/S'$. These crystals of functors restrict to stratifications of functors. By Lemmas 8.1 and 8.2, we have $\mathbf{M}^{\natural}_{\rm crys} \cong \mathbf{M}^{\natural}_{\rm DR}(X/S, n)$, and $\mathbf{R}^{\natural}_{\rm crys} \cong \mathbf{R}^{\natural}_{\rm DR}(X/S, \xi, n)$. The first is universally co-represented by $\mathbf{M}_{\rm DR}(X/S, n)$, and the second is represented by $\mathbf{R}_{\rm DR}(X/S, \xi, n)$. By the previous lemma, we obtain stratifications of schemes $\mathbf{M}_{\rm strat}(X/S, n)$ and $\mathbf{R}_{\rm strat}(X/S, \xi, n)$ on the stratifying site $\mathbf{Inf}^r(S/\mathbf{C})$. By Lemma 8.2, these data are equivalent to the data of isomorphisms

$$\varphi : p_1^* \mathbf{M}_{\rm DR}(X/S, n) \cong p_2^* \mathbf{M}_{\rm DR}(X/S, n)$$

and

$$\varphi : p_1^* \mathbf{R}_{\rm DR}(X/S, \xi, n) \cong p_2^* \mathbf{R}_{\rm DR}(X/S, \xi, n)$$

on $(S \times_{\mathbf{C}} S)^\wedge$, satisfying the cocycle condition on $(S \times_{\mathbf{C}} S \times_{\mathbf{C}} S)^\wedge$. These are the *Gauss-Manin connections.*

We can make the same definitions as above for the category of complex analytic spaces. The algebraic connections induce analytic connections on $\mathbf{R}^{\rm an}_{\rm DR}(X/S, \xi, n)$ and $\mathbf{M}^{\rm an}_{\rm DR}(X/S, n)$. We would like to show that these agree with the connections coming from the Betti realizations.

Recall that the Betti objects $\mathbf{R}_{\rm B}(X/S, \xi, n)$ and $\mathbf{M}_{\rm B}(X/S, n)$ are local systems of schemes over $S^{\rm an}$. The associated spaces $\mathbf{R}^{\rm (an)}_{\rm B}(X/S, \xi, n)$ and $\mathbf{M}^{\rm (an)}_{\rm B}(X/S, n)$ are, by definition, products locally over S. In other words, if $s \in S$ then there exists a neighborhood U of s such that (with the subscript U denoting the inverse image of U)

$$\mathbf{R}^{\rm (an)}_{\rm B}(X/S, \xi, n)_U = U \times \mathbf{R}^{\rm an}_{\rm B}(X_s, \xi(s), n)$$

and

$$\mathbf{M}^{\rm (an)}_{\rm B}(X/S, n)_U = U \times \mathbf{M}^{\rm an}_{\rm B}(X_s, n).$$

A product space of the form $U \times Z$ has an analytic relative integrable connection, given by the natural equalities of objects over $U \times U$

$$p_1^*(U \times Z) = p_2^*(U \times Z) = U \times U \times Z.$$

Thus $\mathbf{R}^{\rm (an)}_{\rm B}(X/S, \xi, n)_U$ and $\mathbf{M}^{\rm (an)}_{\rm B}(X/S, n)_U$ have analytic relative integrable connections. The local product structures over open sets U and V agree over connected components of $U \cap V$, so the connections agree over $U \cap V$. These then glue together to give analytic relative integrable connections on $\mathbf{R}^{\rm (an)}_{\rm B}(X/S, \xi, n)$ and $\mathbf{M}^{\rm (an)}_{\rm B}(X/S, n)$.

Theorem **8.6.** — *The isomorphisms*

$$\mathbf{R}_{\mathrm{B}}^{(\mathrm{an})}(\mathrm{X}/\mathrm{S}, \xi, n) \cong \mathbf{R}_{\mathrm{DR}}^{\mathrm{an}}(\mathrm{X}/\mathrm{S}, \xi, n), \qquad \mathbf{M}_{\mathrm{B}}^{(\mathrm{an})}(\mathrm{X}/\mathrm{S}, n) \cong \mathbf{M}_{\mathrm{DR}}^{\mathrm{an}}(\mathrm{X}/\mathrm{S}, n)$$

identify the connections coming from the locally constant structure of the Betti objects, with the Gauss-Manin connections constructed above for the de Rham objects.

Proof. — It suffices to treat the case where $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ with A an artinian local **C**-algebra of finite type, and X/S is smooth, connected and has a section ξ. Let $s \in \mathrm{S}$ denote the closed point. The Gauss-Manin connections are equivalent to trivializations

$$\mathbf{R}_{\mathrm{DR}}(\mathrm{X}/\mathrm{S}, \xi, n) \cong \mathrm{S} \times \mathbf{R}_{\mathrm{DR}}(\mathrm{X}_s, \xi(s), n)$$

and

$$\mathbf{M}_{\mathrm{DR}}(\mathrm{X}/\mathrm{S}, n) \cong \mathrm{S} \times \mathbf{M}_{\mathrm{DR}}(\mathrm{X}_s, n).$$

In order to show that the associated trivializations of analytic spaces agree with the trivializations

$$\mathbf{R}_{\mathrm{B}}^{(\mathrm{an})}(\mathrm{X}/\mathrm{S}, \xi, n) = \mathrm{S} \times \mathbf{R}_{\mathrm{B}}^{\mathrm{an}}(\mathrm{X}_s, \xi(s), n)$$

and

$$\mathbf{M}_{\mathrm{B}}^{(\mathrm{an})}(\mathrm{X}/\mathrm{S}, n) = \mathrm{S} \times \mathbf{M}_{\mathrm{B}}^{\mathrm{an}}(\mathrm{X}_s, n),$$

it suffices to treat the cases of the representation spaces, since the maps $\mathbf{R}(\mathrm{X}/\mathrm{S}, \xi, n) \to \mathbf{M}(\mathrm{X}/\mathrm{S}, n)$ are universally submersive.

For the representation spaces, it suffices to show that if $f : \mathrm{S}' \to \mathbf{R}_{\mathrm{DR}}(\mathrm{X}/\mathrm{S}, \xi, n)$ is a point with values in an artinian scheme $\mathrm{S}' = \mathrm{Spec}(\mathrm{A}')$, which has constant projection on the second factor in the above product decomposition, then the resulting monodromy representation

$$\pi_1(\mathrm{X}_s, \xi(s)) \to \mathrm{Gl}(n, \mathrm{A}')$$

takes values in $\mathrm{Gl}(n, \mathbf{C})$. The point f corresponds to a vector bundle with integrable relative connection (E, ∇) on $\mathrm{X}' = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}'$, and frame $\beta : \mathrm{E}|_{\xi(\mathrm{S}')} \cong \mathcal{O}_{\mathrm{S}'}^n$. The fact that the projection on the second factor of the product decomposition given by the stratification is trivial, implies that there is an open set $\mathrm{U} \subset \mathrm{X}'$ (containing the image of ξ) and trivializations $\tau : \mathrm{U} \cong \mathrm{S}' \times \mathrm{U}_s$ with $\tau(\xi(\mathrm{S}')) = \mathrm{S}' \times \xi(s)$ and

$$(\mathrm{E}, \nabla, \beta)|_{\mathrm{U}} \cong (p_2 \tau)^*((\mathrm{E}_s, \nabla_s, \beta_s)|_{\mathrm{U}_s}).$$

The local system of relatively constant sections of $(p_2 \tau)^*((\mathrm{E}_s, \nabla_s)|_{\mathrm{U}_s})$ is just the tensor product of the local system of constant sections of $(\mathrm{E}_s, \nabla_s)|_{\mathrm{U}_s}$ with A′. Thus the monodromy representation of $(p_2 \tau)^*((\mathrm{E}_s, \nabla_s, \beta_s)|_{\mathrm{U}_s})$ takes values

$$\pi_1(\mathrm{U}_s, \xi(s)) \to \mathrm{Gl}(n, \mathbf{C}) \subset \mathrm{Gl}(n, \mathrm{A}').$$

Note that the map on fundamental groups is a surjection

$$\pi_1(\mathrm{U}_s, \xi(s)) \to \pi_1(\mathrm{X}_s, \xi(s)) \to 1.$$

The trivialization of $(\mathrm{E}, \nabla, \beta)|_{\mathrm{U}}$ implies that the monodromy representation takes values in $\mathrm{Gl}(n, \mathbf{C})$. □

Remark. — The above proof gives the following criterion: an artinian scheme-valued point f: $S' \to \mathbf{R}_{DR}(X/S, \xi, n)$ has constant projection on the second factor $\mathbf{R}_{DR}(X_s, \xi(s), n)$ if and only if there exists an open set $U \subset X'$ (containing the image of ξ) and trivializations $\tau : U \cong S' \times U_s$ with $\tau(\xi(S')) = S' \times \xi(s)$ and

$$(E, \nabla, \beta)|_U \cong (p_2 \tau)^*((E_s, \nabla_s, \beta_s)|_{U_s}).$$

For, if such an open set and trivializations exist, then the monodromy representation takes values in $Gl(n, \mathbf{C})$. Thus the point f has constant projection on the second factor for the stratification of the Betti spaces. But since the de Rham and Betti spaces are analytically isomorphic, and this isomorphism is compatible with the stratifications, the point f has constant projection on the second factor for the stratification of $\mathbf{R}^{an}_{DR}(X/S, \xi, n)$. Hence it has constant projection for the algebraic stratification.

9. Principal objects

Suppose X is a scheme of finite type over **C**. In what follows, we will use the term *tensor category* to denote an associative commutative **C**-linear tensor category with unit object. A *tensor functor* is a functor together with natural isomorphisms of preservation of the tensor product, compatible with the associative and commutative structures [Sa] [DM].

Suppose G is a complex linear algebraic group. Let Rep(G) denote the tensor category of complex linear representations of G. Let Vect(X) denote the tensor category of vector bundles (considered as locally free sheaves) over X. A morphism $u : E \to F$ of objects in Vect(X) is *strict* if coker(u) is a locally free sheaf. In this case, the kernel and image of u are locally free sheaves.

A *principal right* G-*bundle* over X is a morphism $P \to X$ together with a right action of G on P such that there exists a surjective étale morphism $f : X' \to X$ and a G-equivariant isomorphism

$$P \times_X X' \cong X' \times_{\mathrm{Spec}(\mathbf{C})} G.$$

If P is a principal right G-bundle over X, let $P \times^G V$ be the locally free sheaf in the Zariski topology obtained by descending the sheaf

$$Y \in X^{et} \mapsto \frac{\{(p, v) \in P(Y) \times (V \otimes \mathcal{O}_{X^{et}}(Y))\}}{(pg, v) \sim (p, gv) \text{ for } g \in G(Y)}$$

from the étale topology X^{et} to the Zariski topology. We obtain a functor

$$\rho_P : \mathrm{Rep}(G) \to \mathrm{Vect}(X)$$

by setting $\rho_P(V) = P \times^G V$. This has the following properties: that ρ_P is *strict*, in other words if $u : V \to W$ is a morphism in Rep(G) then $\rho_P(u)$ is a strict morphism in Vect(X);

that ρ_P is *exact*, that is $\rho_P(\ker(u)) = \ker(\rho_P(u))$ and $\rho_P(\operatorname{coker}(u)) = \operatorname{coker}(\rho_P(u))$; and that ρ_P is faithful. Furthermore, for any closed point $x \in X$ the functor $V \mapsto \rho_P(V)_x$ is a fiber functor [Sa] [DM]. Nori has proved the following converse:

Proposition **9.1.** — *Suppose* $\rho : \mathrm{Rep}(G) \to \mathrm{Vect}(X)$ *is a strict exact and faithful tensor functor. Then there exists a principal right* G*-bundle* P *over* X *and an isomorphism of tensor functors* $\rho \cong \rho_P$; *and* P *is unique up to unique isomorphism.*

Proof. — [No]. □

Principal Higgs bundles

Suppose $X \to S$ is a smooth projective morphism to a scheme of finite type over **C**. Let **g** denote the Lie algebra of G with G acting by the adjoint representation. A *principal Higgs bundle* on X over S, for the group G, is a principal right G-bundle $P \to X$ together with a section θ of $(P \times^G \mathbf{g}) \otimes \Omega^1_{X/S}$ such that $[\theta, \theta] = 0$ in $(P \times^G \mathbf{g}) \otimes \Omega^2_{X/S}$. This is the relative version of one of the definitions given in [Si5]. Given such an object and a representation V of G, we get a relative Higgs bundle $\rho_P(V) = P \times^G V$. Say that P is of *semiharmonic type* if the Chern classes of the restrictions of P to fibers X_s are zero in rational cohomology, and if there exists a faithful representation V such that $\rho_P(V)$ restricts to semistable Higgs bundles on the fibers. In this case, the same is true for any other representation (cf. [Si5], remarks after Lemma 6.13). The category of semistable Higgs bundles with vanishing Chern classes (Higgs bundles of semiharmonic type) has a natural structure of tensor category—the tensor product of two semistable Higgs bundles is again semistable [Si5].

Lemma **9.2.** — *The construction* $P \mapsto \rho_P$ *provides an equivalence between the categories of principal Higgs bundles of semiharmonic type for the group* G, *and strict exact faithful tensor functors* ρ *from* Rep(G) *to the category of Higgs bundles of semiharmonic type on* X *over* S.

Proof. — This follows from the previous proposition—see [Si5], remarks after Lemma 6.13. □

Lemma **9.3.** — *Suppose* E *is a Higgs bundle of semiharmonic type on* X *over* S. *Fix a number* k. *There is a projective* S*-scheme* $N(E, k) \to S$ *representing the functor which associates to each* S*-scheme* $f : S' \to S$ *the set of quotient Higgs bundles* $f^*(E) \to F \to 0$ *of rank* k *such that the Chern classes of* F *vanish on fibers of* $X' \to S'$ *(note that any such* F *is a semistable Higgs bundle on* X′ *over* S′, *hence of semiharmonic type). Suppose that the fibers* X_s *are connected, and* $\xi : S \to X$ *is a section. Then the morphism* $N(E, k) \to \mathbf{Grass}_S(\xi^*(E), k)$ *is a closed embedding.*

Proof. — Let p_0 denote the Hilbert polynomial of $\mathcal{O}_X$ over S. Let $\mathbf{Hilb}(E, kp_0)$ denote the Hilbert scheme parametrizing quotient sheaves $E \to F \to 0$ flat over S, with Hilbert polynomial kp_0. Denote the kernel by $O \to K \to E \to F \to 0$. Let

$N(E, k) \subset \mathbf{Hilb}^0(E, kp_0)$ denote the closed subscheme representing the condition that the map $\theta : K \to F \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ is zero (see the first paragraph of the proof of Theorem 3.8, Part I). The points of $N(E, k)$ with values in $f: S' \to S$ correspond to quotient Higgs sheaves $f^*(E) \to F \to 0$ on X' over S', such that F is flat over S' with Hilbert polynomial P. If F is such a quotient, then for any $s \in S'$ the fiber $F_s = F|_{X'_s}$ is a quotient Higgs sheaf of $E_s = E|_{X_s}$ with normalized Hilbert polynomial equal to that of E_s. Let K_s denote the kernel of $E_s \to F_s$. Then K_s is a sub-Higgs sheaf of E_s with the same normalized Hilbert polynomial. By Proposition 6.6, K_s is a strict subbundle with vanishing Chern classes, hence F_s is locally free and has vanishing Chern classes. By Lemma 1.27, Part I, this implies that F is locally free. Thus the points of $N(E, k)$ correspond to quotient Higgs bundles F which are locally free of rank k and have Chern classes restricting to zero on the fibers. This is the desired parametrizing space. Note that $\mathbf{Hilb}(E, kp_0)$ is projective over S and $N(E, k)$ is a closed subset, hence it is also projective.

Suppose X has connected fibers over S and $\xi : S \to X$ is a section. Associating, for each quotient $E \to F \to 0$, the quotient vector bundle $\xi^*(E) \to \xi^*(F) \to 0$, gives a morphism $N(E, k) \to \mathbf{Grass}(\xi^*(E), k)$. It is proper, since $N(E, k)$ is proper over S. Suppose F_1 and F_2 the quotients given by points of $N(E, k)$ (S), such that $\xi^*(F_1) = \xi^*(F_2)$ as quotients of $\xi^*(E)$. Let K_1 denote the kernel of $E \to F_1$. Then K_1 and F_2 are Higgs bundles on X over S with the same normalized Hilbert polynomials. By Proposition 6.6, they both satisfy condition LF(X). The morphism $\psi : K_1 \to F_2$ has $\xi^*(\psi) = 0$, so by Lemma 4.9, Part I, for the case of Λ^{Higgs}, $\psi = 0$. Thus F_2 is a quotient of F_1; similarly in the other direction, F_1 is a quotient of F_2 so $F_1 = F_2$. This shows that the map $N(E, k)$ (S) $\to \mathbf{Grass}(\xi^*(E), k)$ (S) is injective. The same is true for points with values in any S-scheme S'. A morphism which is proper and injective on the level of points is a closed embedding. □

Suppose that the fibers X_s are connected, and $\xi : S \to X$ is a section. Suppose $G \subset H$ is a subgroup. Suppose P is a principal Higgs bundle for the group H which is semistable with vanishing Chern classes, on X over S. Suppose $b : S \to \xi^*(P)$ is an S-valued point. We say that *the monodromy of* (P, b) *is contained in* G if the following condition holds: for every linear representation V of H, and every subspace $W \subset V$ preserved by G, there exists a strict sub-Higgs bundle of semiharmonic type

$$F \subset P \times^H V$$

such that

$$\xi^*(F) = \{ b \} \times W \subset \xi^*(P \times^H V).$$

If S is a point, we define the *monodromy group* Mono(P, b) to be the intersection of all algebraic subgroups $G \subset H$ such that the monodromy of (P, b) is contained in G. Note that the monodromy group jumps *down* under specialization.

Lemma **9.4.** — *Suppose* $G \subset H$. *Suppose* P' *is a principal Higgs bundle of semiharmonic type on* X *over* S, *for the group* G. *Then the principal Higgs bundle* $P = P' \times^G H$ *obtained by* *extending the structure group to* H *is also of semiharmonic type. This construction gives an identification between:* (1) *the set of isomorphism classes of pairs* (P', b') *where* P' *is a principal Higgs bundle of semiharmonic type for the group* G *and* b' *is an* S*-valued point of* $\xi^*(P')$; *and* (2) *the set of isomorphism classes of pairs* (P, b) *where* P *is a principal Higgs bundle of semiharmonic type for the group* H *and* b *is an* S*-valued point of* $\xi^*(P')$, *such that the monodromy of* (P, b) *is contained in* G.

Proof. — The Chern classes of P are induced by those of P′, hence they vanish. To check semistability of P, choose a faithful representation V of H. This restricts to a faithful representation of G, and we have $P \times^H V = P' \times^G V$. By the assumption of semistability of P′, this is semistable, so P is semistable. Our construction gives a functor from the category of objects (1) to the category of objects (2). To go in the opposite direction, let Rep(G, H) denote the category whose objects are pairs (V, W) where W is a representation of H and V is a G-invariant subspace; and whose morphisms are the G-equivariant morphisms between the subspaces V. Forgetting W gives an equivalence of categories $\mathrm{Rep}(G, H) \xrightarrow{\cong} \mathrm{Rep}(G)$. On the other hand, suppose we have a principal H-bundle P with a point $b \in \xi^*(P)(S)$, such that the monodromy is contained in G. By definition, for any $(V, W) \in \mathrm{Rep}(G, H)$ there is a unique sub-Higgs bundle $F(V, W) \subset P \times^H W$ of semiharmonic type with $\xi^*(F(V, W)) = \{ b \} \times V$. Given (V, W) and (V′, W′) and a G-equivariant morphism $f: V \to V'$ we obtain a G-invariant subspace $L \subset W \oplus W'$ giving the graph of the map f. The hypothesis of monodromy in G implies that there exists a sub-Higgs bundle $L(f) \subset F(V, W) \oplus F(V', W')$ which restricts to L on the section ξ. This gives the graph of a morphism $F(V, W) \to F(V', W')$ restricting to f over the section ξ (and the morphism is unique by Lemma 4.9, Part I). We obtain a functor from Rep(G, H) to the category of Higgs bundles of semiharmonic type on X over S, commuting with the functor of taking the fiber along ξ. Composing with the inverse of the above equivalence of categories gives a functor from Rep(G). This has a natural structure of neutral tensor functor (one can define a tensor operation $(V_1, W_1) \otimes (V_2, W_2) = V_1 \otimes V_2, W_1 \otimes W_2)$ on Rep(G, H) as an intermediate in the definition of the tensor structure). By Lemma 9.2, this gives a principal G-bundle P′ as desired. □

Lemma **9.5.** — *Suppose* E *is a Higgs bundle of semiharmonic type, of rank* n *on* X *over* S. *Then the frame bundle* P *of* E *has a natural structure of principal Higgs bundle of semiharmonic type for the group* $\mathrm{Gl}(n, \mathbf{C})$ *on* X *over* S. *The Higgs bundle is recovered as* $E = P \times^{\mathrm{Gl}(n, \mathbf{C})} \mathbf{C}^n$. *This construction provides an identification between the sets of isomorphism classes of* (E, β) *and* (P, b).

Proof. — Define a category $\mathrm{Rep}(\mathrm{Gl}(n, \mathbf{C}), \mathrm{std})$ whose objects are pairs $(V, T^{a,b}(\mathbf{C}^n))$ where the second element refers to the tensor product $(\mathbf{C}^n)^{\otimes a} \otimes ((\mathbf{C}^n)^*)^{\otimes b}$, and $V \subset T^{a,b}(\mathbf{C}^n)$ is a $\mathrm{Gl}(n, \mathbf{C})$-invariant subspace. The morphisms are equivariant morphisms of the subspaces V. This category is equivalent to $\mathrm{Rep}(\mathrm{Gl}(n, \mathbf{C}))$ (and it even has a tensor operation compatible with the tensor product on $\mathrm{Rep}(\mathrm{Gl}(n, \mathbf{C}))$). Suppose

$V \subset T^{a,b}(\mathbf{C}^n)$ is a $\mathrm{Gl}(n, \mathbf{C})$-invariant subspace. Then for any n-dimensional vector space U we obtain a subspace $V \subset T^{a,b}(U)$ which does not depend on the choice of basis. The same construction holds for vector bundles, so we get a subbundle $F \subset T^{a,b}(E)$ with $\beta(F) = V$. The construction of V is also compatible with infinitesimal automorphisms, so the subbundle F is preserved by θ. There is a complementary subspace $V^{\perp}$, and a corresponding complementary subbundle $F^{\perp}$. The tensor product $T^{a,b}(E)$ is also of semiharmonic type [Si5], so any direct factor such as F is of semiharmonic type. Morphisms of representations V give rise to morphisms of the Higgs bundles F, and it is compatible with tensor product, so we obtain a tensor functor from $\mathrm{Rep}(\mathrm{Gl}(n, \mathbf{C}), \mathrm{std})$ to the category of Higgs bundles of harmonic type on X. Lemma 9.2 gives the desired principal bundle P. □

Suppose $G \subset \mathrm{Gl}(n, \mathbf{C})$. Suppose E is a Higgs bundle of semiharmonic type on X over S, of rank n, and suppose $\beta : \xi^*(E) \cong \mathcal{O}_S^n$. Let P denote the frame bundle of E, and b the point corresponding to β. We say that *the monodromy of* (E, β) *is contained in* G if the monodromy of (P, b) is contained in G in the sense defined above. If S is a point, the *monodromy group* $\mathrm{Mono}(E, \beta)$ is again the intersection of all subgroups $G \subset \mathrm{Gl}(n, \mathbf{C})$ such that the monodromy of (E, β) is contained in G.

Theorem **9.6.** — *Suppose* $\xi : S \to X$ *is a section. There is a scheme* $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G)$ *over* S *representing the functor which associates to any* S*-scheme* S′ *the set of pairs* (P, b) *where* P *is a principal Higgs bundle for the group* G *on* $X' = X \times_S S'$ *over* S′, *semistable with vanishing Chern classes, and* $b : S' \to \xi^*(P)$ *is a section over* ξ. *If* $f : G \hookrightarrow H$ *is a closed embedding, then* f *induces a closed embedding* $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G) \hookrightarrow \mathbf{R}_{\mathrm{Dol}}(X/S, \xi, H)$.

Proof. — By Lemma 9.5,

$$\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, \mathrm{Gl}(n, \mathbf{C})) \stackrel{\mathrm{def}}{=} \mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n)$$

does the job for the group $\mathrm{Gl}(n, \mathbf{C})$.

Suppose now that the existence of $\mathbf{R}_{\mathrm{Dol}}(H) = \mathbf{R}_{\mathrm{Dol}}(X/S, \xi, H)$ is known, and that $G \subset H$ is an algebraic subgroup. Suppose that V is a representation of H and W is a subspace preserved by G. Let $(P^{\mathrm{univ}}, b^{\mathrm{univ}})$ denote the universal principal object on $X \times_S \mathbf{R}_{\mathrm{Dol}}(H)$, and let $E^{\mathrm{univ}} = P^{\mathrm{univ}} \times^H V$ denote the universal Higgs bundle associated to the representation V. Let $\mathscr{V} = V \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{R}_{\mathrm{Dol}}(H)}$ and let $\beta^{\mathrm{univ}} : \xi^*(E^{\mathrm{univ}}) \cong \mathscr{V}$ denote the frame given by the point b^{univ}. Let $k = \dim(V) - \dim(W)$, let $\mathscr{W} = W \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{R}_{\mathrm{Dol}}(H)}$ denote the corresponding subobject of $\mathscr{V}$, and let

$$\sigma_{\mathscr{V}/\mathscr{W}} : \mathbf{R}_{\mathrm{Dol}}(H) \to \mathbf{Grass}_{\mathbf{R}_{\mathrm{Dol}}(H)}(\mathscr{V}, k)$$

denote the section corresponding to the quotient $\mathscr{V}/\mathscr{W}$. Let $N(E^{\mathrm{univ}}, k) \subset \mathbf{Grass}_{\mathbf{R}_{\mathrm{Dol}}(H)}(\mathscr{V}, k)$ denote the closed subscheme given by Lemma 9.3 and transported by the frame β^{univ}. Define the closed subscheme

$$C(V, W) \stackrel{\mathrm{def}}{=} \sigma_{\mathscr{V}/\mathscr{W}}^{-1}(N(E^{\mathrm{univ}}, k)) \subset \mathbf{R}_{\mathrm{Dol}}(H).$$

By Lemma 9.3, this subscheme represents the condition on points $g : S' \to \mathbf{R}_{\mathrm{Dol}}(H)$, that there exists a quotient Higgs bundle F′ of harmonic type of $g^*(E^{\mathrm{univ}})$, with $g^*(\beta^{\mathrm{univ}})(\xi^*(F')) = g^*(\mathscr{V}/\mathscr{W})$. This is the same as the condition that there exists a strict sub-Higgs bundle F of harmonic type with $g^*(\beta^{\mathrm{univ}})(\xi^*(F')) = g^*(\mathscr{W})$. Set

$$\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G) = \bigcap_{(V, W)} C(V, W)$$

where the intersection is taken over all representations V of H and subspaces W preserved by G. It is a closed subscheme of $\mathbf{R}_{\mathrm{Dol}}(H)$ which represents the functor associating to an S-scheme S′ the set of (P, b) where P is a principal Higgs bundle of semiharmonic type for the group H on X′ over S′, and b is a point, such that the monodromy of (P, b) is contained in G. By Lemma 9.4, $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G)$ also represents the functor associating to S′ the set of (P′, b') where P′ is a principal Higgs bundle of semiharmonic type for the group G and b' is a point. Every linear algebraic group G is a subgroup of $\mathrm{Gl}(n, \mathbf{C})$ for some n, so we obtain all of the required spaces $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G)$. The last statement is immediate from this construction. □

Remark. — The last statement of the theorem, applied to $G \subset \mathrm{Gl}(n, \mathbf{C})$, gives $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G) \subset \mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n)$, because $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, \mathrm{Gl}(n, \mathbf{C})) = \mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n)$.

Our next task is to study the universal categorical quotients of these representation spaces. Assume from now on that G is *reductive*. Note that G acts algebraically on $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G)$. If $G \subset \mathrm{Gl}(n, \mathbf{C})$ is a faithful representation, then G acts on $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n)$ through its inclusion in $\mathrm{Gl}(n, \mathbf{C})$, and this induces the natural action on the subscheme $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G)$.

Choose a $\mathrm{Gl}(n, \mathbf{C})$-linearized line bundle $\mathscr{L}$ on $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, n)$, such that every point is semistable for the action of $\mathrm{Gl}(n, \mathbf{C})$ (cf. Theorem 4.10, Part I). By Mumford's criterion involving one parameter subgroups [Mu], every point is also semistable for the action of G. Thus every point of the closed subset $\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G)$ is semistable for the action of G with respect to the linearized line bundle $\mathscr{L}|_{\mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G)}$. By [Mu], we may form the universal categorical quotient $\mathbf{M}_{\mathrm{Dol}}(X/S, \xi, G) \overset{\mathrm{def}}{=} \mathbf{R}_{\mathrm{Dol}}(X/S, \xi, G)/G$.

Proposition **9.7.** — *Suppose that* $X \to S$ *is smooth and projective. There exists a space* $\mathbf{M}_{\mathrm{Dol}}(X/S, G)$ *which universally co-represents the functor associating to* $S' \to S$ *the set of isomorphism classes of principal Higgs bundles* P *of harmonic type on* X′ *over* S′ *for the group* G. *If the fibers* X_s *are connected and* $\xi : S \to X$ *is a section, then there is a natural isomorphism between* $\mathbf{M}_{\mathrm{Dol}}(X/S, G)$ *and the universal categorical quotient* $\mathbf{M}_{\mathrm{Dol}}(X/S, \xi, G)$ *constructed above. In this case, the points of* $\mathbf{M}_{\mathrm{Dol}}(X_s, G)$ *parametrize the closed* G-*orbits in* $\mathbf{R}_{\mathrm{Dol}}(X_s, \xi(s), G)$.

Proof. — Choose an étale morphism $S' \to S$ with S′ connected, such that each connected component X'_i of X′ admits a section ξ_i. Then

$$\mathbf{M}_{\mathrm{Dol}}(X'/S', G) = \prod_i \mathbf{M}_{\mathrm{Dol}}(X'_i/S', \xi_i, G)$$

universally co-represents the appropriate functor. If $\{S'_\alpha\}$ is a collection of étale S-schemes covering S, then the collection of spaces $\mathbf{M}_{\mathrm{Dol}}(X'_\alpha/S'_\alpha, G)$ constructed in this way is provided with descent data (since the functors they co-represent are provided with the corresponding descent data). They descend to give $\mathbf{M}_{\mathrm{Dol}}(X/S, G)$ which co-represents the desired functor. □

Theorem **9.8.** — *Suppose* G *is a reductive group and* $f: G \to \mathrm{Gl}(n, \mathbf{C})$ *is a faithful representation. Suppose* $(P, b) \in \mathbf{R}_{\mathrm{Dol}}(X_s, \xi(s), G)$ *maps to* $(E, \beta) \in \mathbf{R}_{\mathrm{Dol}}(X_s, \xi(s), n)$*. Then* (P, b) *is in a closed* G*-orbit in* $\mathbf{R}_{\mathrm{Dol}}(X_s, \xi(s), G)$ *if and only if* (E, β) *in a closed* $\mathrm{Gl}(n, \mathbf{C})$*-orbit in* $\mathbf{R}_{\mathrm{Dol}}(X_s, \xi(s), n)$*, or equivalently the monodromy group of* E *is reductive, or equivalently* E *is semisimple.*

Proof. — The subobjects of (E, β) correspond to the subspaces of $\mathbf{C}^n$ preserved by the monodromy group of E. The monodromy group is reductive if and only if the representation $\mathbf{C}^n$ is completely reducible, thus if and only if E is semisimple. The statement of the theorem is true if $G = \mathrm{Gl}(n, \mathbf{C})$: in Theorem 4.10, Part I, as applied in § 6, we have identified the closed $\mathrm{Gl}(n, \mathbf{C})$-orbits in $\mathbf{R}_{\mathrm{Dol}}(X_s, \xi(s), n)$ as corresponding to the semisimple representations.

Certainly if E is semisimple, then its G-orbit is closed, since the G orbit is a closed subset of the $\mathrm{Gl}(n, \mathbf{C})$-orbit.

Suppose E is not semisimple, so $H = \mathrm{Mono}(E, \beta)$ is not reductive. Let U be the unipotent radical of H. There exists a one parameter subgroup $\mathbf{C}^* \to G$ and a family of morphisms $f_t : H \to G$ for $t \in \mathbf{C}$ such that $f_t(g) = tgt^{-1}$ for $t \in \mathbf{C}^*$, and such that the image $f_0(H)$ is not conjugate to H. To see this, we apply the theorem of Morozov [Mo] —see also [BT]—to conclude that since H is not reductive, it is contained in a proper parabolic subgroup Q, and its radical U intersects the unipotent radical of Q. Now we may assume that Q is defined by a torus $\mathbf{C}^* \to G$. The Levi component of Q is the centralizer of this torus, and the torus acts with positive weights on the unipotent radical of Q. In particular, if $q \in Q$ then the limit $\lim_{t \to 0} tqt^{-1}$ exists. These limits give the map f_0, which completes the family $f_t(q) = tqt^{-1}$ defined for $t \neq 0$. The limits of the elements of the unipotent radical of Q are the identity element, so there is $u \in H$ whose limit is the identity. In particular $f_0(H)$ has dimension smaller than that of H. This is the required family.

The construction of the previous paragraph gives a morphism of group schemes over $\mathbf{A}^1$, $f: H \times \mathbf{A}^1 \to G \times \mathbf{A}^1$. Let (P', b') be the principal Higgs bundle for the group H with $P = P' \times^H G$. The map f gives an associated relative principal Higgs bundle $P^f = P' \times^{H, f} G$ on $X \times \mathbf{A}^1$ over $\mathbf{A}^1$, with an $\mathbf{A}^1$-valued point b^f. For $t \in \mathbf{A}^1$, $\mathrm{Mono}(P^f_t, b^f_t) = f_t(H)$, and furthermore $(P^f_t, b^f_t) \cong (P, tb)$ for $t \neq 0$. Thus (P^f_t, b^f_t) are points in the G-orbit of (P, b), which approach the limit (P^f_0, b^f_0) as $t \to 0$. This limit is not in the same orbit, since its monodromy group, $f_0(H)$, is not conjugate to H. Thus the G-orbit is not closed. □

The de Rham spaces

We can define spaces $\mathbf{R}_{DR}(X/S, \xi, G)$ and $\mathbf{M}_{DR}(X/S, G)$ in the same way as above, and obtain the same results. Suppose $X \to S$ is a smooth projective morphism to a scheme of finite type ovre $\mathbf{C}$. Let $\mathbf{g}$ denote the Lie algebra of G with G acting by the adjoint representation. A *principal bundle with integrable relative connection* on X over S, for the group G, is a principal right G-bundle $P \to X$ together with an integrable connection ∇. For purposes of brevity, we can define an integrable connection as being a G-invariant structure of stratification of schemes for P on X/S in the sense of § 8. Given (P, ∇) and a representation V of G, we get a vector bundle with integrable relative connection $\rho_P(V) = P \times^G V$. The construction $P \mapsto \rho_P$ provides an equivalence between the categories of principal bundles with integrable relative connection, and strict exact faithful tensor functors ρ from Rep(G) to the category of vector bundles with relative integrable connection.

Lemma **9.9.** — *Suppose* E *is a vector bundle with relative integrable connection on* X *over* S. *Fix a number k. There is a projective scheme* $N(E, k) \to S$ *representing the functor which associates to each* S-*scheme* $f : S' \to S$ *the set of quotients* $f^*(E) \to F \to 0$ *compatible with the connection. Suppose that the fibers* X_s *are connected, and* $\xi : S \to X$ *is a section. Then the morphism* $N(E, k) \to \mathbf{Grass}_S(\xi^*(E), k)$ *is a closed embedding.*

Proof. — Let $\Lambda = \Lambda^{DR}$ be the sheaf of rings of all relative differential operators on X over S. We may consider E as a Λ-module. Let p_0 denote the Hilbert polynomial of $\mathcal{O}_X$ over S. The Hilbert scheme $\mathbf{Hilb}(E, kp_0)$ parametrizes quotient sheaves $E \to F \to 0$ flat over S with Hilbert polynomial kp_0. Let $E^{univ} \to F^{univ} \to 0$ denote the universal quotient on $X^{univ} = X \times_S \mathbf{Hilb}(E, kp_0)$, and let $K^{univ} \subset E^{univ}$ denote the kernel. We get a map

$$\Lambda_1^{univ} \otimes_{\mathcal{O}_{X^{univ}}} K^{univ} \to F^{univ}.$$

Let $N(E, k)$ be the closed subscheme representing the condition that this map pulls back to zero. Then $N(E, k)$ parametrizes quotients $E \to F \to 0$ compatible with the action of Λ, such that F is flat with Hilbert polynomial kp_0 over the base. Any such quotient restricts to a sheaf with connection on each fiber, hence to a locally free sheaf. By Lemma 1.27, Part I, F is locally free, so it is a vector bundle with integrable relative connection. From the Hilbert polynomial, it has rank k. Thus $N(E, k)$ represents the desired functor. Furthermore, $N(E, k)$ is projective over S and the natural morphism to $\mathbf{Grass}(\xi^*(E), k)$ is injective on the level of S'-valued points, by an application of Lemma 4.9, Part I. Hence the map to the Grassmanian is a closed embedding. □

Suppose that the fibers X_s are connected, and $\xi : S \to X$ is a section. Suppose $G \subset H$ is a subgroup. Suppose P is a principal bundle with relative integrable connection for the group H on X over S. Suppose $b : S \to \xi^*(P)$ is an S-valued point. We say that *the monodromy of* (P, *b*) is *contained in* G if the following condition holds: for every linear representation V of H, and every subspace $W \subset V$ preserved by G, there exists a strict subbundle preserved by the connection

$$F \subset P \times^{H} V$$

such that

$$\xi^*(F) = \{ b \} \times W \subset \xi^*(P \times^{H} V).$$

If S is a point, we define the *monodromy group* Mono(P, *b*) to be the intersection of all algebraic subgroups $G \subset H$ such that the monodromy of (P, *b*) is contained in G.

We obtain the same result as in Lemma 9.4. Note that the concept of " semi-harmonic type " is not needed, since all Λ^{DR}-modules are automatically *p*-semistable with vanishing rational Chern classes. Suppose $G \subset H$. Suppose P′ is a principal bundle with relative integrable connection on X over S, for the group G. Then $P = P' \times^{G} H$ is a principal bundle with relative integrable connection for the group H. This construction gives an identification between: (1) the set of isomorphism classes of pairs (P′, *b*′) where P′ is a principal bundle with relative integrable connection for the group G and *b*′ is an S-valued point of $\xi^*(P')$; and (2) the set of isomorphism classes of pairs (P, *b*) where P is a principal bundle with relative integrable connection for the group H and *b* is an S-valued point of $\xi^*(P')$, such that the monodromy of (P, *b*) is contained in G.

Suppose E is a vector bundle of rank *n* with integrable relative connection on X over S. Then the frame bundle P of E has a natural structure of principal bundle with integrable relative connection for the group $Gl(n, \mathbf{C})$, and E is recovered as $P \times^{Gl(n, \mathbf{C})} \mathbf{C}^n$. This construction provides an identification between the sets of isomorphism classes of (E, β) and (P, *b*).

Theorem **9.10.** — *Suppose* $\xi : S \to X$ *is a section. There is a scheme* $\mathbf{R}_{DR}(X/S, \xi, G)$ *over* S *representing the functor which associates to any* S-*scheme* S′ *the set of pairs* (P, *b*) *where* P *is a principal bundle with relative integrable connection for the group* G *on* $X' = X \times_S S'$ *over* S′, *and* $b : S' \to \xi^*(P)$ *is a section over* ξ. *If* $f : G \hookrightarrow H$ *is a closed embedding, then f induces a closed embedding* $\mathbf{R}_{DR}(X/S, \xi, G) \hookrightarrow \mathbf{R}_{DR}(X/S, \xi, H)$.

Proof. — The same as the proof of Theorem 9.6. □

The analogues of Proposition 9.7 and Theorem 9.8 also hold.

Relationship with Betti spaces

Suppose $X \to S$ is smooth and projective, with connected fibers, and suppose $\xi : S \to X$ is a section. If G is any linear algebraic group, we obtain a local system of schemes $\mathbf{R}_B(X/S, \xi, G)$ on S^{an}. These are obtained from the fundamental group $\Gamma = \pi_1(X_s, \xi(s))$ by setting $\mathbf{R}(\Gamma, G) = \mathrm{Hom}(\Gamma, G)$; the fundamental group $\pi_1(S, s)$

acts on Γ so it acts on $\mathbf{R}(\Gamma, G)$, and $\mathbf{R}_B(X/S, \xi, G)$ is the corresponding local system of schemes.

If G is reductive then define $\mathbf{M}_B(X/S, G)$ to be the local system of schemes whose fibers are the good quotient $\mathbf{R}(\Gamma, G)/G$ (which exist because the representation space is affine). If $X \to S$ is any smooth and projective morphism, we obtain $\mathbf{M}_B(X/S, G)$ by the same descent as usual.

Recall that the superscript " $^{(an)}$ " denotes the analytic total space associated to a local system of schemes.

Theorem **9.11.** — *We have isomorphisms of complex analytic spaces*

$$\mathbf{R}^{an}_{DR}(X/S, \xi, G) \cong \mathbf{R}^{(an)}_{B}(X/S, \xi, G)$$

and, if G *is reductive,*

$$\mathbf{M}^{an}_{DR}(X/S, G) \cong \mathbf{M}^{(an)}_{B}(X/S, G).$$

These are compatible with the morphisms of functoriality induced by homomorphisms of algebraic groups, and they are equal to those given by Theorem 7.1 *in the case* $G = \mathrm{Gl}(n, \mathbf{C})$. *The monodromy group corresponding to a point in* $\mathbf{R}_{DR}(X_s, \xi(s), G)$ *is equal to the Zariski closure in* G *of the image of the representation parametrized by the corresponding point in* $\mathbf{R}_B(X_s, \xi(s), G)$.

Proof. — Fix an injective homomorphism $G \subset \mathrm{Gl}(n, \mathbf{C})$. Over an analytic base, a relative vector bundle with integrable connection has monodromy contained in G if and only if the corresponding family of representations has image in the subgroup of points with values in G. Thus the subsets

$$\mathbf{R}^{an}_{DR}(X/S, \xi, G) \subset \mathbf{R}^{an}_{DR}(X/S, \xi, \mathrm{Gl}(n, \mathbf{C}))$$

and

$$\mathbf{R}^{(an)}_{B}(X/S, \xi, G) \subset \mathbf{R}^{(an)}_{B}(X/S, \xi, \mathrm{Gl}(n, \mathbf{C}))$$

represent the same functor of analytic spaces $S' \to S^{an}$. Hence they correspond under the isomorphism of Theorem 7.1. We obtain an isomorphism between the moduli spaces by applying Proposition 5.5, Part I. □

Corollary **9.12.** — *If* $G \subset \mathrm{Gl}(n, \mathbf{C})$ *is a closed embedding then the Gauss-Manin connection preserves the subspace* $\mathbf{R}_{DR}(X/S, \xi, G) \subset \mathbf{R}_{DR}(X/S, \xi, n)$. *If* G *is reductive, we obtain a Gauss-Manin connection on the universal categorical quotient* $\mathbf{M}_{DR}(X/S, G)$.

Proof. — The Gauss-Manin connection on the associated analytic space is the same as the connection given by the local trivializations of the Betti spaces. These trivializations are compatible with the subspaces of representations for the group G. Thus the analytic connection preserves the analytic subspace $\mathbf{R}^{an}_{DR}(X/S, \xi, G)$. This implies that the algebraic connection preserves the subspace $\mathbf{R}_{DR}(X/S, \xi, G)$. The connection descends to the universal categorical quotient by Lemma 8.5. □

Principal harmonic bundles

Let J denote the standard metric on $\mathbf{C}^n$. Recall that we defined in § 7 the space $\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, n)$ whose points over $s \in \mathrm{S}$ consist of pairs (E, β) where E is a Higgs bundle on X_s and β is a frame for $\mathrm{E}_{\xi(s)}$ such that there exists a harmonic metric K on E with $\beta(\mathrm{K}_{\xi(s)}) = \mathrm{J}$. Similarly, $\mathbf{R}^{\mathrm{J}}_{\mathrm{DR}}(\mathrm{X/S}, \xi, n)$ was the space whose points over s consist of pairs (E, β) where E is a vector bundle with integrable connection and β is a frame for $\mathrm{E}_{\xi(s)}$ such that there exists a harmonic metric K for E with $\beta(\mathrm{K}_{\xi(s)}) = \mathrm{J}$.

Suppose G is a reductive algebraic group. Fix a maximal compact subgroup $\mathrm{V} \subset \mathrm{G}$. Choose an inclusion $\mathrm{G} \to \mathrm{Gl}(n, \mathbf{C})$ so that the standard metric J is invariant under V; then $\mathrm{V} = \mathrm{G} \cap \mathrm{U}(n)$. Define

$$\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, \mathrm{G}) = \mathbf{R}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, \mathrm{G}) \cap \mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, n).$$

Endow this space with the topology induced by the usual topology of $\mathbf{R}^{\mathrm{an}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, \mathrm{G})$. Define

$$\mathbf{R}^{\mathrm{J}}_{\mathrm{DR}}(\mathrm{X/S}, \xi, \mathrm{G}) = \mathbf{R}_{\mathrm{DR}}(\mathrm{X/S}, \xi, \mathrm{G}) \cap \mathbf{R}^{\mathrm{J}}_{\mathrm{DR}}(\mathrm{X/S}, \xi, n),$$

endowed with the topology induced by the usual topology of $\mathbf{R}^{\mathrm{an}}_{\mathrm{DR}}(\mathrm{X/S}, \xi, \mathrm{G})$. Note that V (with its usual topology) acts continuously on $\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, \mathrm{G})$ and $\mathbf{R}^{\mathrm{J}}_{\mathrm{DR}}(\mathrm{X/S}, \xi, \mathrm{G})$.

Lemma **9.13.** — *The map* $\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, \mathrm{G}) \to \mathbf{M}^{\mathrm{an}}_{\mathrm{Dol}}(\mathrm{X/S}, \mathrm{G})$ *is surjective and proper, and identifies* $\mathbf{M}^{\mathrm{an}}_{\mathrm{Dol}}(\mathrm{X/S}, \mathrm{G})$ *with the topological quotient* $\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi\mathrm{G})/\mathrm{V}$. *The map* $\mathbf{R}^{\mathrm{J}}_{\mathrm{DR}}(\mathrm{X/S}, \xi, \mathrm{G}) \to \mathbf{M}^{\mathrm{an}}_{\mathrm{DR}}(\mathrm{X/S}, \mathrm{G})$ *is surjective and proper, and identifies* $\mathbf{M}^{\mathrm{an}}_{\mathrm{DR}}(\mathrm{X/S}, \mathrm{G})$ *with the topological quotient* $\mathbf{R}^{\mathrm{J}}_{\mathrm{DR}}(\mathrm{X/S}, \xi, \mathrm{G})/\mathrm{V}$.

Proof. — We give the proof for the Dolbeault spaces. Recall that $\mathbf{R}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, \mathrm{G})$ is a closed subset of $\mathbf{R}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, n)$, so $\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, \mathrm{G})$ is a closed subset of $\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, n)$. But $\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, n)$ is proper over $\mathbf{M}_{\mathrm{Dol}}(\mathrm{X/S}, n)$, so $\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, \mathrm{G})$ is proper over $\mathbf{M}_{\mathrm{Dol}}(\mathrm{X/S}, \mathrm{G})$. Furthermore, if $\mathbf{R}^{\mathrm{J}}_{\mathrm{Dol}}(\mathrm{X/S}, \xi, \mathrm{G}) \to \mathbf{M}_{\mathrm{Dol}}(\mathrm{X/S}, \mathrm{G})$ is surjective, then $\mathbf{M}_{\mathrm{Dol}}(\mathrm{X/S}, \mathrm{G})$ is proper over $\mathbf{M}_{\mathrm{Dol}}(\mathrm{X/S}, n)$.

To show surjectivity, suppose $s \in \mathrm{S}$ and suppose q is a point of $\mathbf{M}_{\mathrm{Dol}}(\mathrm{X}_s, \mathrm{G})$. This can be lifted to a point (E, β) in a closed G orbit of $\mathbf{R}_{\mathrm{Dol}}(\mathrm{X}_s, \xi(s), \mathrm{G})$, in which case, by Theorem 9.8, the associated rank n Higgs bundle E is semisimple. Write $\mathrm{E} = \bigoplus \mathrm{E}_i \otimes \mathrm{A}_i$ where E_i are the distinct stable summands of E, and A_i are vector spaces. Choose good metrics K_i for E_i. Then the good metrics for E are those of the form $\sum \mathrm{K}_i \otimes \mathrm{L}_i$ where L_i are any metrics on A_i. The monodromy group fixes the decomposition of E and acts irreducibly on the components E_i.

Choose a metric $\mathrm{K} = \sum \mathrm{K}_i \otimes \mathrm{L}_i$ for E. Let $\mathrm{Mono}(\mathrm{E}, 1_{\xi(s)}) \subset \mathrm{Gl}(\mathrm{E}_{\xi(s)})$ denote the monodromy group induced by the identity frame $1_{\xi(s)} : \mathrm{E}_{\xi(s)} \cong \mathrm{E}_{\xi(s)}$. Let

$$\mathrm{W} = \mathrm{Mono}(\mathrm{E}, 1_{\xi(s)}) \cap \mathrm{U}(\mathrm{E}_{\xi(s)}, \mathrm{K}_{\xi(s)}).$$

We claim that this is a maximal compact subgroup of $\mathrm{Mono}(E, 1_{\xi(s)})$. Let σ denote complex conjugation in $\mathrm{Gl}(E_{\xi(s)})$ with respect to the metric K_x. We will prove that σ fixes $\mathrm{Mono}(E, 1_{\xi(s)})$, and also that every component of $\mathrm{Mono}(E, 1_{\xi(s)})$ contains a fixed point of σ. Then W, being a compact real form which meets every component, will be maximal compact.

Since $\mathrm{Mono}(E, 1_{\xi(s)})$ is reductive, it is equal to the group of elements fixing a subspace of tensors $T \subset E_{\xi(s)}^{\otimes a}(\otimes E_{\xi(s)}^{*})^{\otimes b}$. Furthermore we may assume that T is the space of all tensors so fixed, and hence there is a decomposition of Higgs bundles

$$E^{\otimes a}(\otimes E^{*})^{\otimes b} = (T \otimes \mathcal{O}_X) \oplus F$$

with F not containing any trivial subobjects. In particular, $\mathrm{Mono}(E, 1_{\xi(s)})$ preserves the subspace $F_{\xi(s)}$. Now the harmonic metric K on E induces a harmonic metric on the tensor product, and it follows that the direct sum $(T \otimes \mathcal{O}_X) \oplus F$ is an orthogonal direct sum of bundles with harmonic metrics. For any $g \in \mathrm{Gl}(E_{\xi(s)})$, let g^* denote the adjoint with respect to the metric $K_{\xi(s)}$, defined by the formula $(ge, f) = (e, g^* f)$ (we will suppress reference to the metric $K_{\xi(s)}$ in the notation $(., .)$ for the metric on $E_{\xi(s)}$ or any tensor power thereof). The complex conjugation σ is given by $\sigma(g) = (g^*)^{-1}$. Suppose $g \in \mathrm{Mono}(E, 1_{\xi(s)})$. Then for $t \in T$ and $f \in F_{\xi(s)}$, we have $(g^* t, f) = (t, gf) = 0$, since $gf \in F_{\xi(s)}$. Similarly if $s, t \in T$ then $(g^* t, s) = (t, gs) = (t, s)$. Therefore if $t \in T$, $g^* t = t$. In other words, $g^* \in \mathrm{Mono}(E, 1_{\xi(s)})$. Thus $\sigma(g) = (g^*)^{-1}$ is also in $\mathrm{Mono}(E, 1_{\xi(s)})$. This proves that W is a compact real form of $\mathrm{Mono}(E, 1_{\xi(s)})$.

We still have to prove that it meets every component; this we do by a standard argument. Suppose $g \in \mathrm{Mono}(E, 1_{\xi(s)})$. Then gg^* is a positive definite self adjoint matrix, so it can be raised to any real power, and we get a real one parameter subgroup of $\mathrm{Gl}(n)$ consisting of the self adjoint matrices $(gg^*)^t$, $t \in \mathbf{R}$. Furthermore, it is easy to see that $(gg^*)^t$ preserves any tensor preserved by gg^*, so this one parameter subgroup is in $\mathrm{Mono}(E, 1_{\xi(s)})$. Furthermore, we have $\sigma((gg^*)^t) = (gg^*)^{-t}$. Let $f(t) = g^{-1}(gg^*)^t$; it is in $\mathrm{Mono}(E, 1_{\xi(s)})$. Note that $f(0) = g^{-1}$ and $f(1) = g^*$. On the other hand,

$$\sigma(f(t)) = g^*(gg^*)^{-t} = g^{-1}(gg^*)\,(gg^*)^{-t} = f(1-t).$$

Thus $f(1/2)$ is fixed by σ. We have joined the element g^{-1} to an element of W by a path of elements of $\mathrm{Mono}(E, 1_{\xi(s)})$. This shows that every component of $\mathrm{Mono}(E, 1_{\xi(s)})$ contains an element of W, completing the proof that W is a maximal compact subgroup.

The group W preserves the metric $K_{\xi(s)}$ on $E_{\xi(s)}$, and fixes the factors E_i, so it preserves the metrics K_i on $E_{i, \xi(s)}$. On the other hand, $\mathrm{Mono}(E, 1_{\xi(s)})$ acts irreducibly on $E_{i, \xi(s)}$, and W—being a maximal compact subgroup—does too. Therefore $K_{i, \xi(s)}$ is, up to scalars, the unique metric on $E_{i, \xi(s)}$ preserved by W.

Since W is compact, there exists $g \in G$ such that $gWg^{-1} \subset V$. Then we may replace the point (E, β) by $(E, g\beta) \in \mathbf{R}_{\mathrm{Dol}}(X_s, \xi(s), G)$, so we may assume $W \subset V$. Now J is a W-invariant metric on $E_x = \bigoplus E_{i, \xi(s)} \otimes A_i$. But since $K_{i, \xi(s)}$ is the unique W-invariant metric on $E_{i, \xi(s)}$ up to scalars, and the $E_{i, \xi(s)}$ are distinct irreducible representations of W, there exist metrics L_i' on A_i such that $J = \Sigma K_{i,x} \otimes L_i'$. Thus our point lies in $\mathbf{R}^J_{Dol}(X_s, \xi(s), n)$. Set $K' = \Sigma K_i \otimes L_i'$, and $\beta(K'_{\xi(s)}) = J$. This proves that

$$\mathbf{R}^J_{Dol}(X/S, \xi, G) \to \mathbf{M}_{Dol}(X/S, G)$$

is surjective.

The map is clearly V-invariant, so finally we must prove that two points in $\mathbf{R}^J_{Dol}(X/S, \xi, G)$ which map to the same point in $\mathbf{M}_{Dol}(X/S, G)$ differ by an element of V. Then the properness and surjectivity will imply that $\mathbf{M}_{Dol}(X/S, G)$ is the topological quotient space. Again, we may restrict our attention to the fiber over a point $s \in S$. Any point in $\mathbf{R}^J_{Dol}(X_s, \xi(s), G)$ corresponds to a semisimple object, in other words it is contained in a closed orbit. But the inverse image of a point in $\mathbf{M}_{Dol}(X_s, G)$ contains exactly one closed orbit. Thus if two points map to the same point in $\mathbf{M}_{Dol}(X_s, G)$, we may assume that the two points are (E, β) and $(E, g\beta)$. Then there are two harmonic metrics on E, say $\Sigma K_i \otimes L_i$ and $\Sigma K_i \otimes L_i'$, which map to the metric J via β and $g\beta$ respectively. Note that the stabilizer of E in $Gl(n, \mathbf{C})$ is $Stab(E) = \Pi\, Gl(A_i)$. There is an element $s \in Stab(E)$ such that $gs\beta$ takes the metric $(\Sigma K_i \otimes L_i)_{\xi(s)}$ to J. Thus $gs \in U(n)$, so $g \in U(n).Stab(E)$. We have a unique decomposition

$$Stab(E) = (U(n) \cap Stab(E)).(\exp(\mathbf{p}) \cap Stab(E)),$$

where $\mathbf{gl}(n) = \mathbf{u}(n) \oplus \mathbf{p}$ is the Cartan decomposition. Thus we may write $g = up$ for $u \in U(n)$ and $p \in \exp(\mathbf{p}) \cap Stab(E)$. Furthermore, since $V = G \cap U(n)$, we get a Cartan decomposition $\mathbf{g} = \mathbf{v} \oplus (\mathbf{p} \cap \mathbf{g})$, and we may write $g = vp'$ uniquely for $v \in V$ and $p' \in \exp(\mathbf{p} \cap \mathbf{g})$. It follows that $v = u$ and $p' = p$. In particular, $p' \in \exp(\mathbf{p}) \cap Stab(E)$. Thus $(E, p'\beta) \cong (E, \beta)$ so $(E, g\beta) \cong (E, v\beta)$. Thus our two points differ by an element of the maximal compact group V. This completes the proof for the Dolbeault spaces. The proof for the de Rham spaces is the same. □

Lemma **9.14.** — *The equivalence of categories constructed in* [Si5] *gives homeomorphisms of topological spaces*

$$\mathbf{R}^J_{Dol}(X/S, \xi, G) \cong \mathbf{R}^J_{DR}(X/S, \xi, G) \quad \textit{and} \quad \mathbf{M}_{Dol}(X/S, G) \cong \mathbf{M}_{DR}(X/S, G).$$

Proof. — The equivalence of categories of [Si5] gives an isomorphism of sets

$$\mathbf{R}_{Dol}(X/S, \xi, n) \cong \mathbf{R}_{DR}(X/S, \xi, n).$$

We have seen in Lemma 7.16 that this restricts to a homeomorphism of subspaces

$$\mathbf{R}^J_{Dol}(X/S, \xi, n) \cong \mathbf{R}^J_{DR}(X/S, \xi, n).$$

Furthermore, the equivalence of categories is a tensor functor, so it preserves the monodromy groups. Thus it gives an isomorphism of subsets

$$\mathbf{R}^J_{Dol}(X/S, \xi, G) \cong \mathbf{R}^J_{DR}(X/S, \xi, G).$$

Note that $\mathbf{R}^{J}_{Dol}(X/S, \xi, G)$ and $\mathbf{R}^{J}_{DR}(X/S, \xi, G)$ are respectively closed subsets of $\mathbf{R}^{J}_{Dol}(X/S, \xi, n)$ and $\mathbf{R}^{J}_{DR}(X/S, \xi, n)$, endowed with the subspace topologies. Therefore the above isomorphism gives a homeomorphism of topological spaces

$$\mathbf{R}^{J}_{Dol}(X/S, \xi, G) \cong \mathbf{R}^{J}_{DR}(X/S, \xi, G).$$

Furthermore, this is compatible with the action of V. Thus it descends to a homeomorphism between the quotient spaces which are $\mathbf{M}_{Dol}(X/S, G)$ and $\mathbf{M}_{DR}(X/S, G)$. To finish the proof, note that this homeomorphism is compatible with descent data for going from the case where the fibers are connected and there exists a section, to the general case where the moduli spaces are constructed. □

Corollary **9.15.** — *Suppose* G *and* H *are reductive algebraic groups, and* $G \to H$ *is an injective homomorphism. Then the induced maps between moduli spaces* $\mathbf{M}_{Dol}(X/S, G) \to \mathbf{M}_{Dol}(X/S, H)$ *and* $\mathbf{M}_{DR}(X/S, G) \to \mathbf{M}_{DR}(X/S, H)$ *are proper.*

Proof. — We may assume that $X \to S$ has a section ξ, and that the fibers are connected. Then $\mathbf{R}^{J}_{Dol}(X/S, \xi, G)$ is a closed subset of $\mathbf{R}^{J}_{Dol}(X/S, \xi, H)$. Therefore the map $\mathbf{R}^{J}_{Dol}(X/S, \xi, G) \to \mathbf{M}_{Dol}(X/S, H)$ is proper. But this factors through $\mathbf{M}_{Dol}(X/S, G)$, and the map $\mathbf{R}^{J}_{Dol}(X/S, \xi, G) \to \mathbf{M}_{Dol}(X/S, G)$ is surjective. Therefore the map $\mathbf{M}_{Dol}(X/S, G) \to \mathbf{M}_{Dol}(X/S, H)$ is proper. The same proof works for the de Rham spaces. □

Surprisingly, we obtain a result about representations of any finitely generated group.

Corollary **9.16.** — *Suppose* Υ *is a finitely generated group. Suppose* $G \to H$ *is an injective homomorphism of reductive algebraic groups. The resulting morphism of moduli spaces* $\mathbf{M}(\Upsilon, G) \to \mathbf{M}(\Upsilon, H)$ *is finite.*

Proof. — Suppose X is a connected smooth projective variety with basepoint $x \in X$. The previous corollary implies that $\mathbf{M}_{DR}(X, G) \to \mathbf{M}_{DR}(X, H)$ is proper. By Theorem 9.11, this implies that the map $\mathbf{M}_{B}(X, G) \to \mathbf{M}_{B}(X, H)$ is proper. However, the Betti spaces are affine, and an affine proper map is finite. Thus $\mathbf{M}_{B}(X, G) \to \mathbf{M}_{B}(X, H)$ is finite. If F_g denotes the free group on n generators, and if X is a smooth connected projective curve of genus g with basepoint x, then there is a surjection from $\pi_1(X, x) \to F_g \to 1$ (this is easy to see by drawing a picture of the Riemann surface X^{an} as the surface of a solid with g holes). Thus if Υ is any group generated by g elements, there is a surjection $\pi_1(X, x) \to \Upsilon$. The additional relations in Υ give closed conditions on the representation space, so

$$\mathbf{R}(\Upsilon, G) \subset \mathbf{R}_{B}(X, x, G) \quad \text{and} \quad \mathbf{R}(\Upsilon, H) \subset \mathbf{R}_{B}(X, x, H)$$

are closed equivariant embeddings. Reductivity of the groups G and H implies that the corresponding maps on good quotients

$$\mathbf{M}(\Upsilon, G) \to \mathbf{M}_{B}(X, G) \quad \text{and} \quad \mathbf{M}(\Upsilon, H) \to \mathbf{M}_{B}(X, H)$$

are closed embeddings. This implies that the map $\mathbf{M}(\Upsilon, G) \to \mathbf{M}(\Upsilon, H)$ is finite. □

Lemma **9.17.** — *Suppose* A *is a* **C**-*algebra of finite type, and* N *is a finitely generated* A-*module. Suppose that a reductive algebraic group* G *acts algebraically on* A *and* N. *Then the module of invariants* N^G *is finitely generated over* A^G.

Proof. — [Mu]. □

Corollary **9.18.** — *Suppose* Υ *is a finitely generated group. Suppose* $G \to H$ *is a homomorphism of reductive algebraic groups with finite kernel. Then the resulting morphism of moduli spaces* $\mathbf{M}(\Upsilon, G) \to \mathbf{M}(\Upsilon, H)$ *is finite.*

Proof. — Let $G' \subset H$ denote the image of G. From Corollary 9.16, the map $\mathbf{M}(\Upsilon, G') \to \mathbf{M}(\Upsilon, H)$ is finite. The map $G \to G'$ is finite, and the representation spaces are embedded as closed subsets in products of copies of the groups, so the map $\mathbf{R}(\Upsilon, G) \to \mathbf{R}(\Upsilon, G')$ is finite. The map $G \to G'$ is surjective, so $\mathbf{M}(\Upsilon, G')$ is a good quotient of $\mathbf{R}(\Upsilon, G')$ by the action of G. The previous lemma implies that the map $\mathbf{M}(\Upsilon, G) \to \mathbf{M}(\Upsilon, G')$ is finite. Composing these statements gives the corollary. □

This in turn gives finiteness for the maps of Corollary 9.15.

Corollary **9.19.** — *Suppose* G *and* H *are reductive algebraic groups, and* $G \to H$ *is a homomorphism with finite kernel. Then the induced maps between moduli spaces* $\mathbf{M}_{Dol}(X/S, G) \to \mathbf{M}_{Dol}(X/S, H)$ *and* $\mathbf{M}_{DR}(X/S, G) \to \mathbf{M}_{DR}(X/S, H)$ *are finite.*

Proof. — The map $\mathbf{M}_B^{(an)}(X/S, G) \to \mathbf{M}_B^{(an)}(X/S, H)$ is finite by the previous corollary, and the Dolbeault and de Rham spaces are homeomorphic to these Betti total spaces. Thus $\mathbf{M}_{Dol}^{an}(X/S, G) \to \mathbf{M}_{Dol}^{an}(X/S, H)$ and $\mathbf{M}_{DR}^{an}(X/S, G) \to \mathbf{M}_{DR}^{an}(X/S, H)$ are finite. This implies that the corresponding algebraic maps are finite. □

Limits of the $\mathbf{C}^*$ action

There is an action of $\mathbf{C}^*$ on the category of principal Higgs bundles: $z \in \mathbf{C}^*$ sends (P, φ) to $(P, z\varphi)$. If G is a reductive group, we obtain an action of $\mathbf{C}^*$ on $\mathbf{M}_{Dol}(X/S, G)$. This is compatible with the morphisms of functoriality induced by morphisms of groups, and is equal to the action defined in § 6 in the case $G = Gl(n, \mathbf{C})$.

Corollary **9.20.** — *For any point* $y \in \mathbf{M}_{Dol}(X/S, G)$ *the limit* $\lim_{z \to 0} zy$ *exists, and is a fixed point of the action of* $\mathbf{C}^*$, *in* $\mathbf{M}_{Dol}(X/S, G)$.

Proof. — Corollary 6.12 gives this statement for the group $Gl(n, \mathbf{C})$. Choose a faithful representation $G \subset Gl(n, \mathbf{C})$. The map $\mathbf{C}^* \to \mathbf{M}_{Dol}(X/S, n)$ extends to a map $\mathbf{A}^1 \to \mathbf{M}_{Dol}(X/S, n)$, and by the properness of the maps in Corollary 9.15, the orbit $\mathbf{C}^* \to \mathbf{M}_{Dol}(X/S, G)$ extends to a map $\mathbf{A}^1 \to \mathbf{M}_{Dol}(X/S, G)$. The image of the origin is the desired fixed point of $\mathbf{C}^*$. □

Corollary **9.21.** — *Suppose* X *is a smooth connected projective variety with basepoint* x, *and* G *is a reductive complex algebraic group. Any representation* $\pi_1(X, x) \to G$ *can be deformed to a representation which comes from a complex variation of Hodge structure.*

Proof. — The points in the closed orbit of $\mathbf{R}_{\text{Dol}}(X, x, G)$ lying over fixed points of $\mathbf{C}^*$ correspond to the representations of the fundamental group which come from complex variations of Hodge structure [Si5]. The same proof as for Corollary 6.12 now works. □

10. Local structure

We will now review the deformation theory of Goldman and Millson (descended from Deligne, Schlessinger and Stasheff) [GM]. The cases of $\mathbf{R}_{\text{DR}}$ and $\mathbf{R}_{\text{B}}$ are identical to [GM], and the case of $\mathbf{R}_{\text{Dol}}$ is analogous.

A *differential graded Lie algebra* [GM] is a collection $A = (A^0, A^1, \ldots)$ of $\mathbf{C}$-vector spaces, with differentials $d : A^i \to A^{i+1}$ and a bracket $[\ ,\ ] : A^i \otimes_{\mathbf{C}} A^j \to A^{i+j}$ such that the following axioms hold: $d^2 = 0$; the bracket is graded-anticommutative, $[a, b] = (-1)^{ij+1} [b, a]$ for $a \in A^i$ and $b \in A^j$; the differential and bracket are compatible, $d[a, b] = [da, b] + (-1)^i [a, db]$ if $a \in A^i$; and the Jacobi identity holds with the appropriate signs.

Fix a finite dimensional Lie algebra $\mathbf{g}$. A $\mathbf{g}$-*deformation diagram* is a pair (A, ε) where $A^{\cdot}$ is a differential graded Lie algebra and $\varepsilon : A^0 \to \mathbf{g}$ is a morphism of Lie algebras. Let H^i denote the i-th cohomology of the complex $(A^{\cdot}, d)$. We say that (A, ε) is *finite dimensional* if the spaces H^i are finite dimensional. We say that (A, ε) is *rigid* if the map $\varepsilon : H^0 \to \mathbf{g}$ is injective. Denote by $\mathbf{h}$ the image of H^0 in $\mathbf{g}$, and let $\mathbf{h}^{\perp}$ denote a subspace transverse to $\mathbf{h}$ (for example, if $\mathbf{g}$ is semisimple we can take the perpendicular space with respect to the Killing form).

The main examples are as follows. Suppose X is a connected smooth projective variety over Spec($\mathbf{C}$) with a point $x \in X$. Let E be a Higgs bundle of semiharmonic type of rank n, with a frame $\beta : E_x \cong \mathbf{C}^n$. Let $\mathbf{g} = \mathbf{gl}(n, \mathbf{C})$. Then we can define a $\mathbf{g}$-deformation diagram $(A_{\text{Dol}}(E), \varepsilon)$ with A^i equal to the space of smooth i-forms with coefficients in End E, the differential d given by the operator D'', and the Lie bracket given by the graded commutator of forms. The map ε is evaluation at x composed with the frame β.

Let $G \subset Gl(n)$ be a complex algebraic subgroup, and suppose (E, β) satisfies condition Mono$(E, \beta) \subset G$ (in other words (E, β) represents a point in $\mathbf{R}_{\text{Dol}}(X, x, G)$). Put $\mathbf{g} = \text{Lie}(G)$. Let P be the associated principal Higgs bundle and $\text{Ad}(P) = P \times^G \mathbf{g}$ the adjoint Higgs bundle. We can define a $\mathbf{g}$-deformation diagram $(A_{\text{Dol}}(P), \varepsilon)$ with A^i equal to the space of smooth i-forms with coefficients in Ad(P). The Lie bracket comes from the Lie bracket of $\mathbf{g}$ and the graded commutator of forms, and the augmentation ε is given by evaluation at x using $P_x \cong G$.

Similarly, suppose E is a flat bundle (thought of as a representation of the fundamental group, or equivalently as a holomorphic vector bundle with integrable connection), with frame $\beta : E_x \cong \mathbf{C}^n$. Let $\mathbf{g} = \mathbf{gl}(n, \mathbf{C})$. The $\mathbf{g}$-deformation diagram $(A_B(E), \varepsilon) = (A_{DR}(E), \varepsilon)$ has A^i equal to the space of smooth i-forms with coefficients in End(E), with differential d given by the flat connection D on E, Lie bracket given by graded commutator of forms, and augmentation ε given by evaluation at x.

If the monodromy group is contained in $G \subset Gl(n, \mathbf{C})$ (in other words (E, β) represents a point in $\mathbf{R}_B(X, x, G)$ or $\mathbf{R}_{DR}(X, x, G)$), and P denotes the associated flat principal bundle, then we obtain a Lie(G)-deformation diagram $(A_B(P), \varepsilon) = (A_{DR}(P), \varepsilon)$ where A^i are the spaces of forms with coefficients in Ad(P), the differential is again given by D, the Lie bracket comes from that of Lie(G), and the augmentation is given by evaluation at x.

The deformation theory associated to a deformation diagram

We recall the basic elements of the theory of Goldman and Millson—see [GM] for details. Let Art denote the category of artinian local schemes of finite type over Spec($\mathbf{C}$). An object $S \in$ Art is of the form $S = \mathrm{Spec}(\mathcal{O}_S)$ for a local $\mathbf{C}$-algebra $\mathcal{O}_S$ of finite length. Let $\mathbf{m}_S$ denote the maximal ideal of $\mathcal{O}_S$. Fix a Lie algebra $\mathbf{g}$ and let G be an algebraic group with Lie(G) $= \mathbf{g}$. Let $G^0(\mathbf{g}, S) \subset G(S)$ denote the set of S-valued points sending the closed point to the identity in G. The group $G^0(\mathbf{g}, S)$ depends only on $\mathbf{g}$, not on the choice of G. We have an exponential map from $\mathbf{g} \otimes_{\mathbf{C}} \mathbf{m}_S$ to $G^0(\mathbf{g}, S)$, denoted $u \mapsto e^u$, which is an isomorphism of sets. The formulas giving the group structure of $G^0(\mathbf{g}, S)$ in terms of the exponential isomorphism of sets are universal, applying also to the case of infinite dimensional Lie algebras.

Suppose (A, ε) is a deformation diagram. For $S \in$ Art, we obtain a group $G^0(A^0, S)$ with exponential map $A^0 \otimes_{\mathbf{C}} \mathbf{m}_S \to G^0(A^0, S)$. The Lie algebra A^0 acts on the A^i, and this gives an action of the group $G^0(A^0, S)$ on $A^i \otimes_{\mathbf{C}} \mathbf{m}_S$. We denote the composition of this action with the exponential map by $(u, a) \mapsto e^{-u} a e^u$. There is also an expression $e^{-u} d(e^u) \in A^1 \otimes_{\mathbf{C}} \mathbf{m}_S$. The formulas for these actions are the same as those that one calculates from the terminologies in the case of a finite dimensional Lie algebra.

Given a $\mathbf{g}$-deformation diagram (A, ε) and an artinian scheme $S \in$ Art, let $F(S, A, \varepsilon)$ denote the set of pairs (η, g) with $\eta \in A^1 \otimes_{\mathbf{C}} \mathbf{m}_S$ and $g \in G^0(S)$ such that

$$d(\eta) + \frac{1}{2}[\eta, \eta] = 0.$$

The group $G^0(A^0, S)$ acts on $F(S, A, \varepsilon)$ by the formula

$$e^u : (\eta, g) \mapsto (\mathrm{Ad}(e^{-u})\,\eta + e^{-u}\, d(e^u), e^{-\varepsilon(u)} g).$$

Let $R(S, A, \varepsilon)$ denote the quotient of the set $F(S, A, \varepsilon)$ by the action of $G^0(A^0, S)$.

Lemma **10.1.** — *Suppose the diagram* (A, ε) *is rigid and finite dimensional. Then the functor* $S \mapsto R(S, A, \varepsilon)$ *is pro-represented by a formal scheme* $R(A, \varepsilon)$.

Proof. — [GM]. □

Lemma **10.2.** — *Fix a linear algebraic group* G *and put* $\mathbf{g} = \mathrm{Lie}(G)$. *If* (P', b') *is a principal Higgs bundle of harmonic type (resp. a principal flat bundle) for the group* G, *with a point* $b' \in P'_x$, *then* $(A_{\mathrm{Dol}}(P), \varepsilon)$ *(resp.* $(A_{\mathrm{DR}}(E), \varepsilon)$*) is a rigid and finite dimensional* $\mathbf{g}$*-deformation diagram, and the formal scheme* $R(A, \varepsilon)$ *is naturally isomorphic to the formal completion of the representation space* $\mathbf{R}_{\mathrm{Dol}}(X, x, G)$ *(resp.* $\mathbf{R}_{\mathrm{DR}}(X, x, G)$ *or* $\mathbf{R}_{\mathrm{B}}(X, x, G)$*) at the point corresponding to* (P', b').

Proof. — This is a simple variant of one of the theorems of Goldman and Millson [GM]—theirs is the statement for the space $\mathbf{R}_{\mathrm{B}}(X, x, G)$. Note that the formal completions of $\mathbf{R}_{\mathrm{B}}(X, x, G)$ and $\mathbf{R}_{\mathrm{DR}}(X, x, G)$ at corresponding points are isomorphic, by the analytic isomorphism given in Theorem 7.1. This isomorphism is compatible with the equality of deformation diagrams. We may thus restrict our attention to the case of A_{Dol} and $\mathbf{R}_{\mathrm{Dol}}$.

Suppose H is a linear algebraic group and $N \subset H$ is a normal unipotent subgroup. Fix a principal Higgs bundle of semiharmonic type with frame (P', b') for the group H/N. Let Σ denote the set of triples (P, b, α) where P is a principal Higgs bundle of semiharmonic type for H, b is a frame, and $\alpha : (P, b) \times^{H} (H/N) \cong (P', b')$ is an isomorphism. Choose $(P_0, b_0, \alpha_0) \in \Sigma$. Since H acts on $\mathrm{Lie}(N)$, we obtain a Higgs bundle with Lie algebra structure $P_0 \times^{H} \mathrm{Lie}(N)$. Let $\bar\partial_0$ be the operator giving the holomorphic structure of P_0, and let φ_0 be the Higgs field. Let Σ' denote the set of pairs (u, η) with $u \in N$ and

$$\eta \in A^1(X, P_0 \times^{H} \mathrm{Lie}(N)), \qquad (\bar\partial_0 + \varphi_0)(\eta) + \frac{1}{2}[\eta, \eta] = 0,$$

up to equivalence under the action of $A^0(X, P_0 \times^{H} \mathrm{Lie}(N))$ given by the same formula as above. Then there is a natural isomorphism between Σ' and Σ. The principal Higgs bundle corresponding to (u, η) is $P_\eta = (P_0, \bar\partial_0 + \eta^{0,1}, \varphi_0 + \eta^{1,0})$, and the frame is $b_\eta = b_0 u$. The isomorphism stays the same, $\alpha_\eta = \alpha_0$. These constructions are functorial in terms of the pair (H, N).

Suppose G is a linear algebraic group and S is an artinian local scheme of finite type over $\mathrm{Spec}(\mathbf{C})$. Then we obtain a new group scheme $G(S)$ defined by setting $G(S)(T) = \mathrm{Hom}(S \times T, G)$. There is a morphism $G(S) \to G$, and the kernel $G^0(S)$ is a normal unipotent subgroup. There is a morphism of group schemes over S,

$$\zeta : G(S) \times S \to G \times S$$

equal to the identity in the second factor, and equal to the element of $\mathrm{Hom}(G(S) \times S, G)$ corresponding to the identity in $\mathrm{Hom}(G(S), G(S))$ in the first factor.

Let $p_1 : X \times S \to X$ be the projection on the first factor. If P_S is a principal object for the group G(S) on X, then $p_1^*(P_S)$ is a principal object for the group G(S) on $X \times S$ over S, and

$$\Phi(P_S) \stackrel{\text{def}}{=} p_1^*(P_S) \times^{G(S) \times S, \zeta} (G \times S)$$

is a principal object for the group G on $X \times S$ over S. There is a quasi-inverse: if P is a principal G-bundle on $X \times S$ then put $P(S)(T) = \mathrm{Hom}(S \times T, P)$; this is a principal G(S)-bundle over X with $P = F(P(S))$. If P has some extra structure then P(S) is provided with the same extra structure. Thus, the functor Φ gives an equivalence of categories between principal objects for the group G on $X \times S$, and principal objects for the group G(S) on X. Furthermore, the S-valued points of $\Phi(P_S)|_{\{x\} \times S}$ correspond to the points of $(P_S)_x$. This construction works for principal bundles, principal Higgs bundles, and principal bundles with integrable connection.

Denote by $s_0 \in S$ the closed point. The restriction of $\Phi(P_S)$ to $X \times \{ s_0 \}$ is naturally identified with $P_S \times^{G(S)} G$ (and this identification is compatible with the identification of the S-valued points above $\{ x \} \times S$ given above). Consequently, the construction Φ applied to principal Higgs bundles preserves the property of semiharmonic type.

Applying the previous construction with $H = G(S)$ and $N = G^0(S)$, we obtain a natural identification between: the set of triples (P, b, α) where P is a principal Higgs bundle of harmonic type on $X \times S$ over S, b is an S-valued point of $P|_{\{x\} \times S}$, and $\alpha : (P, b)|_{X \times \{s_0\}} \cong (P', b')$; and the set of elements of $R(S, A_{\mathrm{Dol}}, \varepsilon)$. We obtain an isomorphism of functors of artinian local $\mathbf{C}$-schemes of finite type, giving an isomorphism of formal schemes between $R(A_{\mathrm{Dol}}, \varepsilon)$ and the formal completion of $\mathbf{R}_{\mathrm{Dol}}(X, x, G)$ at (P', b'). □

Remark. — Under the isomorphism of functors given above, the G-orbit of (E, β) goes to the set of elements represented by $(0, g)$.

Remark. — Let $H \subset G$ be the stabilizer of (E, β). Then H acts on $\mathbf{R}_{\mathrm{Dol}}(G)$. Since $H \subset G$, it preserves the bundle $\mathrm{Ad}(P_0)$. Thus H acts by conjugation on the diagram $\mathbf{D}_{\mathrm{Dol}}(E, G)$ (the action on the Lie algebra $\mathbf{g}$ is also by conjugation). Thus H acts on the functor F and the representing formal scheme R(F). Our isomorphism is compatible with these actions of H.

A *morphism* f from a diagram $\mathbf{D}_1$ to a diagram $\mathbf{D}_2$ is a collection of f^i from $A^i(\mathbf{D}_1)$ to $A^i(\mathbf{D}_2)$, such that $f^i \delta_i = \delta_i f^{i-1}$, such that $f(ab) = f(a) f(b)$, and such that $\varepsilon f^0(s) = \varepsilon(s)$. Given such a morphism we get a map of functors $F(S, \mathbf{D}_1) \to F(S, \mathbf{D}_2)$, and hence a map $f : R(\mathbf{D}_1) \to R(\mathbf{D}_2)$.

We say that a morphism f is a *quasi-isomorphism* of diagrams if

$$f^0 : H^0(\mathbf{D}_1) \xrightarrow{\cong} H^0(\mathbf{D}_2)$$

$$f^1 : H^1(\mathbf{D}_1) \xrightarrow{\cong} H^1(\mathbf{D}_2)$$

are isomorphisms, and if

$$f^2 : \mathrm{H}^2(\mathbf{D}_1) \hookrightarrow \mathrm{H}^2(\mathbf{D}_2)$$

is injective.

The fundamental step in the theory of deformations of Goldman-Millson-Deligne-Schlessinger-Stasheff is the following statement [GM].

Proposition **10.3.** — *If $f : \mathbf{D}_1 \to \mathbf{D}_2$ is a quasi-isomorphism of rigid finite-dimensional diagrams then it induces an isomorphism of formal schemes $f : \mathrm{R}(\mathbf{D}_1) \xrightarrow{\cong} \mathrm{R}(\mathbf{D}_2)$.*

Proof. — [GM]. □

We will apply this by using the formality results from ([Si5] § 3). Suppose $(\mathrm{A}, \varepsilon)$ is a diagram where the differentials δ are zero. Let $\mathrm{C} \subset \mathrm{H}^1(\mathrm{A}^\bullet)$ be the quadratic cone which is the zero set of the map from $\mathrm{H}^1(\mathrm{A}^\bullet)$ to $\mathrm{H}^2(\mathrm{A}^\bullet)$ given by $\eta \mapsto \eta \wedge \eta$. Recall that $\mathbf{h}^\perp$ is the perpendicular space of the image of $\mathrm{H}^0(\mathrm{A}^\bullet)$ in $\mathbf{g}$. Goldman and Millson show that the formal scheme $\mathrm{R}(\mathrm{A}, \varepsilon)$ is equal to the formal completion of $\mathrm{C} \times \mathbf{h}^\perp$ at the origin [GM].

Theorem **10.4.** — *Let G be a reductive algebraic group. Suppose (P, p) is a point in a closed orbit in $\mathbf{R}_{\mathrm{Dol}}(\mathrm{X}, \mathrm{G})$ (resp. $\mathbf{R}_{\mathrm{DR}}(\mathrm{X}, \mathrm{G})$). Let C be the quadratic cone in $\mathrm{H}^1(\mathrm{Ad}\,\mathrm{P})$ defined by the map $\eta \mapsto \eta \wedge \eta \in \mathrm{H}^2(\mathrm{Ad}\,\mathrm{P})$. Let C denote the cone defined above for the formal deformation diagram $(\mathrm{A}_{\mathrm{H}}, \varepsilon)$ and, let $\mathbf{h}^\perp$ denote the perpendicular space to the image under β of $\mathrm{H}^0(\mathrm{Ad}\,\mathrm{P})$ in $\mathbf{g}$. Then the formal completion $(\mathbf{R}_{\mathrm{Dol}}(\mathrm{X}, \mathrm{G}), (\mathrm{P}, p))^\wedge$ (resp. $(\mathbf{R}_{\mathrm{DR}}(\mathrm{X}, \mathrm{G}), (\mathrm{P}, p))^\wedge$) is isomorphic to the formal completion $(\mathrm{C} \times \mathbf{h}^\perp, 0)^\wedge$.*

Proof. — Suppose (P, b) is a framed principal Higgs bundle of harmonic type. We get a Higgs bundle $\mathrm{Ad}(\mathrm{P})$ with Lie algebra structure. The Higgs bundle $\mathrm{Ad}(\mathrm{P})$ is a direct sum of stable Higgs bundles with vanishing Chern classes. By the results of [Si5], there is an operator D' on C^∞ $\mathrm{Ad}(\mathrm{P})$-valued forms. Let $(\mathrm{A}_{\mathrm{D}'}(\mathrm{P}), \varepsilon)$ be the diagram with A^i equal to the space of $\mathrm{Ad}(\mathrm{P})$-valued i-forms u such that $\mathrm{D}'(u) = 0$ (in the notation of [Si5]). The map δ is given by $\mathrm{D}'' = \bar\partial + \varphi$ or equivalently by $\mathrm{D} = \mathrm{D}' + \mathrm{D}''$. Let $(\mathrm{A}_{\mathrm{H}}(\mathrm{P}), \varepsilon)$ be the diagram with A^i equal to the space of harmonic forms, which is equal to $\mathrm{H}^i(\mathrm{A}^\bullet_{\mathrm{Dol}}(\mathrm{P}))$. Here the maps δ are zero, in other words $(\mathrm{A}_{\mathrm{H}}(\mathrm{P}), \varepsilon)$ is *formal*. Let $(\mathrm{A}_{\mathrm{DR}}(\mathrm{P}), \varepsilon)$ be the deformation diagram for the flat principal bundle (P, D) corresponding to the principal Higgs bundle P by the correspondence of [Si5]. We have natural morphisms

$$(\mathrm{A}_{\mathrm{D}'}(\mathrm{P}), \varepsilon) \to (\mathrm{A}_{\mathrm{Dol}}(\mathrm{P}), \varepsilon)$$

$$(\mathrm{A}_{\mathrm{D}'}(\mathrm{P}), \varepsilon) \to (\mathrm{A}_{\mathrm{DR}}(\mathrm{P}), \varepsilon)$$

$$(\mathrm{A}_{\mathrm{D}'}(\mathrm{P}), \varepsilon) \to (\mathrm{A}_{\mathrm{H}}(\mathrm{P}), \varepsilon).$$

By [Si5] Lemma 3.2, these are quasi-isomorphisms. By Proposition 10.3 we get isomorphisms of formal schemes

$$R(A_{D'}(P), \varepsilon) \cong R(A_{Dol}(P), \varepsilon)$$
$$R(A_{D'}(P), \varepsilon) \cong R(A_{DR}(P), \varepsilon)$$
$$R(A_{D'}(P), \varepsilon) \cong R(A_{H}(P), \varepsilon).$$

Finally, the formal scheme $R(A_H(P), \varepsilon)$ is isomorphic to the formal completion of the cone $(C \times \mathbf{h}^{\perp}, 0)^{\wedge}$. Now apply Lemma 10.2. □

Remark. — Following Goldman and Millson, we may apply the Artin approximation theorem [Ar] to conclude that the isomorphism of formal completions comes from an isomorphism of analytic or étale neighborhoods.

Remark. — The stabilizer H of (E, β) acts on all of the above spaces and diagrams, and in particular, H acts on the cone C and on $\mathbf{h}^{\perp}$. The quasi-isomorphisms of diagrams are compatible with the action of H. Therefore the isomorphism of formal neighborhoods is compatible with the action of H.

The cone C is affine and H acts linearly, so there is a good quotient C/H.

Proposition **10.5.** — *Suppose* P *is a principal harmonic bundle. The formal completion of the moduli space* $\mathbf{M}_{Dol}(X, G)$ *(resp.* $\mathbf{M}_{DR}(X, G)$, $\mathbf{M}_{B}(X, G)$*) at the point* P *is isomorphic to the formal completion of the good quotient* C/H *of the cone* C *by the action of* H.

Proof. — Apply Luna's étale slice theorem [Lu] to construct an H-stable subscheme $Y \subset \mathbf{R}_{Dol}(G)$ passing through (E, β), and such that the map $H^{\perp} \times Y \to \mathbf{R}_{Dol}(G)$ is locally an isomorphism in the analytic or étale topology. Here $H^{\perp}$ is an H-stable subspace of G passing through the identity, such that $H^{\perp} \times H \to G$ is locally an isomorphism. Now we have an isomorphism $(H^{\perp} \times Y)^{\wedge} \cong (H^{\perp} \times C)^{\wedge}$ of formal schemes, preserving the subscheme $(H^{\perp} \times \{0\})^{\wedge}$. From this we get projections $Y^{\wedge} \to C^{\wedge}$ and $C^{\wedge} \to Y^{\wedge}$. Their composition is a map $Y^{\wedge} \to Y^{\wedge}$ such that the scheme theoretic inverse image of the origin is just the origin. An argument of counting dimensions of the local ring modulo powers of the maximal ideal shows that this must be an isomorphism, so we get an isomorphism $Y^{\wedge} \cong C^{\wedge}$. This is compatible with the action of the group H. Let Y/H and C/H denote the good affine quotients. Now H is reductive, since it is the stabilizer of a point in a closed orbit [Lu], and because of this, we have $(Y/H)^{\wedge} = Y^{\wedge}/H$ and similarly for C. Thus $(Y/H)^{\wedge} \cong (C/H)^{\wedge}$. But Y/H is equal to the moduli space $\mathbf{M}_{Dol}(G)$, locally at E. Thus the formal completion of the moduli space is isomorphic to the formal completion of the affine quotient of the cone C by the action of H. □

We get canonical isomorphisms between the formal completions of the spaces $\mathbf{R}_{Dol}(X, x, G)$ and $\mathbf{R}_{DR}(X, x, G)$, or $\mathbf{M}_{Dol}(X, G)$ and $\mathbf{M}_{DR}(X, G)$, at points corresponding to the same harmonic bundle. These isomorphisms are not related to the identification between the sets of points given by the harmonic theory of [Si5].

Theorem **10.6** (Isosingularity). — *For any point* $y \in \mathbf{R}_{\mathrm{Dol}}(X, x, G)$ *(resp.* $y \in \mathbf{R}_{\mathrm{DR}}(X, x, G)$, $y \in \mathbf{M}_{\mathrm{Dol}}(X, x, G)$, *or* $y \in \mathbf{M}_{\mathrm{DR}}(X, x, G)$*) there exists a point* $z \in \mathbf{R}_{\mathrm{DR}}(X, x, G)$ *(resp.* $y \in \mathbf{R}_{\mathrm{Dol}}(X, x, G)$, $y \in \mathbf{M}_{\mathrm{DR}}(X, x, G)$, *or* $y \in \mathbf{M}_{\mathrm{Dol}}(X, x, G)$*) and étale neighborhoods* U *of* y *and* V *of* z *such that* $(U, y) \cong (V, z)$*; and the local systems corresponding to* y *and* z *have isomorphic semisimplifications.*

Proof. — By the Artin approximation theorem [Ar], it suffices to show that the formal completions at y and z are isomorphic. Suppose first of all that y lies in a closed orbit, so it corresponds to a reductive representation. Then let z be a point in the other space corresponding to the same reductive representation ρ. Then there are natural isomorphisms of cohomology rings $H^{*}_{\mathrm{Dol}}(X, \mathrm{Ad}(\rho)) \cong H^{*}_{\mathrm{DR}}(X, \mathrm{Ad}(\rho))$ [Si5]. Thus the cones that appear in Theorem 10.4 for y and z are isomorphic. The automorphism groups H are also the same in both cases. The formal completions of the representation spaces are both isomorphic to the formal completion of the cone $C \times \mathbf{g}^{\perp}$, and the formal completions of the moduli spaces are isomorphic to C/H. Suppose y does not lie in a closed orbit. Let y' denote a point in the closed orbit adhering to the orbit of y. There exists a point z' in the other representation space, and isomorphic étale neighborhoods U' of y' and V' of z'. There is a point $y_1 \in U'$, mapping to a point in the orbit of y. In particular, the formal completion of U' at y_1 is isomorphic to the formal completion of the representation space at y. Let z_1 denote the point corresponding to y_1 under the isomorphism $U' \cong V'$, and let z denote the image in the other representation space of z_1. The formal completion of V' at z_1 is isomorphic to the formal completion of U' at y_1, so the formal completion of the representation spaces at y and z are isomorphic. We may suppose that y' is in the closure of the orbit Hy_1, so z' is in the closure of the orbit Hz_1. In particular, the closed orbits adhering to the orbits of y and z correspond to the same reductive representations. □

Remark. — If $y \in \mathbf{R}_{\mathrm{DR}}(X, G)$ (resp. $y \in \mathbf{M}_{\mathrm{DR}}(X, G)$) and if z denotes the corresponding point in $\mathbf{R}_{\mathrm{B}}(X, G)$ (resp. $\mathbf{M}_{\mathrm{B}}(X, G)$) then there are étale neighborhoods U of y and V of z, and isomorphisms $(U, y) \cong (V, z)$. This follows directly from the Artin approximation theorem, since the analytic isomorphism of Theorem 7.1 gives an isomorphism of formal neighborhoods.

The Zariski tangent space

We give a result valid for any representation, not necessarily reductive.

Lemma **10.7.** — *Suppose* $(P, p) \in \mathbf{R}_{\mathrm{DR}}(X, G)$. *Then the dimension of the Zariski tangent space to* $\mathbf{R}_{\mathrm{DR}}(X, G)$ *at* (P, p) *is equal to* $h^1_{\mathrm{DR}}(X, \mathrm{Ad}(P)) + \dim(\mathbf{g}) - h^0_{\mathrm{DR}}(X, \mathrm{Ad}(P))$. *The same for* $\mathbf{R}_{\mathrm{Dol}}(X, G)$.

Proof. — Let **D** be the de Rham or Dolbeault deformation diagram corresponding to (P, p). The Zariski tangent space of the representation space is equal to

$R(\mathrm{Spec}(\mathbf{C}[t]/t)^2, \mathbf{D})$. The set $F(\mathrm{Spec}(\mathbf{C}[t]/t^2), \mathbf{D})$ is equal to the set of pairs (ε, g) where $g \in \mathbf{g}$ and $\eta \in A^1$ with $d(\eta) = 0$. The action of $G^0(A^0, \mathrm{Spec}(\mathbf{C}[t]/t^2))$ amounts to changing (η, g) by adding $(d(s), \varepsilon(s))$ for $s \in A^0$. The quotient by this action is $H^1 \oplus (\mathbf{g}/\varepsilon(H^0))$. □

11. Representations of the fundamental group of a Riemann surface

Theorem **11.1.** — *If* X *is a connected smooth projective curve of genus* $g \geqslant 2$, *then the moduli spaces* $\mathbf{M}_B(X, n)$, $\mathbf{M}_{DR}(X, n)$, $\mathbf{M}_{Dol}(X, n)$, *and the representation spaces* $\mathbf{R}_B(X, n)$, $\mathbf{R}_{DR}(X, n)$, *and* $\mathbf{R}_{Dol}(X, n)$, *are normal irreducible varieties.*

Most of the rest of the section is devoted to the proof. First we prove that the schemes are reduced and normal. Note that a normal connected variety is irreducible, so for the second statement it suffices to prove connectedness, which we do afterward. At the end of the section, we give some auxiliary statements about the local structure of the representation space. These were obtained in my original proof of the theorem; they are no longer needed in the present proof but it seemed like a good idea to record them anyway.

Normality

The idea for this part of the proof was suggested by M. Larsen (cf. Corollary 11.6 below). Suppose X is a connected smooth projective curve of genus $g \geqslant 2$. Choose a basepoint $x \in X$.

Lemma **11.2.** — *Every irreducible component of* $\mathbf{R}_B(X, x, n)$ *has dimension greater than or equal to* $2gn^2 - n^2 + 1$. *The Zariski open subset* $\mathbf{R}^s_B(X, x, n)$ *parametrizing irreducible representations is smooth of dimension* $2gn^2 - n^2 + 1$.

Proof. — First note that $\mathbf{R}_B(X, x, n)$ is the subvariety of $\mathrm{Gl}(n, \mathbf{C})^{2g}$ defined by one relation. The relation is a map

$$R : \mathrm{Gl}(n, \mathbf{C})^{2g} \to \mathrm{Sl}(n, \mathbf{C}),$$

and $\mathbf{R}_B(X, x, n) = R^{-1}(e)$. This implies that every irreducible component of $\mathbf{R}_B(X, x, n)$ has dimension $\geqslant 2gn^2 - n^2 + 1$. Suppose ρ is a point in $\mathbf{R}^s_B(X, x, n)$. Let V denote the local system corresponding to ρ and let $\mathrm{Ad}(\rho)$ denote the local system $\mathrm{End}(V)$. Then

$$\mathrm{Tr} : H^i(X, \mathrm{Ad}(\rho)) \to H^i(X, \mathbf{C})$$

are isomorphisms for $i = 0$ and $i = 2$. On the other hand, if $\eta \in H^1(X, \mathrm{Ad}(\rho))$ then $\mathrm{Tr}([\eta, \eta]) = 0$. Therefore the cone C which appears in Theorem 10.4 is equal to all of $H^1(X, \mathrm{Ad}(\rho))$. By Theorems 7.1 and 10.4, $\mathbf{R}_B(X, x, n)$ is smooth at ρ. Finally, the rank of $\mathrm{Ad}(\rho)$ is n^2 so a calculation of Euler characteristics gives

$$\dim(C) = \dim(H^1) = (2g - 2)\, n^2 + 2.$$

In the notation of the previous section, $\dim(\mathbf{h}^\perp) = n^2 - 1$, so the dimension of $\mathbf{R}_B(X, x, n)$ at ρ is $2gn^2 - n^2 + 1$. □

Proposition **11.3.** — *The dimension of any irreducible component of* $\mathbf{R}_B(X, x, n)$ *is equal to* $2gn^2 - n + 1$, *all irreducible components are generically smooth, and* $\mathbf{R}_B(X, x, n)$ *is a complete intersection. The dimension of the subspace of reducible representations has codimension at least two, except in the case* $g = 2$ *and* $n = 2$ *when it has codimension at one.*

Proof. — We suppose that the proposition is known for any $n' < n$. We will prove the proposition for representations of rank n.

For $1 \leqslant k < n$, let P_k denote the parabolic subgroup of $\mathrm{Gl}(n, \mathbf{C})$ consisting of block-upper triangular matrices with 2 blocks, where the first block has size k and the second block has size $n - k$. There is an exact sequence

$$0 \to \mathbf{C}^{k(n-k)} \to P_k \to \mathrm{Gl}(k, \mathbf{C}) \times \mathrm{Gl}(n-k, \mathbf{C}) \to 1,$$

where the kernel represents the abelian group of block upper triangular matrices with the identity matrix in the diagonal blocks.

For each P_k, let $G_k = \mathrm{Gl}(n, \mathbf{C})/P_k$. It is the Grassmanian of k-planes in $\mathbf{C}^n$, with dimension $k(n-k)$. Choose a constructible section $\varphi : G_k \to \mathrm{Gl}(n, \mathbf{C})$. Let $\mathbf{R}_B(X, x, P_k)$ denote the space of representations of $\pi_1(X)$ into P_k. We obtain a constructible family of representations of $\pi_1(X, x)$ into P_k indexed by $G_k \times \mathbf{R}_B(X, x, P_k)$, corresponding to the constructible map $\sigma : G_k \times \mathbf{R}_B(X, x, P_k) \to \mathbf{R}_B(X, x, \mathrm{Gl}(n))$ defined by $\sigma(y, \rho) = \varphi(y)\, \rho \varphi(y)^{-1}$. This has the property that $\sigma(y, \rho)$ is a representation of $\pi_1(X)$ into the conjugate $y P_k y^{-1}$ (this conjugate doesn't depend on the choice of lifting $\varphi(y)$). Let $\mathbf{R}_B^{\mathrm{red}}(X, x, \mathrm{Gl}(n, \mathbf{C}))$ denote the space of reducible representations. Since every reducible representation has a fixed subspace and is therefore conjugate to a representation in some P_k, we have

$$\bigcup_{1 \leqslant k < n} \sigma(G_k \times \mathbf{R}_B(X, x, P_k)) = \mathbf{R}_B^{\mathrm{red}}(X, x, \mathrm{Gl}(n, \mathbf{C})).$$

In particular, the dimension of $\mathbf{R}_B^{\mathrm{red}}(X, x, \mathrm{Gl}(n, \mathbf{C}))$ is bounded by the maximum of the dimensions of $G_k \times \mathbf{R}_B(X, x, P_k)$.

Lemma **11.4.** — *Suppose that Proposition* 11.3 *is known for representations of rank* $n' < n$. *Then for any* $1 \leqslant k < n$, *the dimension of* $G_k \times \mathbf{R}_B(X, x, P_k)$ *is less than or equal to* $2gn^2 - n^2$; *and if* $g \geqslant 3$ *or* $n \geqslant 3$ *then the dimension is less than or equal to* $2gn^2 - n^2 - 1$.

Proof. — We count dimensions, looking at the morphism

$$\mathbf{R}_B(X, x, P_k) \to \mathbf{R}_B(X, x, \mathrm{Gl}(k, \mathbf{C})) \times \mathbf{R}_B(X, x, \mathrm{Gl}(n-k, \mathbf{C}))$$

which associates to a representation ρ its diagonal parts (ρ_1, ρ_2). We would like to know the dimension of the space of representations into P_k which have given diagonal part (ρ_1, ρ_2). Let $\gamma_1, \ldots, \gamma_{2g}$ denote the standard generators of the fundamental group of X, and let $r(\gamma_1, \ldots, \gamma_{2g})$ denote the relation. If we fill in the diagonal parts of the matrices $\rho(\gamma_i)$ according to the given representations ρ_1, ρ_2, then to specify the remaining part of the representation we have to choose a vector $(A_1, \ldots, A_{2g})$ with each A_i in the kernel $\mathbf{C}^{k(n-k)}$ of the above exact sequence. Putting the resulting matrices into the relation gives a map

$$r_0 : \mathbf{C}^{k(n-k)} \times \ldots \times \mathbf{C}^{k(n-k)} \to \mathbf{C}^{k(n-k)}.$$

The kernel of this map is the fiber over (ρ_1, ρ_2), in other words the space of representations with diagonal parts ρ_1, ρ_2. This is the last part of a complex calculating the group cohomology of $\pi_1(X, x)$ with coefficients in the vector space $\mathbf{C}^{k(n-k)}$, so the cokernel of the map is $H^2(\pi_1(X, x), \mathbf{C}^{k(n-k)})$. The action of $\pi_1(X, x)$ on the vector space of coefficients comes from the adjoint action on $\mathrm{Lie}(P_k)$ using the representation ρ. This only depends on ρ_1 and ρ_2. More explicitly it can be seen by expressing $\mathbf{C}^{k(n-k)} = \mathbf{C}^k \otimes \mathbf{C}^{n-k}$, with action on $\mathbf{C}^k$ given by ρ_1 and the action on $\mathbf{C}^{n-k}$ given by ρ_2. By Poincaré duality, the dimension of the H^2 is the same as the dimension of $H^0(\pi_1(X, x), \rho_1^* \otimes \rho_2)$. Thus the dimension of the fiber over (ρ_1, ρ_2) is

$$(2g - 1)\, k(n - k) + h^0(\pi_1(X), \rho_1^* \otimes \rho_2).$$

If ρ_1 and ρ_2 are irreducible and not isomorphic, then $H^0(\pi_1(X, x), \rho_1^* \otimes \rho_2) = 0$. Therefore we can count the dimension of the fiber over (ρ_1, ρ_2) as $(2g - 1)\, k(n - k)$. By induction, the dimension of the space of choices of (ρ_1, ρ_2) is $(2g - 1)\,(k^2 + (n - k)^2) + 2$. The dimension of this part of $G_k \times \mathbf{R}_B(X, x, P_k)$ is

$$\begin{aligned}(2g - 1)\,(k^2 + (n - k)^2 + k(n - k)) + 2 + k(n - k)\\ = (2g - 1)\, n^2 + 1 - ((2g - 2)\, k(n - k) - 1).\end{aligned}$$

In particular, as $g \geqslant 2$ and $n \geqslant 2$, the dimension is at most $2gn^2 - n^2$. If $g \geqslant 3$ or $n \geqslant 3$ then the dimension is at most $2gn^2 - n^2 - 2$.

The set of pairs (ρ_1, ρ_2) such that both representations are reducible has (by induction) dimension bounded by $(2g - 1)\,(k^2 + (n - k)^2)$. For these points we make a coarse counting of the dimension of the fiber over (ρ_1, ρ_2): it is less than $2gk(n - k)$. The dimension of this part of $G_k \times \mathbf{R}_B(X, x, P_k)$ is therefore bounded by

$$\begin{aligned}(2g - 1)\,(k^2 + (n - k)^2) + (2g + 1)\, k(n - k)\\ = (2g - 1)\, n^2 + 1 - ((2g - 3)\, k(n - k) + 1).\end{aligned}$$

The dimension is at most $2gn^2 - n^2 - 1$ for $g \geqslant 2$ and $n \geqslant 2$.

The set of pairs (ρ_1, ρ_2) which are irreducible and isomorphic (hence of rank $k = n - k = n/2$) has dimension less than or equal to $(2g - 1)\, k^2 + 1$. The H^0 has dimension 1, so the fiber has dimension $(2g-1)k^2+1$. The sum of the dimensions is less than or equal to $(2g-1)n^2/2+2$. For $n \geqslant 2$ and $g \geqslant 2$ this is less than or equal to $2gn^2-n^2-4$.

The set of pairs (ρ_1, ρ_2) such that one representation is reducible and one representation is irreducible has dimension bounded by $(2g-1)(k^2+(n-k)^2)+1$. For such a pair, the H^0 discussed above has dimension 0 or 1. Therefore the dimension of the fiber over (ρ_1, ρ_2) is bounded by $(2g-1)k(n-k)+1$. The dimension of this part of the space $G_k \times \mathbf{R}_B(X, x, P_k)$ is bounded by

$$(2g-1)(k^2+(n-k)^2+k(n-k))+2+k(n-k)$$
$$=(2g-1)n^2+1-((2g-2)k(n-k)-1).$$

In this case, note that n must be at least 3. Therefore, the dimension is at most $2gn^2-n^2-2$.

We have shown, in all the cases, that the dimension of $G_k \times \mathbf{R}_B(X, x, P_k)$ is less than or equal to $2gn^2-n^2$, and if $n \geqslant 3$ or $g \geqslant 3$ then the dimension is less than or equal to $2gn^2-n^2-1$. □

We continue with the proof of the proposition. It follows from the lemma that the dimension of the subspace of reducible representations is less than or equal to $(2g-1)n^2$. From the lower bound of Lemma 11.2, no irreducible component can consist entirely of reducible representations. Therefore the open set of irreducible representations is Zariski dense, so the dimension of each component is equal to $(2g-1)n^2+1$. It follows from the equations for $\mathbf{R}_B(X, x, n)$ given in the proof of Lemma 11.2 that $\mathbf{R}_B(X, x, n)$ is a complete intersection. Since $\mathbf{R}_B^s(X, x, n)$ is smooth, each component is generically smooth. Finally note that, except in the case $g=2$ and $n=2$, Lemma 11.4 shows that the dimension of the subspace of reducible representations is less than or equal to $(2g-1)n^2-1$. This proves the proposition. □

Lemma **11.5.** — *The scheme* $\mathbf{R}_B(X, x, n)$ *is smooth outside of a subset of codimension* $\geqslant 2$.

Proof. — This follows from the previous proposition except when $g=2$ and $n=2$. We are reduced to that case, where the space of representations has dimension 13. From the proof of Lemma 11.4, the codimension 1 part of the locus of reducible representations consists of those representations conjugate to an upper triangular representation with distinct diagonal parts. We show that the space of representations is smooth at such points. This statement is invariant under conjugating the representation; so we may fix an upper triangular representation ρ, with diagonal entries $\rho_1 \neq \rho_2$. The space of semisimple reducible representations has dimension 10, so we may assume that ρ is not semisimple. For a 2×2 representation, this implies that there is a unique subrepresentation of rank 1 and a unique quotient. Make the convention that the subrepresentation is ρ_1 and the quotient is ρ_2.

We claim that there are no nonscalar endomorphisms of the representation ρ.

For if f is an endomorphism of rank 1, then the image of f must be the subrepresentation ρ_1, while the coimage must be the quotient ρ_2, contradicting the condition that $\rho_1 \neq \rho_2$. Thus if λ is an eigenvalue of an endomorphism f, then $f - \lambda = 0$ and f is a scalar. Thus, as claimed, $H^0(X, Ad(\rho)) = \{0\}$ (we say that ρ is *simple*).

By Lemma 10.7, the dimension of the Zariski tangent space to $\mathbf{R}_B(X, x, n)$ at ρ is equal to $h^1(Ad(\rho)) + n^2 - h^0(Ad(\rho))$. Since ρ is simple, $h^0(Ad(\rho)) = 1$. By Poincaré duality, $h^2(Ad(\rho)) = 1$, so

$$h^1(Ad(\rho)) = (2g - 2)\, n^2 + 2 = 10,$$

and the dimension of the Zariski tangent space is $10 + 4 - 1 = 13$. Since this is equal to the dimension of any irreducible component of the space, the local ring is regular and $\mathbf{R}_B(X, x, n)$ is smooth at ρ. This completes the proof of the lemma. □

Corollary **11.6.** — *The space of representations* $\mathbf{R}_B(X, x, n)$ *is reduced and normal.*

Proof. — This was pointed out to me by M. Larsen (he refered me to [Ha], Proposition II-8.23). By Proposition 11.3, $\mathbf{R}_B(X, x, n)$ is a complete intersection. The local rings of a complete intersection are Cohen-Macaulay, hence satisfy Serre's condition S_2. The previous lemma shows that the space of representations is regular in codimension 1. By Serre's criterion, $\mathbf{R}_B(X, x, n)$ is reduced and normal. □

Corollary **11.7.** — *The representation spaces* $\mathbf{R}_{Dol}(X, x, n)$, $\mathbf{R}_{DR}(X, x, n)$, *and* $\mathbf{R}_B(X, x, n)$ *are normal varieties of dimension* $2gn^2 - n^2 + 1$. *The moduli spaces* $\mathbf{M}_{Dol}$, $\mathbf{M}_{DR}$, *and* $\mathbf{M}_B$ *are normal varieties of dimension* $2gn^2 - 2n^2 + 2$.

Proof. — We have shown that $\mathbf{R}_B(X, x, n)$ is normal of dimension $2gn^2 - n^2 + 1$. By the isosingularity principle (Theorem 10.6 and the following remark), the same is true for the de Rham and Dolbeault spaces. Good quotients of normal varieties are normal. (This can be seen by proving that if A is a ring which is integrally closed in its field of fractions K and a group acts, then A^G is integrally closed in its field of fractions K^G.) Thus the moduli spaces are normal varieties. To calculate their dimensions, note that there is a Zariski dense open set of points of the representation space where $Sl(n, \mathbf{C})$ acts with finite stabilizer. The dimension of the quotient is the dimension of the representation space minus the dimension of $Sl(n, \mathbf{C})$. □

This corollary provides the first half of the proof of Theorem 11.1. To complete the proof, it suffices to prove that these varieties are connected. Connectedness of the representation spaces is equivalent to connectedness of their universal categorical quotients, the moduli spaces. By Proposition 7.8 and Theorem 7.18, the three moduli spaces are homeomorphic. Thus it suffices to prove that $\mathbf{M}_{Dol}(X, n)$ is connected.

The idea for the proof of connectedness comes from Hitchin's calculation of the cohomology of the moduli space of rank 2 projective bundles with odd degree. In that case, the moduli space is smooth, and Hitchin uses a Morse function, the moment map for the action of S^1, to calculate the cohomology. The lowest stratum is the space of unitary representations, known to be connected by the work of Narasimhan and Seshadri using a lemma of Atiyah. Hitchin deduces from Morse theory that the moduli space is connected and hence irreducible, and in fact he calculates the Betti numbers. It would be good to carry through this program for higher ranks and for the case when there are singularities. We will not attempt this here, but we will do enough to show that the moduli space is connected. Because of the presence of singularities, we will avoid Morse theory and instead proceed by algebraic geometry, using the $\mathbf{C}^*$ action discussed in § 6 (which is the complexification of Hitchin's S^1 action). The relationship between these approaches is that the critical point set of the moment map is equal to the fixed point set of the $\mathbf{C}^*$ action. In ([Si5] § 4), the fixed point set was identified with the set of complex variations of Hodge structure. We remark that in order to apply Hitchin's method to compute the Betti numbers, one would have to be able to compute the Betti numbers of the moduli spaces of variations of Hodge structure.

Actions of $\mathbf{C}^*$

Suppose that Y is a quasiprojective variety on which $\mathbf{C}^*$ acts algebraically. Suppose that L is an ample line bundle with a linearization of the action. Then $\mathbf{C}^*$ acts locally finitely on $H^0(Y, L^{\otimes n})$. Therefore we may choose n and a subspace $V \subset H^0(Y, L^{\otimes n})$ which is preserved by $\mathbf{C}^*$ and which embedds Y into the projective space $\mathbf{P}(V^*)$, so that $\mathbf{C}^*$ acts on this projective space and acts compatibly on the very ample $\mathcal{O}(1)$. This action is compatible with the embedding of Y and with the isomorphism $L^{\otimes n} \cong \mathcal{O}_Y(1)$.

Write $V = \bigoplus V_\alpha$ where the sum is over integers α, and $t \in \mathbf{C}^*$ acts by t^α on V_α. Then the fixed point sets of $\mathbf{C}^*$ on $\mathbf{P}(V^*)$ are the subspaces $\mathbf{P}(V_\alpha^*)$. The fixed point sets in Y are $Y_\alpha = Y \cap \mathbf{P}(V_\alpha^*)$. The action of $t \in \mathbf{C}^*$ on $L|_{Y_\alpha}$ is given by t^α. Let Z be the closure of Y in $\mathbf{P}(V^*)$. Then Z is preserved by $\mathbf{C}^*$, and its fixed point sets are $Z_\alpha = Z \cap \mathbf{P}(V_\alpha^*)$.

If $z \in Z$ then there are unique points $\lim_{t\to 0} tz$ and $\lim_{t\to\infty} tz$ in Z. These are fixed points, hence are in some fixed points set Z_{β^0} and Z_{β^∞} respectively. We will describe the weights $\beta^0(z)$ and $\beta^\infty(z)$ explicitly (see the discussion near the end of § 1, Part I). Lift z to a point $w \in V^*$, and write $w = \sum w_\alpha$ with $w_\alpha \in V_\alpha^*$. Then $\beta^0(z)$ (resp. $\beta^\infty(z)$) is the smallest (resp. largest) integer α such that $w_\alpha \neq 0$.

From this description, if $z \in Z$ then $\beta^0(z) \leqslant \beta^\infty(z)$ and equality holds if and only if z is a fixed point.

Assume that Y has the property that $\lim_{t\to 0} tx$ exists in Y for all $x \in Y$. Let $\beta = \beta^0(Y)$ be the smallest integer such that Y_β is nonempty. Then β is also the smallest integer such that $Z_\beta \neq \emptyset$. If $x \in Y_\alpha$ and there exists $y \in Y$ such that $y \neq x$ and $\lim_{t\to\infty} ty = x$, then $\alpha > \beta$. In particular, we obtain the following criterion.

Lemma **11.8.** — *Suppose that* Y *has the property that* $\lim_{t\to 0} tx$ *exists in* Y *for all* $x \in Y$. *Suppose* $U \subset Y$ *is a connected subset of the fixed point set of* $\mathbf{C}^*$, *and suppose that for any fixed point* x *not in* U, *there exists* $y \neq x$ *in* Y *such that* $\lim_{t\to\infty} ty = x$. *Then* Y *is connected.*

Proof. — Suppose Y' is a connected component of Y not containing U. Let $\beta = \beta^0(Y')$. Choose $x \in Y'_\beta$. By hypothesis there exists $y \neq x$ in Y' such that $\lim_{t\to\infty} ty = x$. On the other hand, $z = \lim_{t\to 0} ty$ is also in Y', say in Y'_α. But y is not a fixed point, so $\alpha < \beta$, contradicting minimality of β. □

Connectedness

Lemma **11.9.** — *Suppose* E *is a stable Higgs bundle of degree zero, fixed up to isomorphism by the action of* $\mathbf{C}^*$. *Suppose that* $\varphi \neq 0$. *Then there is a Higgs bundle* F *not isomorphic to* E, *such that* $\lim_{t\to\infty} tF = E$ *in the moduli space.*

Proof. — Since E is a fixed point, it has the structure of system of Hodge bundles, in other words $E = \bigoplus E^p$ with $\varphi : E^p \to E^{p-1} \otimes \Omega^1_X$. Assume that the indexing is normalized so that $0 \leqslant p \leqslant r$, and $E^0 \neq 0$ and $E^r \neq 0$. Note that $r \geqslant 1$ since $\varphi \neq 0$. Furthermore, note that $\deg(E^0) < 0$ and $\deg(E^r) > 0$, since E is stable of degree zero. Hence $\mathrm{Hom}(E^r, E^0)$ has degree < 0. In particular the Riemann-Roch theorem implies that there exists a nonzero extension class η in $\mathrm{Ext}^1(E^r, E^0) = H^1(\mathrm{Hom}(E^r, E^0))$. For each $t \in \mathbf{C}$, let M_t be the extension

$$0 \to E^0 \to M_t \to E^r \to 0$$

given by the class $t^r \eta$. Let F_t be the Higgs bundle $M_t \oplus \bigoplus_{0<p<r} E^p$. The Higgs field φ is given by the usual maps for $1 < p < r$ and by the compositions

$$E^1 \to E^0 \otimes \Omega^1_X \to M_t \otimes \Omega^1_X$$

and

$$M_t \to E^r \to E^{r-1} \otimes \Omega^1_X.$$

Note that $F_0 = E$. We have isomorphisms $\varphi_t : F_t \cong t^{-1} F_1$ given as follows. On E^p, $0 < p < r$, φ_t is given by multiplication by t^p. On M, φ_t is the isomorphism fitting into the middle of the diagram

$$\begin{array}{ccccc} E^0 & \longrightarrow & M_t & \longrightarrow & E^r \\ {\scriptstyle 1}\downarrow & & {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle t^r} \\ E^0 & \longrightarrow & M_1 & \longrightarrow & E^r. \end{array}$$

Note that $\varphi\theta = t^{-1}\theta\varphi$. Thus we have a family of Higgs bundles F_t with $F_0 = E$ and $F_t = t^{-1} F_1$ for $t \neq 0$. Since E is stable, so are F_t, by the openness of the condition of stability. Hence $\lim_{t-\infty} tF_1 = E$. To complete the verification, we will show that the vector bundles underlying E and F_1 are not isomorphic.

First we show that $M_1 \neq M_0$. Let $A \subset E^r$ be the β-subsheaf, in other words the subsheaf of highest slope, and highest rank among subsheaves of that slope. We may choose η to be a nontrivial extension of A by E^0. Note that the degree of any subsheaf of E^0 is < 0, whereas the slope of A is > 0. Thus if $f : A \hookrightarrow M$ is an inclusion, we must have $p \circ f : A \cong A$ where p is the projection from M to E^r. Thus $(p \circ f)^*(\eta) \neq 0$. But the map f is a splitting of $(p \circ f)^*(\eta)$. This contradiction shows that there is no inclusion of A into M_1, and hence there can be no isomorphism $M_0 \cong M_1$.

The bundle M_1 is a deformation of M_0. By semicontinuity,

$$h^0(\mathrm{Hom}(M_0, M_0)) \geqslant h^0(\mathrm{Hom}(M_0, M_1)).$$

Furthermore, the inequality is strict: for if not then $t \mapsto H^0(\mathrm{Hom}(M_0, M_t))$ would form a vector bundle over the t-line, and $1 \in H^0(\mathrm{Hom}(M_0, M_0))$ could be lifted to $f_t \in H^0(\mathrm{Hom}(M_0, M_t))$ with $f_t \to 1$ as $t \to 0$; then $f_t : M_0 \cong M_t = M_1$ for t near 0, which would contradict the conclusion of the previous paragraph. Now in our situation there exists a vector bundle B (the direct sum of the other Hodge components) such that $E = M_0 \oplus B$ and $F = M_1 \oplus B$. We get

$$h^0(\mathrm{Hom}(M_0, E)) = h^0(\mathrm{Hom}(M_0, M_0)) + h^0(\mathrm{Hom}(M_0, B)),$$

while
$$h^0(\mathrm{Hom}(M_0, F_1)) = h^0(\mathrm{Hom}(M_0, M_1)) + h^0(\mathrm{Hom}(M_0, B)).$$

Thus
$$h^0(\mathrm{Hom}(M_0, E)) > h^0(\mathrm{Hom}(M_0, F_1)),$$

so the vector bundles E and F_1 are not isomorphic. □

Corollary **11.10.** — *The moduli space* $\mathbf{M}_{\mathrm{Dol}}(X, n)$ *is connected.*

Proof. — The ample line bundle L on $\mathbf{M}_{\mathrm{Dol}}(X, n)$ has a linearization of the action of $\mathbf{C}^*$. This is because we constructed $\mathbf{M}_{\mathrm{Dol}}(X, n)$ as the moduli space of some sheaves on the cotangent bundle to X, and the action of $\mathbf{C}^*$ came from the action of multiplication on $T^* X$, so $\mathbf{C}^*$ acts functorially on the Hilbert schemes, the Grassmanians, and the line bundles over the Grassmanians. We apply the criterion of Lemma 11.8.

Let U be the subspace corresponding to Higgs bundles with $\theta = 0$ (these are the ones corresponding to unitary representations). It is isomorphic to the moduli space of vector bundles of rank n on X. The moduli space of vector bundles is projective, so U is a closed subset of $\mathbf{M}_{\mathrm{Dol}}(X, n)$. The subset U is connected—this fact was used by Narasimhan and Seshadri [NS] and comes from a lemma of Atiyah.

Suppose that E is a direct sum of stable components, representing a point in $\mathbf{M}_{\mathrm{Dol}}(X, n) - U$ fixed by $\mathbf{C}^*$. All of the stable components of E are then fixed by $\mathbf{C}^*$. We can write $E = E_1 \oplus E_2$ with E_1 stable and not unitary. Apply Lemma 11.9 to obtain F_1 with $\lim_{t \to \infty} F_1 = E_1$, and $F_1 \neq E_1$. Set $F = F_1 \oplus E_2$. Then $\lim_{t \to \infty} F = E$, but $\mathrm{gr}(F) \neq \mathrm{gr}(E)$. The criterion of Lemma 11.8 now implies that $\mathbf{M}_{\mathrm{Dol}}(X, n)$ is connected. □

Proof of Theorem **11.1.** — From the homeomorphism given by Proposition 7.8 and Theorem 7.18, the varieties $\mathbf{M}_{\mathrm{B}}(X, n)$ and $\mathbf{M}_{\mathrm{DR}}(X, n)$ are also connected. By Corollary 11.7, all the moduli varieties are normal, and a normal connected variety is irreducible. □

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