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TOPOLOGICAL QUANTUM FIELD THEORIES

by Michael ATIYAH

To René Thom on his 65th birthday.

1. Introduction

In recent years there has been a remarkable renaissance in the relation between Geometry and Physics. This relation involves the most advanced and sophisticated ideas on each side and appears to be extremely deep. The traditional links between the two subjects, as embodied for example in Einstein's Theory of General Relativity or in Maxwell's Equations for Electro-Magnetism are concerned essentially with classical fields of force, governed by differential equations, and their geometrical interpretation. The new feature of present developments is that links are being established between *quantum* physics and *topology*. It is no longer the purely *local* aspects that are involved but their *global* counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background. However, the realization that this vague philosophical view-point could be translated into reasonably precise and significant mathematical statements is mainly due to the efforts of Edward Witten who, in a variety of directions, has shown the insight that can be derived by examining the topological aspects of quantum field theories.

The best starting point is undoubtedly Witten's paper [11] where he explained the geometric meaning of super-symmetry. It is well-known that the quantum Hamiltonian corresponding to a classical particle moving on a Riemannian manifold is just the Laplace-Beltrami operator. Witten pointed out that, for super-symmetric quantum mechanics, the Hamiltonian is just the Hodge-Laplacian. In this super-symmetric theory differential forms are bosons or fermions depending on the parity of their degrees. Witten went on to introduce a modified Hodge-Laplacian, depending on a real-valued function f. He was then able to derive the Morse theory (relating critical points of f to the Betti numbers of the manifold) by using the standard limiting procedures relating the quantum and classical theories.

With this model of super-symmetric quantum mechanics rigorously understood, Witten then went on to outline the corresponding ideas for super-symmetric *quantum field theories*. Essentially such quantum field theories should be viewed as the differential geometry of certain infinite-dimensional manifolds, including the associated analysis (e.g. Hodge theory) and topology (e.g. Betti numbers).

Great caution has of course to be used in such infinite-dimensional situations but, taking one's cue from physics on the one hand and topology on the other hand, it is possible to make intelligent guesses and conjectures. There is now ample evidence in favour of many of these conjectures, a number of which have been rigorously established by alternative methods. This applies for example to results concerning " elliptic cohomology " [17] and to the topic I shall discuss in detail in this paper.

Perhaps a few further comments should be made to reassure the sceptical reader. The quantum field theories of interest are inherently *non-linear*, but the non-linearities have a natural origin, e.g. coming from non-abelian Lie groups. Moreover there is usually some scaling or coupling parameter in the theory which in the limit relates to the classical theory. Fundamental topological aspects of such a quantum field theory should be independent of the parameters and it is therefore reasonable to expect them to be computable (in some sense) by examining the classical limit. This means that such topological information is essentially robust and should be independent of the fine analytical details (and difficulties) of the full quantum theory. That is why it is not unreasonable to expect to understand these topological aspects before the quantum field theories have been shown to exist as rigorous mathematical structures. In fact, it may well be that such topological understanding is a necessary pre-requisite to building the analytical apparatus of the quantum theory.

My comments so far have been of a conventional kind, indicating that there may be interesting topological aspects of quantum field theories and that these should be important for the relevant physics. However, we can reverse the procedure and use these quantum field theories as a conceptual tool to suggest new mathematical results. It is remarkable that this reverse process appears to be extremely successful and has led to spectacular progress in our understanding of *geometry in low dimensions*. It is probably no accident that the usual quantum field theories can only be renormalized in (space-time) dimensions $\leqslant 4$, and this is precisely the range in which difficult phenomena arise leading to deep and beautiful theories (e.g. the works of Thurston in 3 dimensions and Donaldson in 4 dimensions).

It now seems clear that the way to investigate the subtleties of low-dimensional manifolds is to associate to them suitable infinite-dimensional manifolds (e.g. spaces of connections) and to study these by standard linear methods (homology, etc.). In other words we use quantum field theory as a refined tool to study low-dimensional manifolds.

Now quantum field theories have, because of the difficulties involved in constructing them, often been described axiomatically. This identifies their essential structural features and postpones the question of their existence. We can apply the same approach, at the topological level, and this leads us to formulate axioms for *topological quantum field theories*. These axioms are considerably simpler than for a full blown theory and they have a certain naturality which makes them plausible objects of interest, independent of any physical interpretation.

In the next section I will therefore present a set of such axioms. Although I will make a few comments on the physical background and notation, these can be ignored and the axioms treated as a basis for a rigorous mathematical theory.

In the third section I will enumerate the examples of theories, satisfying such axioms, which are now known to exist. Much, though not all, of this has been rigorously established by one method or another. The history of these different theories is quite varied so it is certainly helpful to see them all as fitting into a common axiomatic framework.

It will be clear how much this whole subject rests on the ideas of Witten. In formulating the axiomatic framework in § 2, I have also been following Graeme Segal who produced a very similar approach to conformal field theories [10]. Finally it seems appropriate to point out the major role that *cobordism* plays in these theories. Thus René Thom's most celebrated contribution to geometry has now a new and deeper relevance.

2. Axioms for Topological Quantum Field Theories

Before embarking on the axioms it may be helpful to make a few comparisons with standard homology theories. We can describe such a theory as a functor F from the category of topological spaces (or of pairs of spaces) to the category say of Λ-modules, where Λ is some fixed ground ring (commutative, with 1, e.g. $\Lambda = \mathbf{Z}, \mathbf{R}$ or $\mathbf{C}$). This functor satisfies various axioms including

(i) a *homotopy* axiom, described geometrically by using " cylinders " $X \times I$,

(ii) an *additive* axiom asserting that, for disjoint sums, $F(X_1 \cup X_2) = F(X_1) \oplus F(X_2)$.

Note that (ii) implies, for the empty set $\varnothing$,

(ii)′ $F(\varnothing) = 0$.

The theories we shall describe will be somewhat similar, but with the following significant differences:

a) they will be defined only for *manifolds* of *a fixed dimension*,

b) the homotopy axiom is *strengthened* by replacing cylinders with general *cobordisms*,

c) the additive axiom is replaced by a *multiplicative* axiom, and correspondingly the empty set has value Λ rather than 0.

Physically *b)* is related to *relativistic* invariance while *c)* is indicative of the *quantum* nature of the theory.

We come now to the promised axioms. A topological quantum field theory (QFT), in dimension d defined over a ground ring Λ, consists of the following data:

(A) A finitely generated Λ-module $Z(\Sigma)$ associated to each oriented closed smooth d-dimensional manifold Σ,

(B) An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth $(d+1)$-dimensional manifold (with boundary) M.

These data are subject to the following axioms, which we state briefly and expand upon below:

(1) Z is *functorial* with respect to orientation preserving diffeomorphisms of Σ and M,

(2) Z is *involutory*, i.e. $Z(\Sigma^*) = Z(\Sigma)^*$ where Σ^* is Σ with opposite orientation and $Z(\Sigma)^*$ denotes the dual module (see below),

(3) Z is *multiplicative*.

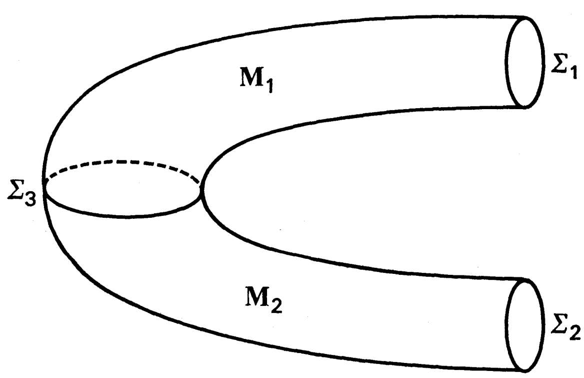
We now elaborate on the precise meaning of the axioms. (1) means first that an orientation preserving diffeomorphism $f: \Sigma \to \Sigma'$ induces an isomorphism $Z(f): Z(\Sigma) \to Z(\Sigma')$ and that $Z(gf) = Z(g)\,Z(f)$ for $g: \Sigma' \to \Sigma''$. Also if f extends to an orientation preserving diffeomorphism $M \to M'$, with $\partial M = \Sigma$, $\partial M' = \Sigma'$, then $Z(f)$ takes $Z(M)$ to $Z(M')$.

The meaning of (2) is clear when Λ is a *field* in which case $Z(\Sigma)$ and $Z(\Sigma)^*$ are dual vector spaces. This is the most important case and, for physical examples $\Lambda = \mathbf{C}$ (or perhaps $\mathbf{R}$). However, there are interesting examples (see § 3) with $\Lambda = \mathbf{Z}$. In this case the relation between $Z(\Sigma)$ and $Z(\Sigma^*)$ is like that between integer homology and cohomology. The duality can be formalized by considering free chain complexes but we shall not pursue this in detail. Instead *we shall take* Λ *to be a field* and the case $\Lambda = \mathbf{Z}$ can essentially be replaced by the fields $\mathbf{Q}$, $\mathbf{Z}_p$.

The multiplicative axiom (3) asserts first that, for disjoint unions,

(3a) $$Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2).$$

Moreover if $\partial M_1 = \Sigma_1 \cup \Sigma_3$, $\partial M_2 = \Sigma_2 \cup \Sigma_3^*$ and $M = M_1 \cup_{\Sigma_3} M_2$ is the manifold obtained by glueing together the common Σ_3-component (see figure)



then we require:

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle \tag{3b}$$

where $\langle\ ,\ \rangle$ denotes the natural pairing

$$Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2).$$

Note that when $\Sigma_3 = \varnothing$ so that M is the disjoint union of M_1 and M_2 then (3*b*) reduces to the obvious multiplicative requirement

$$Z(M) = Z(M_1) \otimes Z(M_2). \tag{3c}$$

Our multiplicative axiom, involving (3*b*), is therefore a very strong axiom. It asserts that Z(M) can be computed (in many different ways) by "cutting M in half" along any Σ_3.

An equivalent way of formulating (3*b*) is to decompose the boundary M into two components (possibly empty) so that

$$\partial M = \Sigma_1 \cup \Sigma_0^*;$$

then $Z(M) \in Z(\Sigma_0)^* \otimes Z(\Sigma_1) = \mathrm{Hom}(Z(\Sigma_0), Z(\Sigma_1))$. We can therefore view any *cobordism* M between Σ_0 and Σ_1 as inducing a linear transformation

$$Z(M) : Z(\Sigma_0) \to Z(\Sigma_1).$$

Axiom (3*b*) asserts that this is *transitive* when we compose bordisms.

Note that the multiplicative axiom (3*a*) shows that when $\Sigma = \varnothing$, the vector space $Z(\Sigma)$ is idempotent. It is therefore zero or canonically isomorphic to the ground field Λ. To exclude the trivial case we should then add a non-triviality axiom

$$Z(\varnothing) = \Lambda \text{ (for } \varnothing \text{ the empty } d\text{-dimensional manifold).} \tag{4a}$$

Similarly when $M = \varnothing$ axiom (3*b*) shows that $Z(\varnothing) \in \Lambda$ is an idempotent. To exclude the trivial case $Z(\varnothing) = 0$ we impose

$$Z(\varnothing) = 1 \text{ (for } \varnothing \text{ the empty } d + 1\text{-dimensional manifold).} \tag{4b}$$

Again the multiplicative axiom in its cobordism guise clearly shows that, for a cylinder $\Sigma \times I$, the element

$$Z(\Sigma \times I) \in \mathrm{End}(Z(\Sigma))$$

is an idempotent σ and more generally it acts as the identity on the subspace of $Z(\Sigma)$ spanned by all elements $Z(M)$ with $\partial M = \Sigma$. If we replace $Z(\Sigma)$ by its image under the idempotent σ it is easy to see that the axioms are still satisfied. Moreover, this new theory contains essentially all the interesting information of the old one since elements in the kernel of σ play no real role. Thus it is reasonable to assume $\sigma = 1$, i.e. to impose a further non-triviality axiom:

$$Z(\Sigma \times I) \text{ is the identity.} \tag{4c}$$

Let us now deduce a few elementary consequences of our axioms. Apply axiom (1) with $M = M' = \Sigma \times I$ and

$$F : M \to M'$$

a homotopy f_t of maps $\Sigma \to \Sigma$. We deduce the

homotopy invariance of $Z(f) : Z(\Sigma) \to Z(\Sigma)$.

This implies therefore that the group $\Gamma(\Sigma)$ of components of the orientation preserving diffeomorphisms of Σ acts on $Z(\Sigma)$.

Next let us note that when M is a *closed* $(d+1)$-dimensional manifold so that $\partial M = \varnothing$, then

$$Z(M) \in Z(\varnothing) = \Lambda$$

is a constant (element of the ground field). Thus the theory produces in particular *invariants* of closed $(d+1)$-manifolds. Moreover if we cut M along Σ into two parts M_1, M_2 so that we get two vectors

$$Z(M_1) \in Z(\Sigma),$$
$$Z(M_2) \in Z(\Sigma)^*,$$

then, as a special case of the multiplicative axiom (3*b*), we get

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle.$$

Thus the invariant for a closed manifold can be computed in terms of such a decomposition.

If we view $Z(M)$, for closed M, as a numerical invariant of M, then for a manifold with boundary we should think of $Z(M) \in Z(\partial M)$ as a " relative " invariant. The whole theory is then concerned with these invariants and their formal properties.

If we form the product manifold $\Sigma \times S^1$ by identifying opposite ends of the cylinder $\Sigma \times I$ then our axioms imply that

$$\begin{aligned} Z(\Sigma \times S^1) &= \mathrm{Trace}(\mathrm{Id} \mid Z(\Sigma \times I)) \\ &= \dim Z(\Sigma \times I). \end{aligned}$$

More generally let $f : \Sigma \times \Sigma$ be an orientation preserving diffeomorphism, and identify opposite ends of $\Sigma \times I$ by f. This gives a manifold Σ_f and our axioms imply

$$Z(\Sigma_f) = \mathrm{Trace}\, Z(f)$$

where $Z(f)$ is the induced automorphism of $Z(\Sigma)$.

The observant reader will have noticed that our involutory axiom refers only to reversing the orientation of the d-dimensional manifolds Σ. Nothing has been said so far about the effect of orientation reversal on the $(d+1)$-dimensional manifolds M. In particular our axioms give no relation as yet between the invariants $Z(M)$ and $Z(M^*)$ for closed manifolds. Such a relation may or may not exist depending on the theory (as we shall see in § 3). However, in many interesting cases there is such a relation. This depends on the *additional assumption* that our vector spaces $Z(\Sigma)$ have a non-degenerate *hermitian* structure (relative to a conjugation on Λ). This gives a natural isomorphism

$$Z(\Sigma)^* \to \overline{Z(\Sigma)}$$

where $\overline{V}$ denotes V with the conjugate action of Λ. We can now consider the extra *hermitian axiom*

(5) $$Z(M^*) = \overline{Z(M)}.$$

If $\partial M = \Sigma_0^* \cup \Sigma_1$ so that $Z(M)$ can be viewed as a linear transformation between hermitian vector spaces:

$$Z(M) : Z(\Sigma_0) \to Z(\Sigma_1),$$

then axiom (5) is equivalent to

(5′) $Z(M^*)$ is the adjoint of $Z(M)$.

In particular for a *closed* manifold M (5) asserts that the numerical invariant $Z(M)$ changes to its conjugate under orientation reversal. Unless all values are real (fixed by conjugation) then these invariants can " detect " orientation.

Note that for a manifold M with boundary Σ we can always form the *double* $M \cup_\Sigma M^*$ which is a closed manifold. Axiom (5) shows that

$$Z(M \cup_\Sigma M^*) = |\, Z(M)|^2$$

where on the right we compute the norm in the hermitian (possibly indefinite) metric.

These axioms can be modified in various important ways and this is necessary for many of the interesting examples. Let me briefly indicate the kind of modification that can be incorporated.

(1) the vector spaces $Z(\Sigma)$ may be mod 2 graded with appropriate signs then inserted,
(2) the manifolds Σ, M may carry more structure, e.g. a spin structure, a framing or some distinguished homology classes,
(3) one can consider a " relative " theory for a pair $X_{d-r} \subset \Sigma_d$ consisting of Σ and a submanifold X. This will have to couple together topological QFTs in dimensions d and $d - r$,
(4) we might want to allow Σ_d to have a boundary: this is closely related to (3) above with $r = 1$.

A more serious modification is

(5) allow $Z(\Sigma)$ to be infinite-dimensional.

The axioms in this case certainly need significant changes. For example certain quantities become infinite (e.g. $Z(\Sigma \times S^1) = \dim Z(\Sigma)$).

This list is not meant to be exhaustive particularly if we move away from requiring our QFT to be strictly topological. For example conformal field theories are clearly related to the ideas here (see § 3).

So far in this section we have deliberately refrained from attempting to describe the physical interpretation or terminology (except for the acronym QFT). This was meant to emphasize the mathematical nature of the presentation in order to encourage mathematicians to take these ideas seriously. However, we should now rectify the situation by briefly indicating the physical background.

Σ is meant to indicate the physical space (e.g. $d = 3$ for standard physics) and the extra dimension in $\Sigma \times I$ is "imaginary" time. The space $Z(\Sigma)$ is the Hilbert space of the quantum theory and a physical theory, with a Hamiltonian H, will have an evolution operator e^{itH} or an "imaginary time" evolution operator e^{-tH}. The main feature of *topological* QFTs is that $H = 0$, which implies that there is no real dynamics or propagation, along the cylinder $\Sigma \times I$. However, there can be non-trivial "propagation" (or tunneling amplitudes) from Σ_0 to Σ_1 through an intervening manifold M with $\partial M = \Sigma_0^* \cup \Sigma_1$: this reflects the topology of M.

If $\partial M = \Sigma$, then the distinguished vector $Z(M)$ in the Hilbert space $Z(\Sigma)$ is thought of as the *vacuum state* defined by M. For a closed manifold M the number $Z(M)$ is the vacuum-vacuum expectation value. In analogy with statistical mechanics it is also called the partition function.

The reader may wonder how a theory with zero Hamiltonian can be sensibly formulated. The answer lies in the Feynman path-integral approach to QFT. This incorporates relativistic invariance (which caters for general $(d + 1)$-dimensional "space-times") and the theory is formally defined by writing down a suitable Lagrangian—a functional of the classical fields of the theory. A Lagrangian which involves only first derivatives in time formally leads to a zero Hamiltonian, but the Lagrangian itself may have non-trivial features which relate it to the topology.

For a fuller understanding of topological QFTs from the physical view-point the reader should consult the papers of Witten on the subject.

3. Examples

We shall list a number of interesting examples of topological QFTs in dimensions $d \leqslant 3$. The description will inevitably be brief and it should be emphasized that there are many points that have yet to be fully investigated in some of these theories. Some parts exist in a fully rigorous mathematical sense while other parts have as yet only been treated formally. Nevertheless the general picture is very convincing and there seems little doubt that the essential features are correct.

Naturally the theories in general increase in difficulty with the dimension d. We shall begin with the apparently trivial case of $d = 0$ and progress up to $d = 3$.

(3.0) $d = 0$

Space Σ now consists of finitely many (say n) points. To a single point we must associate a vector space $V = Z$ (point), and to n points we associate the n-fold tensor product: $V^{\otimes n} = V \otimes V \otimes \ldots \otimes V$. The symmetric group S_n (diffeomorphisms of n points) then acts on $V^{\otimes n}$. Thus the classical theory of representations of S_n appears as a basic ingredient in theories for $d = 0$.

The question now arises as to the origin of the vector space V, the Hilbert space of the quantum theory. A standard way to get the quantum Hilbert space is first to give a classical symplectic manifold (or phase space) and then to quantize this. In particular an interesting class of examples arise from *compact Lie groups* G and their homogeneous symplectic manifolds; these are co-adjoint orbits, generically copies of the flag manifold. If we take " integral " orbits for which the symplectic structure comes from a line-bundle, then quantizing leads to the irreducible representations V of G. This is the physical interpretation of the Borel-Weil theorem which is usually formulated in algebraic-geometric language. The Lagrangian of these theories is the classical action (holonomy of the line-bundle).

Thus topological QFTs with $d = 0$ arise naturally in relation to classical representation theory of Lie groups and the symmetric group. In this low dimension there is no interesting topology, only quantum symmetries.

(3.1) $d = 1$

There are two rather different types of theory in this dimension, both of which are linked to the Lie group theories in dimension zero. We describe these in turn.

a) Floer/Gromov theory

Here the classical phase space consists of paths, in a compact symplectic manifold X, with appropriate boundary conditions. To fit the formal framework we set up in § 2 we should consider periodic boundary conditions given by closed loops in X. Holonomy round such loops, used in (3.0) as a Lagrangian, is now used to modify the Hamiltonian as in Witten [11]. For a closed surface M the invariant Z(M) of the theory is the number of pseudo-holomorphic maps $M \to X$ in the sense of Gromov [5] (e.g. they are ordinary holomorphic maps if X is a Kähler manifold). If this number is infinite, i.e. if there are " moduli ", then we must fix further data on M. This can be done by picking some points P_i and then looking at holomorphic maps $f: M \to X$ with $f(P_i)$ constrained to lie on a fixed hyperplane.

Witten [14] has written down the relevant Lagrangian for this theory. Floer [3] has given a rigorous treatment, based on Witten's Morse theory ideas, for the case when the boundary conditions are not periodic but instead require the initial and end-points of paths to lie on two fixed Lagrangian sub-manifolds. This is a case when Σ is an interval, rather than a circle, and is a modification of the type (4) above. The Floer theory is naturally Mod 2 graded and is defined over the integers.

b) Holomorphic Conformal Field Theories

These are not strictly *topological* QFTs in our sense since they depend on a complex structure, and the Hilbert spaces are infinite-dimensional. However, they are closely related to topological QFTs. They have been axiomatized by G. B. Segal [10] and, as mentioned in § 1, his axioms motivated our version.

There are conformal field theories related to compact Lie groups G in which the classical phase space consists of co-adjoint orbits of (a central extension of) the loop group LG. Quantizing these produces the Hilbert spaces of the theory as irreducible (projective) representations of LG. The whole theory is very similar to that in (3.0). The group $\mathrm{Diff}_+(S^1)$ now substitutes for the symmetric group and plays an important role (see [9] for details). The partition function in such theories depends on complex structures: it is not purely topological.

(3.2) $d = 2$

There are a number of similar theories in this dimension. We begin with perhaps the most interesting and well-developed theory.

a) Jones/Witten theory [12]

Here the classical phase space, associated to a closed surface Σ, is the moduli space of flat G-bundles over Σ. The Lagrangian is an integer multiple of the Chern-Simons function of a G-connection on a 3-manifold (which has to be " framed "). The integer multiple k, called the level, is a parameter of the theory and $k \to \infty$ gives the classical limit.

This theory can be naturally coupled with the $d = 0$ theory to produce a " relative " theory of the type indicated at the end of § 2. The details have been developed by Witten who shows that the partition function for a (framed) link in the 3-sphere is just the value of the Jones polynomials [8] for a suitable root of unity. The theory can be defined over the relevant cyclotomic field.

Instead of coupling the $d = 2$ theory to $d = 0$ we can couple it to the $d = 1$ conformal theory in *b)* above, by considering Riemann surfaces with boundary.

b) Casson theory

Here the " Hilbert " spaces of the theory are essentially the *homology* of the moduli spaces of flat G-bundles over Σ (whereas in the Jones/Witten theory one takes a holomorphic analogue). The theory is mod 2 graded and the hermitian forms are indefinite (being given by Poincaré duality). Witten has recently written down a Lagrangian for this Casson theory. The invariant Z, for homology 3-spheres is the original Casson invariant. The theory is defined over the integers. As yet there are details, related to the singularities of the moduli-spaces, which have not been fully worked out.

c) Johnson theory

Recently D. Johnson [7] has developed a theory imitating Casson, but using Reidemeister torsion as an essential ingredient. Witten [16] has shown that Johnson's theory can be explained as a topological QFT for a Chern-Simons Lagrangian. However, the group involved is not a compact Lie group G but the semi-direct product IG of G with its Lie algebra (viewed simply as abelian group). The classical phase space is the moduli space of flat IG-bundles and is not compact, so that the Hilbert spaces of this theory are not finite-dimensional. Thus the partition function may sometimes be infinite.

d) " Thurston " theory

Witten has recently considered [15] a Chern-Simons theory for the non-compact group SL(2, **C**) in relation with gravitational theories in $(2 + 1)$-dimensions, and hence with hyperbolic 3-manifolds. This should make contact with Thurston's work in due course. The classical phase space is again the (non-compact) moduli space of flat SL(2, **C**) bundles over a surface Σ. The Hilbert spaces are again infinite-dimensional.

Remark. — Moduli spaces of flat SL(2, **C**) bundles (and their generalizations) have been considered in a remarkable paper by N. J. Hitchin [6]. Hitchin shows that this moduli space is naturally fibered by *Abelian varieties* and that the moduli space of flat SU(2)-bundles appears as a component of the most degenerate fibre. On the basis of Hitchin's paper it seems likely that all the theories described in this sub-section can, in a sense, be reduced to the abelian case. This is now being investigated and should have major consequences.

(3.3) $d = 3$ *(Floer/Donaldson theory)*

Donaldson [2] has defined integer invariants of smooth 4-manifolds by using moduli spaces of SU(2)-instantons. These invariants are polynomials on the second homology. Thus we should consider 4-manifolds with extra data (consisting of an element of the symmetric algebra of H_2). Witten [13] has produced a super-symmetric Lagrangian which formally reproduces the Donaldson theory. Witten's formula can be understood as an infinite-dimensional analogue of the Gauss-Bonnet theorem (as I shall explain elsewhere).

The Hamiltonian version of the theory has been developed by Floer [4] in terms of the space of connections on a 3-manifold. Floer uses the Chern-Simons function (the Lagrangian of the Jones/Witten theory) to modify the Hamiltonian (see the remarks in (3.1) *a)*). The abelian groups he defines have a mod 2 grading.

I have described this Floer/Donaldson theory at greater length in [1]. Here I just wish to emphasize that it formally fits some version of the axioms of § 2 and that there is a Lagrangian formulation.

This theory is defined over the integers. It does not satisfy any axiom of type (5) of § 2. The Donaldson invariants of a 4-manifold with its two different orientations have no obvious relation to each other. Instantons and anti-instantons are quite different, particularly when the signature of the 4-manifold is non-zero.

Witten [14] has also shown how one can couple the $d = 3$ and $d = 1$ theories together: this is quite analogous to the coupling between $d = 2$ and $d = 0$ in the Jones theory.

Conclusion. — These examples, which have natural geometric origins, and cover many of the most interesting topics in low-dimensional geometry show that topological QFTs have real relevance to geometry. There are still many technical problems to be solved (e.g. the Casson invariant has so far only been treated for $G = SU(2)$) and there are many intriguing questions. For example the Chern-Simons function appears in both $d = 2$ and $d = 3$ theories, but playing quite a different role each time. What is the significance of this?

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